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Solvability of a Nonlinear Coupled System of n Fractional Differential Equations

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Abstract In this paper, we study a nonlinear coupled system of $n-$fractional differential equations. Applying Banach contraction principle and Schaefer's fixed point theorem, new existence and uniqueness results are established. We also give some concrete examples to illustrate the possible application of the established analytical results.

Keywords Caputo derivative; fixed point; nonlinear coupled system; existence; uniqueness

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1 Introduction

Fractional differential equations have gained considerable importance due to their various applications in visco-elasticity, electro-chemistry and many physical problems (see [1–2], see also [3–4]). So far, there have been several fundamental works on the fractional derivative and fractional differential equations, see [3,5–10] and references therein. Moreover, the study of systems of fractional order is also important as such systems occur in various problems of applied nature, for instance, see [11–13]. Recently, many people have studied the existence and uniqueness for solutions of some systems of nonlinear fractional differential equations, reader can see [10,13–16] and references cited therein.

This paper deals with the existence and uniqueness of solutions for the following system of n fractional differential equations:

$$\begin{cases} D^{\alpha_1}x_1(t) = f_1(t, x_1(t), x_2(t), ..., x_n(t)), t \in J, \\ D^{\alpha_2}x_2(t) = f_2(t, x_1(t), x_2(t), ..., x_n(t)), t \in J, \\ . \\ . \\ . \\ D^{\alpha_n}x_n(t) = f_n(t, x_1(t), x_2(t), ..., _n(t)), t \in J, \\ x_i(0) = \gamma_i \int_0^{\eta_i} A_i(s)\, x_i(s)\, ds, 0 < \eta_i < 1, i = 1, 2, ..., n, \end{cases} \tag{1}$$

where D^{α_i} denote the Caputo fractional derivatives, $0 < \alpha_i < 1$, $J = [0, 1]$, $\gamma_i \in \mathbb{R}$, A_i are continuous functions and f_i are some functions that will be specified later.

The rest of this paper is organized as follows: In section 2, we give some preliminaries and lemmas. In Section 3, we establish new conditions for the uniqueness of solutions and for the existence of at least one solution of problem (1). The first main result is based on Banach contraction principle and the second on Schaefer fixed point theorem. In the last section, some examples are discussed to illustrate the application of the established analytical results.

2 Preliminaries

The following notations and preliminary facts will be used throughout this paper:

Definition 1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[0,\infty)$ is defined as:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau; \alpha > 0, t > 0. \quad (2)$$

$$J^0 f(t) = f(t), \quad (3)$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

Definition 2 The Caputo derivative of order α of $f \in C^n([0,\infty[)$ is defined as:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, n-1 < \alpha, n \in N^*. \quad (4)$$

For more details about fractional calculus, we refer the reader to [17, 18]. For $i = 1, 2, ..., n$, we introduce the spaces

$$X_i = \{x_i(t), i = 1, 2, ..., n : x_i \in C(J, \mathbb{R})\}$$

endowed with the norm

$$\|x_i\|_{X_i} = \sup_{t \in J} |x_i|.$$

It is clear that for each $i = 1, 2, ..., n$, $\left(X_i, \|.\|_{X_i}\right)$ is a Banach space. The product space $\left(X_1 \times X_2 \times ... \times X_n, \|.\|_{X_1 \times X_2 \times ... \times X_n}\right)$ is also a Banach space with norm

$$\|(x_1, x_2, ..., x_n)\|_{X_1 \times X_2 \times ... \times X_n} = \max_{t \in J} \left(\|x_1\|_{X_1}, \|x_2\|_{X_2}, ..., \|x_n\|_{X_n}\right)$$

We give the following lemmas [19]:

Lemma 1 For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha} x = 0$ is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \quad (5)$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1, n = [\alpha] + 1$.

Lemma 2 Let $\alpha > 0$. Then we have

$$J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \quad (6)$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.

