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L(2,1)-Labeling: An Algorithmic Approach to Cycle Dominated Graphs

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Abstract Let G be a connected, undirected graph. Distance two labeling or a L(2,1)-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x) - f(y)|$ is greater than or equal to 2 if x and y are adjacent and $|f(x) - f(y)|$ is greater than or equal to 1 if x and y are at distance 2, for all x and y in $V(G)$. The L(2,1)-labeling number $\lambda(G)$ of G is the smallest number k such that G has an L(2,1)-labeling f with max $\{f(v) : v \in V(\mathrm{G})\ \}$is equal to k. In this paper, we present algorithms to get L(2,1)-labeling of cycle dominating graphs like Diamond graphs, nC_4 with a common vertex and Books B_n and hence we find the λ - number of these graphs.

Keywords Channel assignment; transmitters; L(2,1)-labeling; Books; Distance two labeling

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1 Introduction

In the channel assignment problem, a channel is assigned to each television or radio transmitters located at various places so that any interference in the communication can be avoided.

F. Roberts proposed the problem of efficiently assigning radio channels to transmitters at several locations, using non-negative integers to represent channels, so that close locations receive different channels, and channels for very close locations are at least two apart such that these channels would not interfere with each other. In other words, this is assigning integer frequencies to radio transmitters such that transmitters that are one unit of distance apart receive frequencies that differ by at least two and transmitters that are two units of distance apart receive frequencies that differ by at least one.

The mathematical abstraction of the above concept is distance two labeling or L(2,1)-labeling . A L(2,1)-labeling of a graph G is an assignment f from the vertex set $V(\mathrm{G})$ to the set of non-negative integers such that $|f(x) - f(y)| \geq 2$ if x and y are adjacent and $|f(x) - f(y)| \geq 1$ if x and y are at distance 2, for all x and y in $V(G)$. The L(2,1)-labeling number $\lambda(G)$ of G is the smallest number k such that G has an L(2,1)-labeling f with max $\{f(v) : v \in V(G)\} = k$.

L(2,1)-labeling can be generalized into $L(p, q)$-labeling for arbitrary non-negative integers p and q and it can be seen that L(1,0)-labeling is only the classical vertex coloring. Many results on trees are available in this area and strict bounds of λ are found for trees. Therefore, this paper is focused on cycle dominated graphs like Diamond graphs. A Diamond graph is a collection of $C_{4\mathrm{s}}$, more than two, with consecutive $C_{4\mathrm{s}}$ having a common vertex such that common vertices are diametrically opposite. A Book is a collection of $C4s$ with a common edge.

2 Basic Results

Here, we present some basic important results on $L(2,1)$-labeling or distant two labeling found in [1], [2] and [3].

Result 1 Let P_n be a path on n vertices. Then (i) $\lambda(P_2) = 2$, (ii) $\lambda(P_3) = \lambda(P_4) = 3$ and (iii) $\lambda(P_n) = 4$, for $n \geq 5$.

Result 2 Let C_n be a cycle of length n. Then $\lambda(C_n) = 4$, for any n.

Result 3 Let Q_n be the n-cube. Then, for all $n \geq 5$, $n+3 \leq \lambda(Q_n) \leq 2n+1$.

Result 4 Let T be a tree with maximum degree $\Delta \geq 1$. Then $\lambda(T)$ is either $\Delta+1$ or $\Delta+2$.

Result 5 The λ-number of a star $K_{1,\Delta}$ is $\Delta+1$, where Δ is the maximum degree.

Result 6 The λ-number of a complete graph K_n is $2n-2$.

Result 7 Let G be a graph with maximum degree $\Delta \geq 2$. If G contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta+2$.

Result 8 Let G be a lobster cactus then $\lambda(G) = \Delta+1$ or $\Delta+2$, where Δ is the maximum degree.

