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Super Ostrowski homotopy continuation method for solving polynomial system of equations

[1]**Hafizudin Mohamad Nor,** [2]**Amirah Rahman,** [3]**Ahmad Izani Md. Ismail and** [4]**Ahmad Abd. Majid**
[1,2,3,4]Universiti Sains Malaysia
School of Mathematical Sciences, 11800 Gelugor, Penang.
e-mail: [1]hafiz97my2001@yahoo.com, [2]amirahr@usm.my, [3]izani@cs.usm.my, [4]majid@cs.usm.my

Abstract Homotopy continuation methods (HCMs) can be used to find the solutions of polynomial equations. The advantages of HCMs over classical methods such as the Newton and bisection methods are that HCMs are able to resolve divergence and starting value problems. In this paper, we develop Super Ostrowski-HCM as a technique to overcome the starting value problem. We compare the performance of this proposed method with Ostrowski-HCM. The results provide evidence of the superiority of Super Ostrowski-HCM.

Keywords Numerical method; Polynomial equations; Ostrowski homotopy continuation method.

2010 Mathematics Subject Classification 65H05; 65H10; 65H20.

1 Introduction

One difficulty associated with solving systems of scalar nonlinear equations numerically is the choice of starting value. Lee and Chiang [1] stated that a good starting point is hard to find. The user should have a sufficient knowledge of the location of roots to determine an appropriate starting value. An inappropriate starting value will cause the numerical method used to diverge or converge slowly. Consider the solution of a system of polynomial equations

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}, \tag{1}$$

where $\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_n(\mathbf{x}))^t$ and $\mathbf{x} = (x_1, x_2, ..., x_n).f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_n(\mathbf{x})$ are called coordinate functions of $\mathbf{F}(\mathbf{x})$[2]. Several classical methods can be employed to solve a system of polynomial equations. They include Newton's method, Steepest Descent method [2], Broyden's method [3,4,5] and Brent's method [6,7]. However, the aforementioned methods require starting values that must be close to the intended root for a particular application.

Homotopy continuation methods are classified as global methods. Gritton et al. [8] defined global methods as those that use an arbitrary starting value to find a solution. By choosing an arbitrary starting value, one need not worry about the location of actual roots before performing the computations. The question arises, is that we can choose all numbers to be the starting value without having any problem. In this paper, we investigate the issue of starting value with regard to homotopy continuation method (HCM).

2 Starting value problem of Ostrowski-HCM

Burden and Faires [2] defined starting value problem for nth-order system of first-order differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(t, x_1, x_2, x_3, ..., x_n),\\ \frac{dx_2}{dt} &= f_2(t, x_1, x_2, x_3, ..., x_n),\\ \frac{dx_3}{dt} &= f_3(t, x_1, x_2, x_3, ..., x_n),\\ &\vdots\\ \frac{dx_n}{dt} &= f_n(t, x_1, x_2, x_3, ..., x_n),\end{aligned} \tag{2}$$

for $a \leq t \leq b$ can be defined as the solution to the differential equations that satisfies a set of given initial condition $\mathbf{x}(a) = \tau$. Chapra and Canale [9] stated n conditions are required when dealing with nth-order differential equations to obtain a unique solution $\mathbf{x}$. When the differential equations (2) are reduced to zeroth-order, we are dealing with the solution of a system of polynomial equations (1). Therefore, starting value problem of polynomial equations can be defined as a unique solution $\mathbf{x}$ to a system of polynomial equations that satisfies a given starting value $\mathbf{x}_0$.

Noor and Waseem [10], Nikkhah-Bahrami and Oftadeh [11], Faires and Burden [12], Kotsireas [13] and Nor *et al.* [14] focused on starting value which is reasonably close to a root. However, there is still not yet investigation of large of starting value which are somewhat far away from a root.

Ostrowski-HCM was developed by using Ostrowski's method as a basis. Ostrowski's method was introduced by Alexander Markowich Ostrowski [15,16] by extending Newton's method. The variants of Ostrowski's method were widely developed such as in Kou *et al.* [17], Grau *et al.* [18], Sharma and Guha [19] and Chun and Ham [20] for solving scalar nonlinear equations. Ostrowski [15] and Grau-Sanchez *et al.* [21] focused on systems of nonlinear equations. The formula for Ostrowski's method is:

$$\begin{aligned}\mathbf{y}_i &= \mathbf{x}_i - A_i^{-1}\mathbf{F}(\mathbf{x}_i),\\ \mathbf{x}_{i+1} &= \mathbf{y}_i - A_i^{-1}\mathbf{O}_j(\mathbf{x}_i),\end{aligned} \qquad i = 0, 1, 2, ..., k-1, \quad j = 1, 2, ..., n. \tag{3}$$