We prove also the following auxiliary lemma:

Lemma 3 Let $g \in C([0,1])$. The solution of the equation

$$D^{\alpha}x(t) = g(t), t \in J, 0 < \alpha < 1, \tag{7}$$

subject to the condition

$$x(0) = \gamma \int_0^{\eta} A(s)\, x(s)\, ds, 0 < \eta < 1, \tag{8}$$

is given by:

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\, ds \\ &+ \frac{\gamma}{1-\gamma \int_0^{\eta} A(s)\, ds} \int_0^{\eta} A(s) \left[\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} g(\tau)\, d\tau \right] ds \end{aligned} \tag{9}$$

provided that $1-\gamma \int_0^{\eta} A(s)\, ds \neq 0$.

Proof By lemmas 3 and 4, the general solution of (6) is given by the following formula

$$x(t) = \int_0^t \frac{(t-s)}{\Gamma(\alpha)} g(s)\, ds - c_0. \tag{10}$$

According to (7), we get

$$c_0 = \frac{-\gamma}{1-\gamma \int_0^{\eta} A(s)\, ds} \int_0^{\eta} A(s)\, J^{\alpha} g(s)\, ds. \tag{11}$$

Substituting the value of c_0 in (9), we obtain the desired quantity (8). □

3 Main Results

We begin by introducing the quantities:

$$\begin{aligned} M_i &= \frac{1}{\Gamma(\alpha_i+1)} + \frac{|\gamma_i| \sup_{0 \leq s \leq 1} |A_i(s)|\, \eta^{\alpha_i+1}}{\left|1-\gamma_i \int_0^{\eta_i} A_i(s)\, ds\right| \Gamma(\alpha_i+1)}, i = 1, ..., n, \\ M &= \max_{i=1,\ldots,n} M_i. \end{aligned}$$

We impose also the following hypotheses:

(H₁): The functions $f_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are continuous.

(H₂): There exist nonnegative functions$\{m_{i,j}\}_{i,j=1,\ldots,n}$ such that for all $t \in [0,1]$ and

$\overline{x}=(x_1,...,x_n)\,,\overline{y}=(y_1,...,y_n)\in\mathbb{R}^n$, we have

$$\begin{array}{lll}\left|f_1(t,\overline{x})-f_1(t,\overline{y})\right| & \leq & \sum_{i=1}^{n} m_{1,i}(t)\left|x_i-y_i\right|,\\ \left|f_2(t,\overline{x})-f_2(t,\overline{y})\right| & \leq & \sum_{i=1}^{n} m_{2,i}(t)\left|x_i-y_i\right|,\\ & & .\\ & & .\\ & & .\\ \left|f_n(t,\overline{x})-f_n(t,\overline{y})\right| & \leq & \sum_{i=1}^{n} m_{n,i}(t)\left|x_i-y_i\right|,\end{array}$$

where

$$m=\max_{i,j=1,...,n}\left\{\sup_{0\leq t\leq 1} m_{i,j}(t)\right\}.$$

($\mathbf{H_3}$): There exist positive constants $L_i, i=1,...,n,$ such that

$$\left|f_i(t,\overline{x})\right|\leq L_i,$$

for each $t\in J$ and all $\overline{x}\in\mathbb{R}^n$.

Our first result is based on Banach contraction principle. We have:

Theorem 1 *Suppose $\gamma_i\int_0^{\eta_i}A_i(s)\,ds\neq 1$ for all $i=1,...,n$, and assume that the hypothesis **(H_2)** holds. If*

$$M_{mn}<1, \tag{12}$$

then the system (1) *has a unique solution on J.*

Proof Consider the operator $T:X_1\times X_2\times...\times X_n\to X_1\times X_2\times...\times X_n$ defined by:

$$T(x_1,...,x_n)(t)=\left(T_1(x_1,...,x_n)(t)\,,T_2(x_1,...,x_n)(t)\,,...,T_n(x_1,...,x_n)(t)\right),$$

where

$$\begin{aligned}T_1(x_1,...,x_n)(t)\quad:&=\int_0^t\frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}f_1(\tau,x_1(\tau),...,x_n(\tau))\,d\tau+\frac{\gamma_1}{1-\gamma_1\int_0^{\eta_1}A_1(s)\,ds}\\ &\times\int_0^{\eta_1}A_1(s)\left[\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}f_1(\tau,x_1(\tau),...,x_n(\tau))\,d\tau\right]ds,\end{aligned}$$

