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A Note About Configuration of A Group

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Abstract In 2001, Rosenblatt and Willis defined the concept of configuration of a group to give a condition for amenability of groups. In this paper, we study the relation between configuration and commutator subgroup G' of G and prove that if G_1 and G_2 are two finitely generated groups with the same configuration set, then $\frac{G_1}{G_1'} \cong \frac{G_2}{G_2'}$ and if G_1' and G_2' are finite, then $G_1' \cong G_2'$. Also, we prove that if two free finitely generated Burnside groups of finite exponent have the same configuration set, then they must be isomorphic.

Keywords Configuration; Amenability; Commutator Subgroup; Burnside Group

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1 Introduction

In [1], Rosenblatt and Willis defined the concept of configuration of groups for the first time and applied it to prove that "weak convergence is not strong convergence for amenable groups". The definition of the configuration of a finitely generated group G is given in the following.

Definition 1 Suppose that G is a finitely generated group. Let $\mathcal{E} = \{E_1, \ldots, E_m\}$ be a finite partition for G and $\mathfrak{g} = (g_1, \ldots, g_n)$ be an ordered set of generators for G. A configuration corresponding to $\mathcal{E}$ and $\mathfrak{g}$ is an $(n+1)$-tuple $C = (c_0, c_1, \ldots, c_n)$ such that $c_j \in \{1, \ldots, m\}$ for all $j \in \{0, 1, \ldots, n\}$ and there is an element $x_0 \in G$ with $x_0 \in E_{c_0}$ such that $x_j = g_j x_0 \in E_{c_j}$ for each $j \in \{1, \ldots, n\}$. The element x_0 is called a base point of C and the elements $x_j, j = 1, \ldots, n$ are called branch points of C. The set of all such configurations C corresponding to $\mathcal{E}$ and $\mathfrak{g}$ is denoted by $Con(\mathfrak{g}, \mathcal{E})$. Also, the set of all configuration sets of G will be denoted by $Con(G)$.

A group property **P** is said to be characterized by configuration if a finitely generated group G_1 has **P** and if $Con(G_1) = Con(G_2)$ for another finitely generated group G_2, then G_2 has also **P**. The being finite property, periodic property and Abelian property can be characterized by configuration (see [2]) and more generally the group property of being nilpotent of class c can be characterized by configuration (see [3]). In the following theorem, the amenability of groups is characterized by configuration.

Theorem 1 *Suppose that $Con(G_1) = Con(G_2)$, then we have:*

(i) *G_1 is amenable if and only if G_2 is amenable.*

(ii) *G_1' is amenable if and only if G_2' is amenable; where G' is commutator subgroup of G.*

(iii) *$\tau(G_1) = \tau(G_2)$; where $\tau(G)$ is Tarski number of G.*

Proof See [4]. □

2 Configuration and commutator subgroup

In this section, we study the relation between configuration and commutator subgroup. Those groups whose commutator subgroup has finite index, are important to us. In [5] some of these groups are studied; for a simple example let G be a finitely generated groups in which every element has finite order (periodic groups), then $\frac{G}{G'}$ is finite. In section 3 we will introduce a class of groups which satisfy this property.

Theorem 2 *Let G_1 and G_2 be finitely generated groups and $Con(G_1) = Con(G_2)$, then:*

$$|G_1| = |G_2| \quad and \quad \frac{G_1}{G_1'} \cong \frac{G_2}{G_2'}.$$

Proof Let G_1 be infinite. Thus it is countable and $|G_1| = \aleph_0 = |G_2|$. To prove the next assertion we use Proposition 2.3 of [3]; since $G_1' \trianglelefteq G_1$, then there exists $N_2 \trianglelefteq G_2$ such that $\frac{G_1}{G_1'} \cong \frac{G_2}{N_2}$. Therefore, $\frac{G_2}{N_2}$ is Abelian, then $G_2' \subseteq N_2$. So we have:

$$\frac{G_1}{G_1'} \overset{\varphi}{\cong} \frac{\frac{G_2}{G_2'}}{\frac{N_2}{G_2'}}.$$