Griggs and Yeh [1] posed a conjecture that $\lambda(G) \leq \Delta^2$ for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G, and it is proved that $\lambda(G) \leq \Delta^2+2\Delta$ at the same time. Chang and Kuo [2] proved that $\lambda(G) \leq \Delta^2+\Delta$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G. Kral and Skrekovski [4] proved that $\lambda(G) \leq \Delta^2+\Delta-1$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G. Goncalves [5] proved that $\lambda(G) \leq \Delta^2+\Delta-2$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G. In spite of all efforts, the conjecture posed by Griggs and Yeh is still open.

3 Main Results

Definition 1 A Diamond graph is a collection of $C_{4\mathrm{s}}$, more than two, with consecutive $C_{4\mathrm{s}}$ having a common vertex such that common vertices are diametrically opposite.

Algorithm 1 .

Input : A Diamond graph.
Output : Distance two labeling of a Diamond graph.

Step 1: Arrange the vertices of a Diamond graph as w_0, w_1,w_2,...,w_n, u_1, u_2,...,u_n, v_1,v_2,...,v_n. Also arrange the edges as $w_i u_{i+1}$, $w_i v_{i+1}$, $i = 0,1,2,\ldots, n-1$ and $w_i u_i$, $w_i v_i$, $i = 1,2,\ldots, n$.

Step 2: Label $w_i s$ with 0, 5, 3 consecutively and repeat this labeling until we reach w_n.

Step 3: Label $u_i\, s$ with 3, 0, 5 consecutively and repeat this labeling until we reach u_n.

Step 4: Label $v_i s$ with 2, 1, 6 consecutively and repeat this labeling until we reach v_n.

The justification of the desired output is proved in Theorem 1.

Theorem 1 *For a Diamond graph G, $\lambda(G) = 6$ if $n > 3$ and $\lambda(G) = 5$ if $n = 3$, where n is the number of $C_{4\mathrm{s}}$ in the Diamond graph.*

Proof Consider the Diamond Graph G.
Let $V(G) = \{w_0, w_1, w_2, \ldots, w_n,\ u_1,\ u_2, \ldots,\ u_n,\ v_1, v_2, \ldots, v_n\}$.

Case 1: $n > 3$

Define f: $V\ (G) \to N \cup \{0\}$ such that

$$\begin{aligned} f(w_i) &= 0 \text{ if } i \bmod 3 = 0 \\ &= 5 \text{ if } i \bmod 3 = 1 \\ &= 3 \text{ if } i \bmod 3 = 2 \end{aligned}$$

$$\begin{aligned} f(u_{\mathrm{i}}) &= 3 \text{ if } i \bmod 3 = 1 \\ &= 0 \text{ if } i \bmod 3 = 2 \\ &= 5 \text{ if } i \bmod 3 = 0 \end{aligned}$$

$$\begin{aligned} f(v_i) &= 2 \text{ if } i \bmod 3 = 1 \\ &= 1 \text{ if } i \bmod 3 = 2 \\ &= 6 \text{ if } i \bmod 3 = 0 \end{aligned}$$

Where d(s,t) is the distance between between s_i and t_j and that is, $d(s,t) = 1$, we have $|f(s_i) - f(t_j)| = 2$ or 3 or 4 or 5 or 6, by construction of f.

We note that $|d(w_i\,) - d(w_{i+1})| = 2$, $|d(u_i\,) - d(u_{i+1})| = 2$, $|d(u_i\,) - d(v_{i+1})| = 2$, $|d(v_i\,) - d(v_{i+1})| = 2$, $|d(v_i\,) - d(u_{i+1})| = 2$.

Since the labels at the vertices at distance 2 are all distinct, when $d(s_i,\, t_j) = 2$, we have $|f(s_i\,) - f(t_j)| \geq 1$.

Hence f is a distance two labeling and $\lambda(G) \leq 6$.

Since the maximum degree of $G$is 4, $\lambda(G) \geq 5$.

Suppose, $\lambda(G) = 5$, then we use the labels 0, 1, 2, 3, 4 and 5.