and for all $i=1,...,n,$

$$\begin{aligned}T_i(x_1,...,x_n)(t)\quad:&=\int_0^t\frac{(t-\tau)^{\alpha_i-1}}{\Gamma(\alpha_i)}f_i(\tau,x_1(\tau),...,x_n(\tau))\,d\tau+\frac{\gamma_i}{1-\gamma_i\int_0^{\eta_i}A_i(s)\,ds}\\ &\times\int_0^{\eta_i}A_i(s)\left[\int_0^s\frac{(s-\tau)^{\alpha_i-1}}{\Gamma(\alpha_i)}f_i(\tau,x_1(\tau),...,x_n(\tau))\,d\tau\right]ds.\end{aligned}$$

We shall prove that T is contractive:
For $\overline{x}=(x_1,...,x_n),\overline{y}=(y_1,...,y_n)\in X_1\times X_2\times...\times X_n$ and for each $t\in J$, we have:

$$\begin{aligned}
& |T_1(x_1,...,x_n)(t)-T_1(y_1,...,y_n)(t)| \\
= \; & |\int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} f_1(\tau,x_1(\tau),...,x_n(\tau))\,d\tau \\
& +\frac{\gamma_1}{1-\gamma_1\int_0^{\eta_1}A_1(s)\,ds}\int_0^{\eta_1}A_1(s)\times\left[\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} f_1(\tau,x_1(\tau),...,x_n(\tau))\,d\tau\right]ds \\
& -\int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} f_1(\tau,y_1(\tau),...,y_n(\tau))\,d\tau
\end{aligned}$$

$$+\frac{\gamma_1}{1-\gamma_1\int_0^{\eta_1}A_1(s)\,ds}\int_0^{\eta_1}A_1(s)\left[\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} f_1(\tau,y_1(\tau),...,y_n(\tau))\,d\tau\right]ds\,|$$

Thus,

$$\begin{aligned}
& |T_1(x_1,...,x_n)(t)-T_1(y_1,...,y_n)(t)| \\
\leq \; & \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}\times\sup_{0\leq\tau\leq1}|f_1(\tau,x_1(\tau),...,x_n(\tau))-f_1(\tau,y_1(\tau),...,y_n(\tau))| \\
& +\frac{|\gamma_1|\sup_{0\leq s\leq1}|A_1(s)|}{\left|1-\gamma_1\int_0^{\eta_1}A_1(s)\,ds\right|}\int_0^{\eta_1}\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}d\tau ds \\
& \times\sup_{0\leq\tau\leq1}|f_1(\tau,x_1(\tau),...,x_n(\tau))-f_1(\tau,y_1(\tau),...,y_n(\tau))|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& |T_1(x_1,...,x_n)(t)-T_1(y_1,...,y_n)(t)| \\
\leq \; & \frac{1}{\Gamma(\alpha_1+1)}\times\sup_{0\leq\tau\leq1}|f_1(\tau,x_1(\tau),...,x_n(\tau))-f_1(\tau,y_1(\tau),...,y_n(\tau))| \\
& +\frac{|\gamma_1|\sup_{0\leq s\leq1}|A_1(s)|\,\eta_1^{\alpha_1+1}}{\left|1-\gamma_1\int_0^{\eta_1}A_1(s)\,ds\right|\Gamma(\alpha_1+2)} \\
& \times\sup_{0\leq\tau\leq1}|f_1(\tau,x_1(\tau),...,x_n(\tau))-f_1(\tau,y_1(\tau),...,y_n(\tau))|.
\end{aligned}$$