Similarly,

$$\frac{G_2}{G_2'} \overset{\psi}{\cong} \frac{\frac{G_1}{G_1'}}{\frac{N_1}{G_1'}},$$

where $N_1 \trianglelefteq G_1$ with $\frac{G_2}{G_2'} \cong \frac{G_1}{N_1}$.
Now consider the following diagram:

$$\frac{G_1}{G_1'} \xrightarrow{\pi_1} \frac{\frac{G_1}{G_1'}}{\frac{N_1}{G_1'}} \xrightarrow{\psi} \frac{G_2}{G_2'} \xrightarrow{\pi_2} \frac{\frac{G_2}{G_2'}}{\frac{N_2}{G_2'}} \xrightarrow{\varphi} \frac{G_1}{G_1'},$$

where π_1 and π_2 are natural epimorphisms. Thus we have a surjective endomorphism on $\frac{G_1}{G_1'}$. Since $\frac{G_1}{G_1'}$ and $\frac{G_2}{G_2'}$ are finitely generated Abelian groups, then they are Hopfian. So this surjective endomorphism is injective and thus π_1 is an isomorphism. This implies that $G_2' \cong N_2$. Therefore

$$\frac{G_1}{G_1'} \cong \frac{G_2}{G_2'}.$$ □

Remark 1 Let G be a group. Let $\mathcal{F}(G)$ denote the set of isomorphism classes of finite quotients of G and $\mathcal{A}(G)$ denote the set of isomorphism classes of Abelian quotients of G. We say that groups G and H have isomorphic finite quotients if $\mathcal{F}(G) = \mathcal{F}(H)$ and we say that they have isomorphic Abelian quotients if $\mathcal{A}(G) = \mathcal{A}(H)$. It is shown in [2] and [3] two finitely generated groups have isomorphic finite quotients and isomorphic Abelian quotients if they have the same configuration set.

For a group G, let $Z(G)$ denote the center of G.

Lemma 1 Let G_1 and G_2 be finitely generated groups with finite commutator subgroups such that $Con(G_1) = Con(G_2)$. Then

(i) $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$.

(ii) $G_1' \cong G_2'$.

(iii) $Z(G_1) \cong Z(G_2)$.

Proof The proof of (i) follows from Remark 1 and Theorem 2.1 of [6]. Part (ii) can be concluded from the part (i) easily.
By using (i), $Z(G_1)\times\mathbb{Z} \cong Z(G_2)\times\mathbb{Z}$. But $Z(G_1)$ and $Z(G_2)$ are Abelian and then they must be isomorphic. This proved (iii). □

The following theorem is a generalization of Corollary 3.4 of [3].

Theorem 3 *Let G_1, G_2 and F be finitely generated groups with finite commutator subgroups such that $Con(G_1) = Con(G_2 \times F \times \mathbb{Z})$. Then $G_1 \cong G_2 \times F \times \mathbb{Z}$.*

Proof Since G_1 and $G_2 \times F \times \mathbb{Z}$ have finite commutator subgroups, then by Lemma 1 part (i), we have $G_1 \times \mathbb{Z} \cong G_2 \times F \times \mathbb{Z} \times \mathbb{Z}$. So $G_1 \cong G_2 \times F \times \mathbb{Z}$ by Lemma 1 in [7]. □

Question 1 Here there is a natural question; let G_1 and G_2 be finitely generated groups and $Con(G_1) = Con(G_2)$. Does it follow that $G_1' \cong G_2'$? Or in other words, can we say that the finiteness of G_1' implies the finiteness of G_2'?
Note that if G_1 or G_2 is periodic then the other is periodic too and by Theorem 2 we have $|G_1'| = |G_2'|$. On the other hand, there are two finite non-Abelian periodic groups G_1 and G_2 such that:

$$Z(G_1) \cong Z(G_2), \quad G_1' \cong G_2' \quad \text{and} \quad \frac{G_1}{G_1'} \cong \frac{G_2}{G_2'},$$

but

$$G_1 \ncong G_2 \quad \text{and} \quad Con(G_1) \neq Con(G_2).$$

As an example, set $G_1 = D_8$ (dihedral group of order 8) and $G_2 = Q_8$ (quaternion group of order 8).