where A_i is the Jacobian matrix of $\frac{\partial f_j}{\partial x_k}$ evaluated at point $\mathbf{x}$ and $\mathbf{O}_j(\mathbf{x}) = \frac{f_j(\mathbf{x}_i)f_j(\mathbf{y}_i)}{f_j(\mathbf{x}_i)-2f_j(\mathbf{y}_i)}$. The additional Ostrowski's function can also be written as

$$\mathbf{O}_j(\mathbf{x}_i) = \begin{pmatrix} \dfrac{f_1(\mathbf{x}_i)f_1(\mathbf{y}_i)}{f_1(\mathbf{x}_i) - 2f_1(\mathbf{y}_i)} \\ \dfrac{f_2(\mathbf{x}_i)f_2(\mathbf{y}_i)}{f_2(\mathbf{x}_i) - 2f_2(\mathbf{y}_i)} \\ \vdots \\ \dfrac{f_n(\mathbf{x}_i)f_n(\mathbf{y}_i)}{f_n(\mathbf{x}_i) - 2f_n(\mathbf{y}_i)} \end{pmatrix}, \quad i = 0, 1, 2, ..., k-1, \quad j = 1, 2, ..., n. \tag{4}$$

Equation (3) is also known as double-Newton's method [22, 23]. We now consider the starting value problem of Ostrowski-HCM. The stopping criterion used is $\|\mathbf{F}(\mathbf{x}_k)\|_\infty < \varepsilon$, where $\varepsilon = 10^{-3}$.

Example 2.1 Consider the following system of polynomial equations in Kotsireas [13]:

$$\begin{aligned} f_1(x,y) &= x^2 + y^2 - 25 = 0, \\ f_2(x,y) &= x^2 - y - 5 = 0, \end{aligned} \tag{5}$$

where the exact solutions are (x_1, y_1) = (3,4), (x_2, y_2) = (-3,4) and $(x_3, y_3) = (0, -5)$. Kotsireas [13] used the parametric plot to illustrate the location of exact solutions. Graphically, it can be shown in Figure 1.

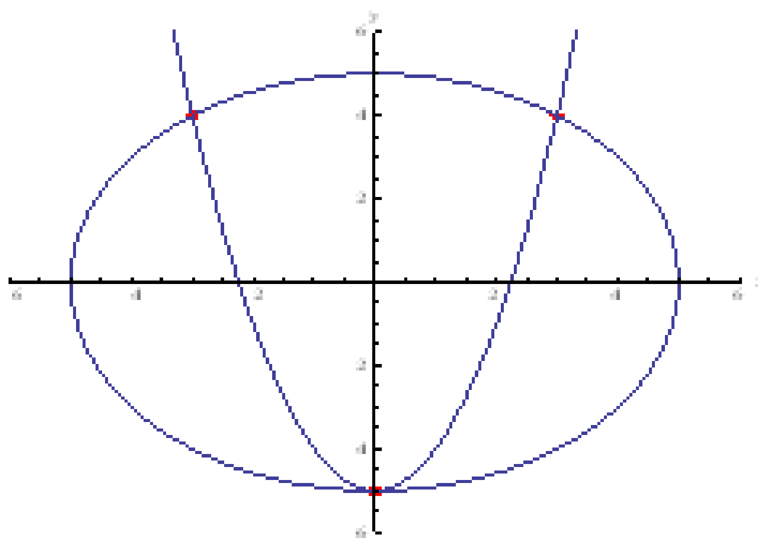

Figure 1: Location of exact solutions for equation (5) using parametric plot [13]

The intersection points between two curves are the location of exact solutions. Palancz *et al.* [24] used homotopy path to illustrate the location of roots. Graphically, it can be shown in Figure 2. We now observe the performance of Ostrowski-HCM and the results are shown in Tables 1 to 3 by varying the value of the starting values.

Homotopy path moves from the starting points (blue) to the real solutions (red). The solutions can be tracked by using only one initial value which is located in the complex domain.

Tables 1 to 3 show the performance of Ostrowski-HCM when starting from three different starting values: (1) the starting value is close to exact solutions, (2) the starting value which is far from exact solutions, and (3) the starting value is very far from exact solutions.

Table 1 indicates that Ostrowski-HCM performs well if the starting value is close to the exact solutions. However, Table 2 and Table 3 indicate that Ostrowski-HCM needs 35 iterations and 331 iterations to converge to the exact solutions when it starts from $(x_0, y_0) = (-100, -100)$ and $(x_0, y_0) = (1000, 1000)$ respectively. The performance of Ostrowski-HCM

Table 1: Performance of Ostrowski-HCM for equation (5) with starting value $(x_0, y_0) = (2.5, 3.5)$

Iterations	t	Approximate solution $(\tilde{x}, \tilde{y})$	$\mathbf{F}(\tilde{x}, \tilde{y})$
0	0	$(2.5, 3.5)$	$(-6.5, -2.25)$
1	0.5	(2.91568,4.01067)	$(-0.413327, -0.509495)$
2	1	(3,4.00001)	$(0.0000506821, -3.07095 \times 10^{-6})$