Using $\mathbf{(H_2)}$, we can write:

$$\begin{aligned}
& |T_1(x_1,...,x_n)(t)-T_1(y_1,...,y_n)(t)| \\
\leq \; & \frac{1}{\Gamma(\alpha_1+1)}+\frac{|\gamma_1|\sup_{0\leq s\leq1}|A_1(s)|\,\eta_1^{\alpha_1+1}}{\left|1-\gamma_1\int_0^{\eta_1}A_1(s)\,ds\right|\Gamma(\alpha_1+2)}m\sum_{i=1}^n|x_i(t)-y_i(t)|.
\end{aligned} \tag{13}$$

With some simple calculations, we obtain

$$|T_1(\overline{x})(t) - T_1(\overline{y})(t)| \leq M_1 mn \|\overline{x} - \overline{y}\|_{X_1 \times X_2 \times \ldots \times X_n}. \tag{14}$$

Hence, we have

$$\|T_1(\overline{x}) - T_1(\overline{y})\|_{X_1} \leq M_1 mn \|\overline{x} - \overline{y}\|_{X_1 \times X_2 \times \ldots \times X_n}. \tag{15}$$

With a similar method as before, for $i = 2, ..., n$, we can write

$$\|T_i(\overline{x}) - T_i(\overline{y})\|_{X_i} \leq M_i mn \|\overline{x} - \overline{y}\|_{X_1 \times X_2 \times \ldots \times X_n}. \tag{16}$$

Thanks to (15) and (16) yields the following inequality

$$\|T(\overline{x}) - T(\overline{y})\|_{X_1 \times X_2 \times \ldots \times X_n} \leq Mmn \|\overline{x} - \overline{y}\|_{X_1 \times X_2 \times \ldots \times X_n}. \tag{17}$$

Consequently by (12), we conclude that T is contractive. As a consequence of Banach fixed point theorem, we deduce that T has a unique fixed point which is a solution of (1). □

The second main result is the following theorem:

Theorem 2 *Suppose that for all* $i = 1, 2, ..., n$, $\gamma_i \int_0^{\eta_i} A_i(s)\,ds \neq 1$ *and assume that the hypotheses* ***(H₁)*** *and* ***(H₃)*** *are satisfied. Then, the system* (1) *has at least one solution on* J.

Proof We use Scheafer fixed point theorem to prove that T has at least one fixed point on $X_1 \times X_2 \times ... \times X_n$:

step 1: T is continuous on $X_1 \times X_2 \times ... \times X_n$ in view of $\mathbf{(H_1)}$

step 2: The operator T maps bounded sets into bounded sets in $X_1 \times X_2 \times ... \times X_n$. For $\sigma > 0$ we take $(x_1, ..., x_n) \in B_\sigma$ such that:
$B_\sigma := \left\{(x_1, ..., x_n) \in X_1 \times X_2 \times ... \times X_n, \|(x_1, ..., x_n)\|_{X_1 \times X_2 \times \ldots \times X_n} \leq \sigma\right\}$. Then, for each $t \in J$, we have:

$$\begin{aligned} |T_1(x_1, ..., x_n)(t)| \leq & \int_0^t \frac{(t-\tau)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \times \sup_{0 \leq \tau \leq 1} |f_1(\tau, x_1(\tau), ..., x_n(\tau))| \\ & + \frac{|\gamma_1| \sup_{0 \leq s \leq 1} |A_1(s)|}{\left|1 - \gamma_1 \int_0^{\eta_1} A_1(s)\,ds\right|} \int_0^{\eta_1} \int_0^s \frac{(s-\tau)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} d\tau ds \\ & \times \sup_{0 \leq \tau \leq 1} |f_1(\tau, x_1(\tau), ..., x_n(\tau))|. \end{aligned} \tag{18}$$

Thanks to $\mathbf{(H_3)}$, we obtain

$$|T_1(x_1, ..., x_n)(t)| \leq \frac{L_1}{\Gamma(\alpha_1 + 1)} + \frac{L_1 |\gamma_1| \sup_{0 \leq s \leq 1} |A_1(s)|}{\left|1 - \gamma_1 \int_0^{\eta_1} A_1(s)\,ds\right| \Gamma(\alpha_1 + 2)}. \tag{19}$$