3 Burnside groups

The Burnside problem, posed by Burnside in 1902 and one of the oldest and most influential questions in group theory, asks whether a finitely generated group in which every element has finite order must necessarily be a finite group. The problem has many variants that differ in the additional conditions imposed on the orders of the group elements. Nevertheless, the general answer to Burnside's problem turned out to be negative. Golod [8] provided a counter-example to the Burnside problem. Also, the restricted Burnside problem (RBP), formulated in the 1930, asks another related question; are there only finitely many finite

m-generator groups of exponent n up to isomorphism? In 1994, Zelmanov said yes to this question.
The group $B(m,n)$ is known as the m-generator Burnside group of exponent n which is defined as follows.

Definition 2 Let F_m denote the free group of rank m on set $\{x_1,...,x_m\}$. For a fixed n, let F_m^n denote the subgroup of F_m generated by g^n for all $g \in F_m$. Then F_m^n is a normal subgroup of F_m and we define the Burnside group $B(m,n)$ to be the quotient group $\frac{F_m}{F_m^n}$.

In 1968 (see [9]) Novikov and Adyan presented the group $B(m,n)$ by defining relations of the form

$$B(m,n) =< x_1,...,x_m; w^n = 1 > .$$

It is known that $B(m,n)$ is infinite (see [9]) and non-amenable (see [10]) for $m \geq 2$ and an odd integer $n \geq 665$. On the other hand, since a group with finite exponent and finite commutator subgroup is clearly finite, then $B(m,n)$ is finite if and only if $B'(m,n)$ is finite.

Lemma 2 If $G = B(m,n)$ then $\frac{G}{G'} \cong \mathbb{Z}_n^m = \underbrace{\mathbb{Z}_n \times ... \times \mathbb{Z}_n}_{m}$.

Proof First, note that $\frac{G}{G'} = \frac{F_m}{F'_m F_m^n}$. Define $\varphi : F_m \to \mathbb{Z}_n^m$ such that $\varphi(x_i) = (0,...,1,...,0)$ for $i = 1,...,m$ and $1 \in \mathbb{Z}_n$ is located in the i-th place.
The epimorphism is given from Von Dycks' theorem. On the other hand, $ker(\varphi) = F'_m F_m^n$; indeed, this epimorphism has an Abelian image with exponent n, then $F'_m F_m^n$ is contained in $ker(\varphi)$.

Also, if $\varphi(x_{i_1}^{\epsilon_1}...x_{i_s}^{\epsilon_s}) = (0,...,0)$ then the sum of exponents $x_{i_1},...,x_{i_s}$ must be in the form nk_j. This implies that $ker(\varphi) \subseteq F'_m F_m^n$. □

Theorem 4 *The following statements are equivalent:*

(i) $Con(B(m,n)) = Con(B(m',n'))$.

(ii) $m = m', n = n'$.

(iii) $B(m,n) \cong B(m',n')$.

Proof It is sufficient to show $(i) \Rightarrow (ii)$. Since $B(m,n)$ and $B(m',n')$ satisfy in semigroup laws $w^n = 1$ and $w^{n'} = 1$ respectively, and $Con(B(m,n)) = Con(B(m',n'))$ then from Theorem 5.1 of [2], $n = n'$. Also, Lemma 2 and Theorem 2 imply that $|\mathbb{Z}_n^m| = |\mathbb{Z}_n^{m'}|$. Thus $m = m'$. □

Question 2 Let G be finitely generated group such that $Con(G) = Con(B(m,n))$ for some m,n. Does it follow that $G \cong B(m,n)$? Note that if G is a group of exponent n and m-generated and further G and $B(m,n)$ have isomorphic finite quotients, we can't say that $G \cong B(m,n)$. For this, take the intersection of all the normal subgroups of finite index in $B(m,n)$. From Zelmanov's solution of the RBP there are only finite number of them. Thus, the intersection is of finite index and the quotient satisfies our request, but it is finite (and of course generally, $B(m,n)$ is infinite).

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