Table 2: Performance of Ostrowski-HCM for equation (5) with starting value $(x_0, y_0) = (-100, -100)$

Iterations	t	Approximate solution $(\tilde{x}, \tilde{y})$	$\mathbf{F}(\tilde{x}, \tilde{y})$
0	0	$(-100, -100)$	$(19975, 10095)$
1	1/35	$(18.8485, -21.4375)$	(789.831,371.703)
2	2/35	$(19.4408, -42.8368)$	(2187.94,415.783)
3	3/35	$(-17.2683, 106.036)$	(11516.9,187.159)
4	4/35	$(-46.2786, -32.1872)$	(3152.73,2168.9)
⋮	⋮	⋮	⋮
34	34/35	$(0.0556136, -4.97184)$	$(-0.277745, -0.0250702)$
35	1	$(0.0209382, -5)$	(0.000397552,0.000434321)

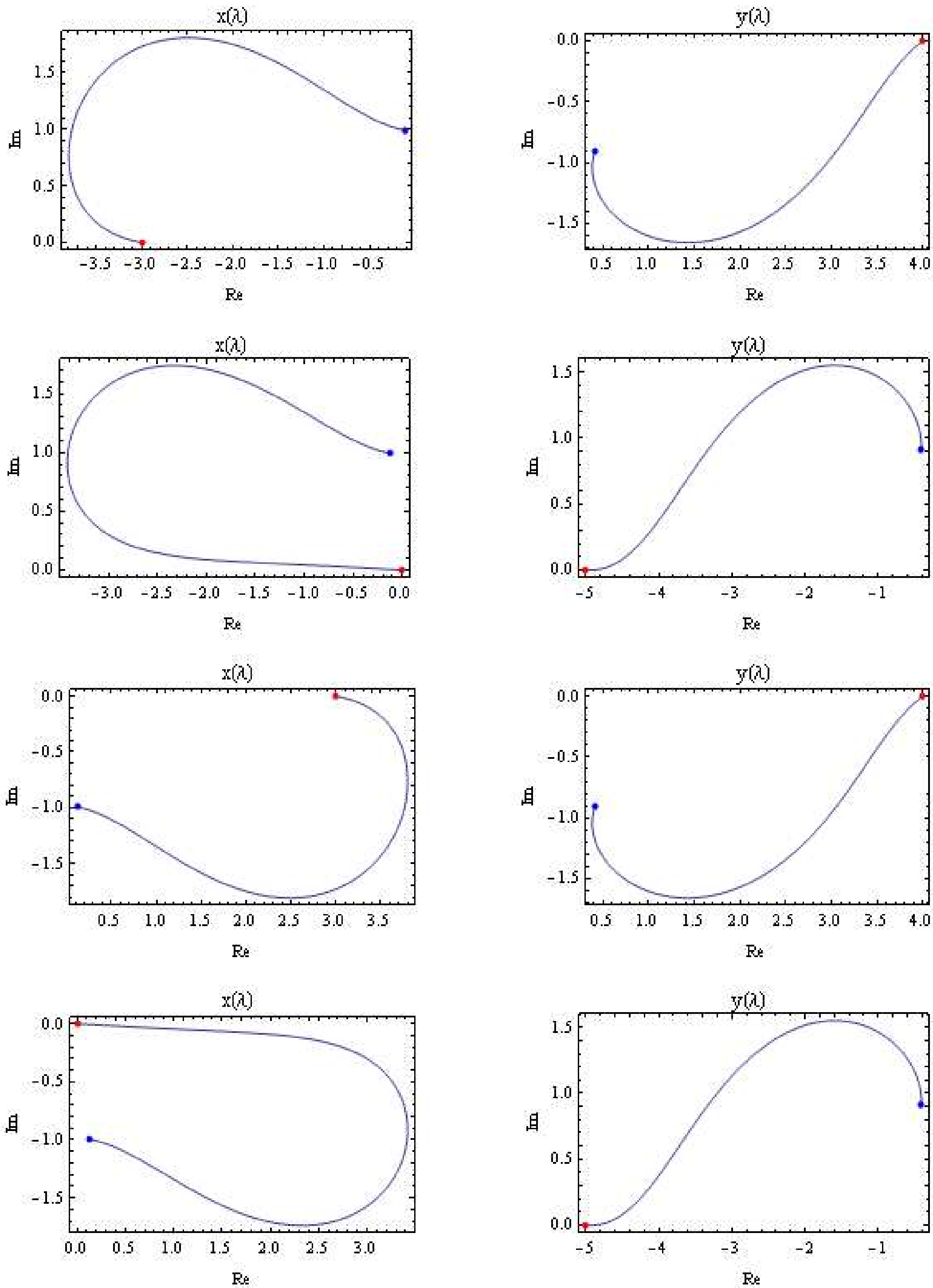

Figure 2: Location of exact solutions for equation (5) using homotopy path [24]

Table 3: Performance of Ostrowski-HCM for equation (5) with starting value $(x_0, y_0) = (1000, 1000)$