Thus,

$$|T_1(x_1, ..., x_n)(t)| \leq L_1 M_1, t \in J. \tag{20}$$

Consequently,

$$\|T_1(x_1,...,x_n)\|_{X_1} \leq L_1 M_1. \tag{21}$$

Similarly, for all $i=2,...,n$, we can write

$$\|T_i(x_1,...,x_n)\|_{X_i} \leq L_i M_i. \tag{22}$$

Consequently, we obtain,

$$\|T(x_1,...,x_n)\|_{X_1\times X_2\times...\times X_n} \leq M \max\{L_i\}_{i=1}^n. \tag{23}$$

Therefore,

$$\|T(x_1,...,x_n)\|_{X_1\times X_2\times...\times X_n} < \infty. \tag{24}$$

Step 3: The equi-continuity of T: Let us take $(x_1,...,x_n)\in B_\sigma$, $t_1,t_2\in J$, such that $t_1<t_2$. We have:

$$\begin{aligned} & |T_1(x_1,...,x_n)(t_2) - T_1(x_1,...,x_n)(t_2)| \\ \leq & \int_0^{t_1} \frac{(t_2-\tau)^{\alpha_1-1}-(t_1-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} \times \sup_{0\leq\tau\leq 1} |f_1(\tau,x_1(\tau),...,x_n(\tau))| \\ & + \int_{t_1}^{t_2} \frac{(t_2-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} \times \sup_{0\leq\tau\leq 1} |f_1(\tau,x_1(\tau),...,x_n(\tau))|. \end{aligned} \tag{25}$$

Thanks to $\mathbf{(H_3)}$, we can write

$$|T_1(x_1,...,x_n)(t_2) - T_1(x_1,...,x_n)(t_2)| \leq \frac{L_1}{\Gamma(\alpha_1+1)}(t_2^{\alpha_1}-t_1^{\alpha_1}). \tag{26}$$

Thus,

$$\|T_1(x_1,...,x_n) - T_1(x_1,...,x_n)\|_{X_1} \leq \frac{L_1}{\Gamma(\alpha_1+1)}(t_2^{\alpha_1}-t_1^{\alpha_1}). \tag{27}$$

Analogously, for all $i=2,...,n$, we can write

$$\|T_1(x_1,...,x_n) - T_1(x_1,...,x_n)\|_{X_i} \leq \frac{L_i}{\Gamma(\alpha_i+1)}(t_2^{\alpha_i}-t_1^{\alpha_i}). \tag{28}$$

And then,

$$\|T_1(x_1,...,x_n) - T_1(x_1,...,x_n)\|_{X_1\times X_2\times...\times X_n} \leq \max\left\{\frac{L_i}{\Gamma(\alpha_i+1)}\right\}_{i=1}^n (t_2^{\alpha_i}-t_1^{\alpha_i}). \tag{29}$$

This implies that $\|T_1(x_1,...,x_n) - T_1(x_1,...,x_n)\|_{X_1\times X_2\times...\times X_n} \to 0$ as $t_2\to t_1$:

By Arzela-Ascoli theorem, we conclude that T is a completely continuous operator.

Step 4: We shall prove that the set Ω defined by

$$\Omega = \{(x_1,...,x_n)\in X_1\times X_2\times...\times X_n, (x_1,...,x_n) = \lambda T(x_1,...,x_n), 0<\lambda<1\}.$$

is bounded. Let $(x_1,...,x_n)\in\Omega$, then $(x_1,...,x_n)=\lambda T(x_1,...,x_n)$, for some $0<\lambda<1$. Thus, for each $t\in J$, we have:

$$\begin{aligned} x_1(t) &= \lambda T_1(x_1,...,x_n)(t) \\ x_2(t) &= \lambda T_2(x_1,...,x_n)(t) \\ &. \\ &. \\ &. \\ x_n(t) &= \lambda T_n(x_1,...,x_n)(t). \end{aligned}$$