Now consider the sub-graph of the diamond graph which is given below.

Each w_i , $i = 1, 2, 3$ should receive the label 0 or 5 since the label difference of w_i with each of its adjacent vertices is greater than or equal to 2. Without loss of generality, we assume that the label of w_1 is 0. Then, the label of w_2 should be 5. Then the adjacent vertices of w_1 must receive the labels 2, 3, 4 and 5. In particular v_2 and u_2 should receive

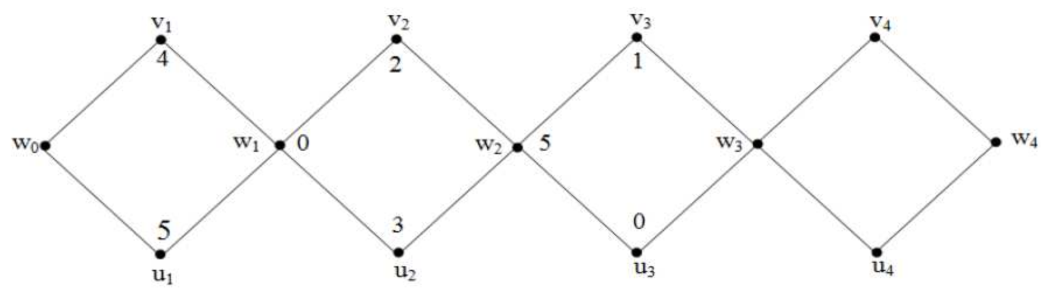

Figure 1: Diamond Graph

the labels 2 and 3, since the label of w_2 is 5. Hence v_3 and u_3 should receive the labels 1 and 0. Since f is $L(2,1)$-labeling, v_3 must receive 3 or 4, which is not possible. Hence $\lambda(G)$ >5. Hence $\lambda(G) = 6$.

Case 2: $n = 3$

Define f: $V(G) \to N \cup \{0\}$such that $f\ (w_0) = 1$, $f\ (w_1) = 0$, $f\ (w_2) = 5$, $f\ (w_3) = 3$, $f(v_1) = 4$, $f(v_2) = 2$,$f(v_3) = 0$, $f(u_1) = 5$, $f(u_2) = 3$ and $f(u_3) = 1$.

Clearly f is a $L(2,1)$-labeling and hence $\lambda(G) \leq 5$. Since this graph has maximum degree 4, $\lambda(G) \geq 5$. Hence $\lambda(G) = 5$. □

Now consider the graph, the collection of $C_{4\mathrm{s}}$, all of which have a common vertex, that is, nC_4 with a common vertex.

Algorithm 2 .

Input : nC_4 with a common vertex.
Output : Distance two labeling of the graph, nC_4 with a common vertex.

Step 1: Let the common vertex be u and the vertices of the i^{th} cycle be u, v_{i_1}, v_{i_2} and v_{i_3}, i = 1,2,..., n.

Step 2: Label the common vertex u with 0.

Step 3: Starting with v_{11}, label the vertices which are adjacent with u with the labels 2,3,..., $2n$+1.

Step 4: Label v_{12} with the label 5 and all other $v_{i_2}\,s$ with 1.

The justification of the desired output is proved in Theorem 2.

Theorem 2 *The λ - number of nC_4 with a common vertex is 2n+ 1, n >1.*

Proof Consider the graph, nC_4 with a common vertex. Let the common vertex be u and the vertices of the i^{th} cycle be u, v_{i_1}, v_{i_2} and v_{i_3}, $i = 1,2,\ldots,n$.

Define f: $V\ (G) \to N \cup \{0\}$ such that

$$f(u) = 0$$

$$f(v_{i_1}) = 2i,\ i = 1,2,\ldots,n$$

$$f(v_{Pi_3}) = f(v_{i_1}) + 1,\ i = 1,2,\ldots,n$$

$$f(v_{12}) = 5$$

$$f(v_{\text{i2}}) = 1,\ i \neq 1.$$

We note that $d(u,\ v_{ij}) = 1$, $i = 1,2,\ldots,n$ and $j = 1,\ 3$.