Iterations	t	Approximate solution $(\tilde{x}, \tilde{y})$	$\mathbf{F}(\tilde{x}, \tilde{y})$
0	0	$(1000, 1000)$	$(1999975, 998995)$
1	1/331	(503.381,186.942)	(288315,253201)
2	2/331	(82.2808,2026.83)	$(4.11477 \times 10^6, 4738.31)$
3	3/331	(317.552,551.667)	(405150,100282)
4	4/331	(262.071,138.983)	(87972.7,68537.3)
5	5/331	(300.932,-372.486)	(229281,90927.9)
⋮	⋮	⋮	⋮
329	329/331	(3.88009,4.00008)	(6.05571,6.05498)
330	330/331	(3.46682,4.00018)	(3.02028,3.01866)
331	1	(3.00016,4)	(0.000992422,0.000975747)

converges slowly when the starting value is very far from the any of isolated exact solutions. Besides that, the behavior of approximate solutions $(\mathbf{x}, \mathbf{y})$ and the value of function $\mathbf{F}(\tilde{x}, \tilde{y})$ become inconsistent when the parameter t decreases and the number of iterations increase. Motivated by this, we develop a new scheme which is expected to be able to solve the starting value problem.

3 Solution of starting value problem

The super Ostrowski-HCM that we propose is a combination of Ostrowski-HCM, quadratic Bezier homotopy function, linear fixed point function and a technique from Palancz *et al.* [24], which were defined by Nor *et al.* [14,25,26]. The details are explained in the following subsections.

3.1 Ostrowski-HCM

The formula for Ostrowski-HCM as in Nor *et al.* [14] is as follows

$$\begin{aligned} \mathbf{y}_i &= \mathbf{x}_i - \left[D_{\mathbf{x}} H(\mathbf{x}_i, t)\right]^{-1} \mathbf{H}(\mathbf{x}_i, t), \\ \mathbf{x}_{i+1} &= \mathbf{y}_i - \left[D_{\mathbf{x}} H(\mathbf{x}_i, t)\right]^{-1} \mathbf{O}_j(\mathbf{x}_i, t), \end{aligned} \qquad i = 0, 1, 2, ..., k-1. \tag{6}$$

where

$$\mathbf{O}_j(\mathbf{x}_i, t) = \frac{H_j(\mathbf{x}_i, t) H_j(\mathbf{y}_i, t)}{H_j(\mathbf{x}_i, t) - 2H_j(\mathbf{y}_i, t)}, \quad j = 1, 2, ..., n.$$

$$\mathbf{H}(\mathbf{x}_i, t) = (1-t)\mathbf{G}(\mathbf{x}) + t\mathbf{F}(\mathbf{x}), \quad t \in [0, 1]. \tag{7}$$

The function $\mathbf{H}(\mathbf{x}, t)$ is referred to as the homotopy function, $D_{\mathbf{x}}\mathbf{H}(\mathbf{x}, t)$ is the derivative of the homotopy function with respect to variable $\mathbf{x}$ and $\mathbf{O}_j(\mathbf{x}_i, t)$ is referred to the Ostrowski's homotopy function.

3.2 Quadratic Bezier homotopy function

Quadratic Bezier homotopy function was discussed in detail in Nor *et al.* [25]. This new homotopy function is an extension of standard homotopy function. The standard homotopy function is as follows:

$$\mathbf{H}(\mathbf{x}, t) = (1-t)\mathbf{G}(\mathbf{x}) + t\mathbf{F}(\mathbf{x}). \tag{8}$$

To improve the accuracy of solutions, the authors in [25] modified the standard homotopy function to

$$\mathbf{H}_2(\mathbf{x}, t) = (1-t)^2\mathbf{G}(\mathbf{x}) + 2t(1-t)\mathbf{H}(\mathbf{x}, t) + t^2\mathbf{F}(\mathbf{x}), \tag{9}$$

by using recursive construction technique. An in-depth study of this construction (9) was discussed in [25]. Nor *et al.* [27] developed QBHF but the construction of a new homotopy function in [27] was for solving scalar of nonlinear equations. The results in [25] and [27] indicated that QBHF performs better than standard homotopy function in terms of accuracy.