Then,

$$\begin{aligned} \frac{1}{\lambda}|x_1(t)| \leq & \int_0^t \frac{(t-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)} \times \sup_{0\leq\tau\leq 1}|f_1(\tau,x_1(\tau),...,x_n(\tau))| \\ & + \frac{|\gamma_1|\sup_{0\leq s\leq 1}|A_1(s)|}{\left|1-\gamma_1\int_0^{\eta_1}A_1(s)\,ds\right|}\int_0^{\eta_1}\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}d\tau ds \\ & \times \sup_{0\leq\tau\leq 1}|f_1(\tau,x_1(\tau),...,x_n(\tau))|. \end{aligned} \tag{30}$$

Thanks to $\mathbf{(H_3)}$, we can write

$$\frac{1}{\lambda}|x_1(t)| \leq \frac{L_1}{\Gamma(\alpha_1+1)} + \frac{L_1|\gamma_1|\sup_{0\leq s\leq 1}|A_1(s)|\,\eta_1^{\alpha_1+1}}{\left|1-\gamma_1\int_0^{\eta_1}A_1(s)\,ds\right|\Gamma(\alpha_1+2)}. \tag{31}$$

Therefore,

$$|x_1(t)| \leq \lambda L_1\left[\frac{1}{\Gamma(\alpha_1+1)} + \frac{|\gamma_1|\sup_{0\leq s\leq 1}|A_1(s)|\,\eta_1^{\alpha_1+1}}{\left|1-\gamma_1\int_0^{\eta_1}A_1(s)\,ds\right|\Gamma(\alpha_1+2)}\right]. \tag{32}$$

Hence,

$$|x_1(t)| \leq \lambda L_1 M_1, t\in J. \tag{33}$$

Thus,

$$\|x_1\|_{X_1} \leq \lambda L_1 M_1. \tag{34}$$

With the same arguments as before and using $\mathbf{(H_3\ )}$, we can state that

$$\|x_i\|_{X_i} \leq \lambda L_i M_i. \tag{35}$$

Thanks to (34) and (35), we obtain

$$\|(x_1,x_2,...,x_n)\|_{X_1\times X_2\times...\times X_n} \leq \lambda\max\{L_iM_i\}_{i=1}^n. \tag{36}$$

Hence,

$$\|(x_1,x_2,...,x_n)\|_{X_1\times X_2\times...\times X_n} < \infty.$$

This shows that Ω is bounded. As consequence of Schaefer's fixed point theorem, we deduce that T at least a fixed point, which is a solution of the fractional differential system (1). □

4 Example

To illustrate our main results, we present the following examples:

Example 1: Consider the following fractional differential system:

$$\begin{cases} D^{\alpha_1}x_1(t) = \frac{e^{-t}\sin(x_1(t)+x_2(t)+x_3(t))}{16(\pi t^2+1)} + 2t^2+1, t\in[0,1], \\ D^{\alpha_2}x_2(t) = \frac{|x_1(t)|+|x_2(t)|+|x_3(t)|}{(\pi t+20)(1+|x_1(t)|+|x_2(t)|+|x_3(t)|)} + e^t, t\in[0,1], \\ D^{\alpha_3}x_3(t) = \frac{\sin(x_1(t))+\sin(x_2(t))+\sin(x_3(t))}{(t^2+t+20)} + e^{-t}, t\in[0,1], \\ x_i(0) = \gamma_i\int_0^{\eta_i}\frac{s^i}{16}x_i(s)\,ds, (i=1,2,3), \end{cases} \tag{37}$$

with $\alpha_i = \frac{1}{2}$, $\eta_i = \frac{1}{4}$, $(i=1,2,3)$, $\gamma_1 = -16$, $\gamma_2 = -24$, $\gamma_3 = -64$ and $A_i(t) = \frac{t^i}{16}, t\in[0,1]$.