Since the label of u is 0 and the labels of v_{ij}, $i = 1, 2, \ldots\ ,n$ and $j = 1,\ 3$ are 2,3,..., $2n+ 1$, we have $|f(u) - f(v_{ij}\,)| \geq 2$, $i = 1,2,\ldots,n$ and $j = 1, 3$.

Also $d(v_{i_1},\ v_{i_2}) = 1$ and $d(v_{i_3},\ v_{i_2}) = 1$. Since $f(v_{11}) = 2$, $f(v_{13}) = 3$ and $f(v_{12}) = 5$, $f(v_{i_2}) = 1$, $i \neq 1$ and $f(v_{ij})$, $i \neq 1$ and $j \neq 2$ are 4,5,..., $2n+ 1$, we have $|f(v_{\text{ij}}) - f(v_{i_2}\,)| \geq 2$, $i = 1,2,\ldots,n$ and $j = 1,3$.

For $i = 1,2,\ldots,n$ and $j = 2$, $d(u,\ v_{ij}) = 2$ and for i, $\alpha = 1,2,\ldots,n$ and j, $k = 1,\ 3$, $d(v_{ij},\ v_{\alpha k}) = 2$ (i, α, j, k all are not equal) .

Since $f(u) = 0$ and for $i = 1,2,\ldots,n$ and $j = 2$, $f(v_{ij})$ are 1 or 5, we have $|f(u) - f(v_{ij})| \geq 1$, $i = 1,2,\ldots,n$ and $j = 2$.

Also, all $f(v_{ij})$ are distinct for $i = 1,2,\ldots,n$ and $j = 1,3$ we have $|f(v_{ij}) - f(v_{\alpha k})| \geq 1$, i, $\alpha = 1,2,\ldots,n$ and j, $k = 1,3$ (i, α, j, k all are not equal). Hence $f$is a distance two labeling and $\lambda(G) \leq 2n+ 1$.

Since the maximum degree of G is $2n$, $\lambda(G) \geq 2n+ 1$. Hence $\lambda(G) = 2n+ 1$. □

Definition 2 A Book B_n is the product of the star $K_{1,\,n}$ with K_2.

Next we present an algorithm to get a distance two labeling of a Book B_n. A Book B_n has $2n+ 2$ vertices, $3n+ 1$ edges, n pages and maximum degree $n+ 1$.

Algorithm 3 .

Input : A Book B_n.
Output : Distance two labeling of B_n.

Step 1: Let the vertices of the n^{th} page of B_n be v, v_{n}, w_{n} and w where v and w lie on the common edge vw.

Step 2: Label v with the label 0 and w with the label $n+ 2$.

Step 3: Label v_i with $i+ 1$, $i = 1,2,\ldots,n$.

Step 4: Label w_i] with $i+ 3$, $i = 1,2,\ldots,n - 3$ and w_{n-2} with the label 1, w_{n-1} with 2 and w_n with 3.

The justification of the desired output is proved in Theorem 3.

Theorem 3 *The λ-number of a Book B_n is $n+ 2$, $n \geq 4$.*

Proof Consider a Book B_n. It has $2n+2$ vertices, $3n+1$ edges, n pages and maximum degree $n+1$. Let the vertices of the n^{th} page of B_n be v, v_{n}, w_{n} and w where v and w lie on the common edge vw.