3.3 Linear fixed point function

Similar to the development of QBHF, Nor *et al.* [26] introduced a new auxiliary homotopy function known as linear fixed point (LFP) function. The auxiliary homotopy function or starting function, denoted as $\mathbf{G}(\mathbf{x})$, plays an important role to initiate the computation. As discussed in Nor *et al.* [14,25,26] , when homotopy function $\mathbf{H}(\mathbf{x}, t)$ or $\mathbf{H}_2(\mathbf{x}, t)$ is evaluated at $t = 0$, we have

$$\begin{aligned} \mathbf{H}(\mathbf{x}, 0) &= \mathbf{G}(\underset{\sim}{\rightarrow} \mathbf{x}), \\ &= \mathbf{H}_2(\mathbf{x}, 0). \end{aligned} \tag{10}$$

Similarly, when we developed new auxiliary homotopy function $\mathbf{G}(\mathbf{x}, t)$, we have

$$\begin{aligned} \mathbf{H}(\mathbf{x}, 0) &= \mathbf{G}(\mathbf{x}, 0), \\ &= \mathbf{H}_2(\mathbf{x}, 0). \end{aligned} \tag{11}$$

One of advantages of using LFP over standard of auxiliary function is that $\mathbf{G}(\mathbf{x}, t)$ always moves whilst $\mathbf{G}(\mathbf{x})$ is always fixed for every increment parameter t [26]. This situation accelerates the speed of convergence to the solution of equations.

A familiar auxiliary homotopy function that is widely used is the fixed point function. The fixed point function is

$$\mathbf{G}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0. \tag{12}$$

Nor *et al.* [26] developed

$$\mathbf{G}(\mathbf{x}, t) = (1-t)(\mathbf{x} - \mathbf{x}_0) + t\mathbf{F}(\mathbf{x}). \tag{13}$$

Nor *et al.* [26] demonstrated several systems of equations to investigate the advantages of LFP. The results obtained showed that LFP was better than the standard auxiliary homotopy function.

3.4 Super Ostrowski-HCM

We combine the formulae developed in Nor *et al.* [14, 25, 26] to become a new scheme. The improved homotopy function is

$$\mathbf{H}_2(\mathbf{x},t) = (1-t)^2\mathbf{G}(\mathbf{x},t) + 2t(1-t)\left[(1-t)\mathbf{G}(\mathbf{x},t) + t\mathbf{F}(\mathbf{x})\right] + t^2\mathbf{F}(\mathbf{x}). \tag{14}$$

We use the technique from Palancz *et al.* [24] to accelerate the convergence of Ostrowski-HCM

$$x_{i+1} = \text{NewtonRaphson}(H(x_i,t_{i+1}),(x,x_i)). \tag{15}$$

We modify this technique to

$$\mathbf{x}_{i+1} = \text{Ostrowski'sMethod}(\mathbf{H}_2(\mathbf{x}_i,t_{i+1}),(\mathbf{x},\mathbf{x}_i)), \quad i = 0,1,2,...,k-1. \tag{16}$$

However, we only iterate twice for every t_{i+1} to reduce the number of iterations involved and to enhance the accuracy of solutions. Finally, our proposed global procedures becomes

$$\begin{aligned} \mathbf{y}_i &= \mathbf{x}_i - \left[D_{\mathbf{x}}\mathbf{H}_2(\mathbf{x}_i,t)\right]^{-1}\mathbf{H}_2(\mathbf{x}_i,t), \\ \mathbf{x}_{i+1} &= \mathbf{y}_i - \left[D_{\mathbf{x}}\mathbf{H}_2(\mathbf{x}_i,t)\right]^{-1}\mathbf{O}_2^{(j)}(\mathbf{x},t), \end{aligned} \quad i = 0,1,2,...,k-1, j = 1,2,...,n, \tag{17}$$

where $\mathbf{H}_2(\mathbf{x}_i,t)$ is the improved of homotopy function as (9) and $\mathbf{O}_2(\mathbf{x},t)$ can be written as

$$\mathbf{O}_2^{(j)}(\mathbf{x}_i,t) = \begin{pmatrix} \dfrac{H_2^{(1)}(\mathbf{x}_i,t)H_2^{(1)}(\mathbf{y}_i,t)}{H_2^{(1)}(\mathbf{x}_i,t) - 2H_2^{(1)}(\mathbf{y}_i,t)} \\ \dfrac{H_2^{(2)}(\mathbf{x}_i,t)H_2^{(2)}(\mathbf{y}_i,t)}{H_2^{(2)}(\mathbf{x}_i,t) - 2H_2^{(2)}(\mathbf{y}_i,t)} \\ \vdots \\ \dfrac{H_2^{(n)}(\mathbf{x}_i,t)H_2^{(n)}(\mathbf{y}_i,t)}{H_2^{(n)}(\mathbf{x}_i,t) - 2H_2^{(n)}(\mathbf{y}_i,t)} \end{pmatrix}. \tag{18}$$

All of this combination is named Super Ostrowski homotopy continuation method. We re-solve Example 2.1 using Super Ostrowski-HCM. The results are as in Table 4, Table 5 and Table 6. The tables show that Super Ostrowski-HCM performs better and converge faster than Ostrowski-HCM as the numbers of iterations have been vastly reduced. Super Ostrowski-HCM needs less than 10 iterations when starting from three starting values: $(x_0,y_0) = (2.5,3.5)$, $(x_0,y_0) = (-100,-100)$ and $(x_0,y_0) = (1000,1000)$. It is observed that Super Ostrowski-HCM can resolve the starting value problem which is faced by Ostrowski-HCM.