For $(u_1,v_1,z_1),(u_2,v_2,z_2)\in\mathbb{R}^3, t\in[0;1]$, we have

$$\begin{aligned} f_1(t,u,v,z) &= \frac{e^{-t}\sin(u+v+z)}{16(\pi t^2+1)} + 2t^2+1, \\ f_2(t,u,v,z) &= \frac{|u|+|v|+|z|}{(\pi t+20)(1+|u|+|v|+|z|)} + e^t, \\ f_3(t,u,v,z) &= \frac{\sin(u)+\sin(v)+\sin(z)}{(t^2+t+20)} + e^{-t}, \end{aligned}$$

and

$$\begin{aligned} |f_1(t,u_2,v_2,z_2) - f_1(t,u_1,v_1,z_1)| &\le \frac{e^{-t}}{16(\pi t^2+1)}(|u_2-u_1|\,|v_2-v_1|\,|z_2-z_1|), \\ |f_2(t,u_2,v_2,z_2) - f_2(t,u_1,v_1,z_1)| &\le \frac{1}{(\pi t+20)}(|u_2-u_1|\,|v_2-v_1|\,|z_2-z_1|), \\ |f_2(t,u_2,v_2,z_2) - f_2(t,u_1,v_1,z_1)| &\le \frac{1}{(t^2+t+20)}(|u_2-u_1|\,|v_2-v_1|\,|z_2-z_1|), \end{aligned}$$

So we take $m_{11}(t) = m_{12}(t) = m_{13}(t) = \frac{e^{-t}}{16(\pi t^2+1)}$, $m_{21}(t) = m_{22}(t) = m_{23}(t) = \frac{1}{(\pi t+20)}$, $m_{31}(t) = m_{32}(t) = m_{33}(t) = \frac{1}{(t^2+t+20)}$, and then, we obtain $m = \max\{m_{ij}\}_{i,j=1}^3 = \frac{1}{16}$. On the other hand,$\gamma_i\int_0^{\eta_i}A_i(s)\,ds \neq 1$, $i=1,2,3$, and

$$\begin{aligned} M_1 &= \frac{2}{\sqrt{\pi}} + \frac{16}{99\sqrt{\pi}} = 1,219, \\ M_2 &= \frac{2}{\sqrt{\pi}} + \frac{32}{129\sqrt{\pi}} = 1,269, \\ M_1 &= \frac{2}{\sqrt{\pi}} + \frac{512}{771\sqrt{\pi}} = 1,504. \end{aligned}$$

Hence, we obtain

$$Mmn = 1,504\times\frac{1}{16}\times 3 = 0,282 < 1.$$

The conditions of Theorem 6 hold. Therefore, the problem (37) has a unique solution on $[0,1]$.

Example 2: Consider the following problem:

$$\begin{cases} D^{\frac{1}{4}}x_1(t) = \frac{e^{-t}}{2+\sin(x_1(t))+\cos(x_2(t)+x_3(t))}, t \in [0,1], \\ D^{\frac{2}{5}}x_2(t) = \frac{e^{-2t}\sin(x_1(t))}{2+\cos(x_2(t)+x_3(t))}, t \in [0,1], \\ D^{\alpha_1}x_3(t) = e^{-2t}\sin(x_1(t)) + \cos(x_2(t)+x_3(t)), t \in [0,1], \\ x_i(0) = -\sqrt{2}\int_0^{\eta_i} \exp(is)\, x_i(s)\, ds, (i = 1,2,3), \end{cases} \tag{38}$$

For this example, we have $\alpha_1 = \frac{1}{4}, \alpha_2 = \frac{2}{5}, \alpha_3 = \frac{2}{7}$, $\gamma_1 = \gamma_2 = \gamma_3 = -\sqrt{2}$ We take $\eta_1 = \frac{4}{5}$, $\eta_2 = \eta_3 = \frac{1}{5}$, $A_i(t) = \exp(it); (i = 1,2,3)$, and for $(u,v,z) \in \mathbb{R}^3, t \in [0,1]$, we have

$$\begin{aligned} f_1(t,u,v,z) &= \frac{e^{-t}}{2+\sin(u)+\cos(v+z)}, \\ f_2(t,u,v,z) &= \frac{e^{-2t}\sin(u)}{2+\cos(v+z)}, \\ f_3(t,u,v,z) &= e^{-2t}\sin(u)+\cos(v+z). \end{aligned}$$

It is clear that the conditions of Theorem 7 hold. Then the problem (38) has at least one solution on $[0,1]$.

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