Define f: $V\ (B_n) \to N \cup \{0\}$ such that

$$f(v) = 0$$

$$f(w) = n+2$$

$$f(v_i) = i+1,\ i = 1,2,\ldots,n$$

$$f(w_i) = i+3,\ i = 1,2,\ldots,n-3$$

$$f(w_{n-2}) = 1$$

$$f(w_{n-1}) = 2$$

$$f(w_n) = 3$$

We note that $d(v, v_i) = 1$, $d(w, w_i) = 1$, $d(v_i, w_i) = 1$, $i = 1,2,\ldots,n$ and $d(v, w) = 1$. Now,

$$|f(v) - f(v_{\text{i}})| = \text{i} + 1 \geq 2,\ \ i = 1,2,\ldots,n.$$

$$f(w) - f(w_i)|] = n + 2 - (i + 3) = n - i - 1 \geq 2,\ \ i = 1,\ 2,\ \ldots,\ n - 3.$$

$$|f(w) - f(w_{n-2})| = n + 2 - 1 = n + 1,$$

$$|f(w) - f(w_{n-1})| = n + 2 - 2 = n.$$

$$|f(w) - f(w_n)| = n + 2 - 3 = n - 1.$$

Consider

$$|f(v_{\text{i}}) - f(w_i)| = i + 3 - (i + 1) = 2,\ i = 1,2,\ldots,\ n - 3.$$

$$|f(v_{n-2}) - f(w_{n-2})| = n - 1 - 1 = n - 2,$$

$$|f(v_{n-1}) - f(w_{n-1})| = n - 2 = n - 2.$$

$$|f(v_n) - f(w_n)| = n + 1 - 3 = n - 2.$$

Also

$$|f(v) - f(w)| = n + 2.$$

Thus, $|f(s_i) - f(t_i)| \geq 2$ when $d(s_i, t_i) = 1$, s_i, $t_i \in V(B_n)$.

We note that $d(v,\ w_i) = 2$, $d(w,\ v_i) = 2$, $i = 1,2,\ldots,n$, $d(v_i,\ v_j) = 2, i \neq j$, i, $j = 1,2,\ldots,n$ and $d(w_i,\ w_j) = 2, i \neq j, i$, $j = 1,2,\ldots,n$.

When $d(v,\ w_i) = 2$, $i = 1,2,\ldots,n$,

$$|f(v) - f(w_i)| = i + 3 \geq 1,\ \ i = 1,2,\ldots,n - 3.$$

$$|f(v) - f(w_{n-2})| = 1,$$

$$|f(v) - f(w_{n-1})| = 2,$$

$$|f(v) - f(w_n)| = 3.$$

When $d(w, v_i) = 2$, $i = 1,2,\ldots,n$,

$$|f(w) - f(v_i)| = n + 2 - (i + 1) = n - i + 1,\ i = 1, 2, \ldots, n.$$

When $d(v_i, v_j) = 2$, $i \neq j$, $i, j = 1,2,\ldots, n$, v_i, v_j receive different labels and hence

$$|f(v_i) - f(v_j)| \geq 1.$$

When $d(w_i, w_j) = 2$, $i \neq j$,i, $j = 1, 2, \ldots, n$, w_i, w_j receive different labels and hence

$$|f(w_i) - f(w_j)| \geq 1.$$

Thus, $|f(s_i) - f(t_i)| \geq 1$ when $d(s_i, t_i) = 2$, $s_i, t_i \in V(B_n)$. Hence $f$is a distance two labeling and $\lambda(B_n) \leq n+ 2$.

Since the maximum degree of B_n is $n+ 1$, $\lambda(B_n) \geq n + 2$. Hence $\lambda(B_n) = n+ 2$. □

4 Conclusion

In this paper, we have found the λ-number for the families of cycle dominating graphs like Diamond graphs, nC_4 with a common vertex and Books. The generalization of $L(2, 1)$-labelings to $L(p, q)$-labelings will attract more researchers. We have generalized these results and some more into $L(p, q)$-labelings. These problems will be more interesting if we look them at the algorithmic point of view. It is known to be NP-hard for general graphs [1], but only a few graph classes are known to have polynomial time algorithms for this problem. For example, the L(2,1)-labeling number of paths, cycles, wheels can be determined within polynomial time [1]. We believe that this paper will create an interest towards $L(p, q)$-labelings.

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