4 Numerical experiments and discussion

We test four examples of system of polynomial equations. The comparative study involves standard Ostrowski-HCM and Super Ostrowski-HCM. The performances of both methods are measured by the range of starting value and the number of iterations (NOI) needed to solve the system of equations. The stopping criterion used is

$$\|\mathbf{F}(\mathbf{x}_k)\|_\infty < \varepsilon, \tag{19}$$

where $\varepsilon = 10^{-3}$.

Table 4: Performance of super Ostrowski-HCM for equation (5) with starting value $(x_0, y_0) = (2.5, 3.5)$

Iterations	t	Approximate solution $(\tilde{x}, \tilde{y})$	$\mathbf{F}(\tilde{x}, \tilde{y})$
0	0	$(2.5, 3.5)$	$(-6.5, -2.25)$
1	1	$(3.0000000010576, 4.000000006225)$	$(5.61 \times 10^{-8}, 1.21 \times 10^{-10})$

Table 5: Performance of super Ostrowski-HCM for equation (5) with starting value $(x_0, y_0) = (-100, -100)$

Iterations	Approximate Solution $(\tilde{x}, \tilde{y})$	$\mathbf{F}(\tilde{x}, \tilde{y})$
0	$(-100, -100)$	$(19975, 10095)$
1	$(-5.97694, -7.76859)$	(71.0748,38.4924)
2	(-1.48746,-5.10653)	(3.28916,2.31906)
3	$(-0.34214, -5.0193)$	(0.310444,0.136361)
4	$(0.0427401, -5.00036)$	(0.00539642,0.00218368)
5	(2.95256,3.98892)	$(-0.370909, -0.271339)$
6	$(-0.12185, -5.00279)$	(0.0427771,0.0176397)
7	$(0.0155802, -5.00003)$	(0.0005668,0.000275147)

Table 6: Performance of super Ostrowski-HCM for Eq. (5) with starting value

Iterations	Approximate Solution $(\tilde{x}, \tilde{y})$	$\mathbf{F}(\tilde{x}, \tilde{y})$
0	$(1000, 1000)$	$(1999975, 998995)$
1	(62.9921,62.1387)	(7804.23,3900.87)
2	(16.2309,15.5624)	(480.63,242.881)
3	(4.86538,5.01062)	(23.7783,13.6613)
4	(3.02308,4.02287)	(0.322549,0.116163)
5	$(2.99999649, 4.00002164)$	(0.00015203,-0.0000426916)

Example 4.1 Consider the following system of polynomial equations in [13] :

$$\begin{aligned} f_1(x,y) &= x^2 + y^2 - 25 = 0, \\ f_2(x,y) &= x^2 - y - 5 = 0, \end{aligned} \tag{20}$$

where the exact solutions are $(x_1, y_1) = (3{,}4)$, $(x_2, y_2) = (-3{,}4)$ and $(x_3, y_3) = (0, -5)$. The results are shown in Table 7 by varying the values of starting value. The table shows that

1. Super Ostrowski-HCM uses fewer iterations than Ostrowski-HCM.
2. Ostrowski-HCM was a large number of iterations unless the starting value is close to the actual roots.
3. All the starting values proved to be successful for Super Ostrowski-HCM.

Table 7: Comparison between the NOI when using Ostrowski-HCM and our proposed procedure for equation (20)

Starting Value (x_0, y_0)	Location of starting value with respect to the exact solution	Ostrowski-HCM	Super Ostrowski-HCM
$(2.5, 3.5)$	Close	2	1
$(-3.5, 4.5)$	Close	3	1
$(-10, -10)$	Far	12	5
$(10, 10)$	Far	5	2
$(-100, -100)$	Far	35	7
$(100, 100)$	Far	35	3
$(-1000, -1000)$	Very far	138	5
$(1000, 1000)$	Very far	331	5

Example 4.2 Consider the following system of polynomial equations in [28] :

$$\begin{aligned} f_1(x,y) &= x^2 - 2x - y + 0.5 = 0, \\ f_2(x,y) &= x^2 + 4y^2 - 4 = 0, \end{aligned} \tag{21}$$

where the exact solutions are $(x_1, y_1) = (1.900676726367066, 0.3112185654192943)$ and $(x_2, y_2) = (-0.2222145550597218, 0.993808418599834)$. The results are shown in Table 8 by varying the values of the starting value. The table shows that

1. Super Ostrowski-HCM uses fewer iterations than Ostrowski-HCM.
2. Ostrowski-HCM was a large number of iterations unless the starting value is close to the actual roots.
3. All the starting values proved to be successful for Super Ostrowski-HCM.

Example 4.3 Consider the following example in [10]:

$$\begin{aligned} f_1(x,y,z) &= x^2 + y^2 + z^2 - 1 = 0, \\ f_2(x,y,z) &= 2x^2 + y^2 - 4z = 0, \\ f_3(x,y,z) &= 3x^2 - 4y^2 + z^2 = 0, \end{aligned} \tag{22}$$

Table 8: Comparison between the NOI when using Ostrowski-HCM and our proposed procedure for equation (21)

Starting Value (x_0, y_0)	Location of starting value with respect to the exact solution	Ostrowski-HCM	Super Ostrowski-HCM
$(2, \frac{1}{2})$	Close	2	1
$(-10, -10)$	Far	32	5
$(10, 10)$	Far	39	3
$(-100, -100)$	Far	159	11
$(100, 100)$	Far	239	9
$(-1000, -1000)$	Very far	274	17
$(1000, 1000)$	Very far	1831	9

where $(x, y, z) = (0.69828860997151, \text{-}0.62852429796021, 0.342564189689569)$ is an exact solutions. The results are shown in Table 9 by varying the value of starting values. The table shows that

1. Super Ostrowski-HCM uses fewer iterations than Ostrowski-HCM.
2. Ostrowski-HCM was a large number of iterations unless the starting value is close to the actual roots.
3. All the starting values proved to be successful for Super Ostrowski-HCM.

Table 9: Comparison between the NOI when using Ostrowski-HCM and our proposed procedure for equation (22)

Starting Value (x_0, y_0)	Location of starting value with respect to the exact solution	Ostrowski-HCM	Super Ostrowski-HCM
$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	Close	3	1
$(-10, -10, -10)$	Far	56	10
$(10, 10, 10)$	Far	38	3
$(-100, -100, -100)$	Far	318	17
$(100, 100, 100)$	Far	319	5
$(-1000, -1000, -1000)$	Very far	1734	18
$(1000, 1000, 1000)$	Very far	$+\infty$	9

Example 4.4 Consider the following example in [29] :

$$\begin{aligned} f_1(w,x,y,z) &= x + 10y = 0, \\ f_2(w,x,y,z) &= \sqrt{5}(z - w) = 0, \\ f_3(w,x,y,z) &= (y - 2z)^2 = 0, \\ f_4(w,x,y,z) &= \sqrt{10}(x - w)^2 = 0, \end{aligned} \tag{23}$$

where $(w,x,y,z) = (0,0,0,0)$ is an exact solutions. The results are shown in Table 10 by varying the values of the starting value. The table shows that

1. Super Ostrowski-HCM uses fewer iterations than Ostrowski-HCM.
2. Ostrowski-HCM was a large number of iterations unless the starting value is close to the actual roots.
3. All the starting values proved to be successful for Super Ostrowski-HCM.

Table 10: Comparison between the NOI when using Ostrowski-HCM and our proposed procedure for equation (23)

Starting Value (x_0, y_0)	Location of starting value with respect to the exact solution	Ostrowski-HCM	Super Ostrowski-HCM
$(0.001, -0.001, 0.001, -0.001)$	Close	1	1
$(1, 4, 1, 2)$	Not close	126	4
$(10, -10, 10, -10)$	Far	$+\infty$	6
$(-10, 10, -10, 10)$	Far	$+\infty$	7
$(100, -100, 100, -100)$	Far	$+\infty$	12
$(-100, 100, -100, 100)$	Far	$+\infty$	12
$(1000, -1000, 1000, -1000)$	Very far	$+\infty$	28
$(-1000, 1000, -1000, 1000)$	Very far	$+\infty$	28

The superior performance of the Super Ostrowski-HCM may be due to the combination of the superior accuracy of homotopy continuation method, homotopy function and auxiliary homotopy function as developed by Nor *et al.* [14, 25, 26] respectively. The technique of Palancz *et al.* [24] may also have contributed to the superior performance of Super Ostrowski-HCM.

5 Conclusion

The results from Tables 7 – 10 indicate that Super Ostrowski-HCM performs better than the standard of Ostrowski-HCM. Ostrowski-HCM performs well if the selected starting value is close to the actual roots of equations but requires more iterations when the selected starting value is far away from the exact solutions. The problem arises when the user does not know their chosen starting value is close or not to the exact solutions. An starting value that is not close to the exact solution gives poor results when using Ostrowski-HCM in the examples considered. This means an arbitrary starting value using Ostrowski-HCM does not guarantee the acceleration of convergence. The main contribution in this paper is the proposed method can resolve the problem associated with the starting values which are faced by standard of Ostrowski-HCM. The aforementioned problem can be resolved by using the Super Ostrowski-HCM which only needs a few iterations to converge to the solutions of a system of polynomial equations.

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References

[1] Lee, J. and Chiang, H. Convergent regions of the Newton homotopy method for nonlinear systems: theory and computational applications, *Fundamental Theory and Applications.* 2001. 48: 51 – 66.

[2] Burden, R. L. and Faires, J. D. *Numerical Analysis.* 9^{th} International Edition. Singapore: Brooks/Cole, Cencag Learning . 2011.

[3] Broyden, C. G. A class of methods for solving nonlinear simultaneous equations. *Mathematics of Computation.* 1965. 19 (92): 577 – 593.

[4] Broyden, C. G. Quasi-Newton methods and their applications to function minimisation. *Mathematics of Computation.* 1967. 21(99): 368 – 381.

[5] Broyden, C. G. A new method for solving nonlinear simultaneous equations, *Computing Journal*. 1969. 12: 94 – 99.

[6] Brent, R. P. On the Davidenko-Branin method for solving simultaneous nonlinear equations. *IBM Journal of Research and Development*. 1972: 434 – 436.

[7] Brent, R. P. Some efficient algorithms for solving systems of nonlinear equations. *Journal of Numerical Analysis.* 1973. 10(2): 327 – 344.

[8] Gritton, K. S., Seader, J. D. and Lin,W. Global homotopy continuation procedures for seeking all roots of a nonlinear equation. *Computers and Chemical Engineering*. 2001. 25: 1003 – 1019.

[9] Chapra, S. C. and Canale, R. P. *Numerical Methods for Engineers with Software and Programming Applications.* 4^{th} Edition. New York: McGraw-Hill Higher Education. 2003.

[10] Noor, M.A. and Waseem, M. Some iterative methods for solving a system of nonlinear equations. *Computers and Mathematics with Applications*. 2009. 57: 101-106.

[11] Nikkhah-Bahrami, M. and Oftadeh, R. An effective iterative method for computing real and complex roots of systems of nonlinear equations. *Applied Mathematics and Computation.* 2009. 215: 1813 – 1820.

[12] Faires, J. D. and Burden, R. L. *Numerical Method.* 3rd International Edition. Singapore: Brooks/Cole Cencage Learning. 2003.

[13] Kotsireas, I. S. Homotopies and polynomial system solving 1, basic principles. *ACM SIGSAM Bulletin*. 2001. 35(1): 19 – 32.

[14] Nor, H. M., Rahman A., Md. Ismail, A. I. and Majid, A. A. Numerical solution of polynomial equations using Ostrowski homotopy continuation method. *Matematika*. 2014. 30: 47 – 57.

[15] Ostrowski, M. *Solutions of Equations and System of Equations*. New York: Academic Press. 1960.

[16] Ostrowski, M. *Solution of Equations in Euclidean and Banach Space* 3rd Edition. New York :Academic Press. 1973.

[17] Kou, J., Li, Y. and Wang, X. Some variants of Ostrowski's method with seventh-order convergence. *Journal of Computational and Applied Mathematics* . 2007. 209: 153-159.

[18] Grau, M. and Diaz-Barrero, J.S. An improvement to Ostrowski root-finding method, *Applied Mathematics and Computation*. 2006. 173: 450-456.

[19] Sharma, J. R. and Guha, R. K. A family of modified Ostrowski methods with accelerated sixth order convergence. *Applied Mathematics and Computations*. 2007. 190: 111-115.

[20] Chun, C. and Ham, Y. Some sixth-order variants of Ostrowski root-finding methods. *Applied Mathematics and Computation*. 2007. 193 : 389-394.

[21] Grau-Sanchez, M., Grau, A. and Noguera M. Ostrowski type methods for solving systems of nonlinear equations. *Applied Mathematics and Computation*. 2011. 218: 2377-2385.

[22] Rafiq, A. A note on "New classes of iterative methods for nonlinear equations" and "Some iterative methods free from second derivatives for nonlinear equations" , *Applied Mathematics and Computation*. 2007. 193: 572 – 576.

[23] Hu, Z., Guocai, L. and Tian, L. An iterative method with ninth-order convergence for solving nonlinear equations. *International Journal Contemp. Math. Sciences*. 2011. 6: 17 – 23.

[24] Palancz, B., Awange, J. L., Zaletnyik, P. and Lewis, R. H. Linear homotopy solution of nonlinear systems of equations in geodesy. *Journal of Geodesy.*2010. 84: 79 – 95.

[25] Nor, H. M., Md. Ismail, A. I. and Majid, A. A. Quadratic Bezier homotopy function for solving polynomial equations, *Matematika.* 2013. 29: 159 – 171.

[26] Nor, H. M., Md. Ismail, A. I. and Majid, A. A. Linear fixed point function for solving system of polynomial equations. In *Proceedings of the 1602 AIP The 3*rd *International Conference on Mathematical Sciences*. 2014. 105 – 112.

[27] Nor, H. M., Md. Ismail, A. I. and Majid, A. A. A New Homotopy Function for Solving Nonlinear Equations. In *Proceedings of the 1557 AIP International Conference on Mathematical Sciences and Statistics 2013, February 5 – 7*. 2013. 21-25.

[28] Rafiq, A. and Awais, M. Convergence on the homotopy continuation method. *Int. J. of Appl. Math. and Mech*. 2008. 4(6) :62 – 70.

[29] Morgan, A. P. A method for computing all solutions to systems of polynomial equations, *ACM Transaction on Mathematical Software*. 1983. 9(1): 1 – 17.