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Oscillation and Fixed Points of Derivatives of Solutions of Some Linear Differential Equations

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Abstract In this paper, we investigate the relationship between solutions and their derivatives of the differential equation $f^{(k)} + Af = 0$ for $k \geqslant 2$ and meromorphic functions of infinite iterated p-order, where A is a meromorphic function of finite iterated p-order $\rho_p(A) = \rho$. We also study the oscillation theory of derivatives of the nonhomogeneous differential equation $f^{(k)} + Af = F$ for $k \geqslant 2$.

Keywords Linear differential equations; meromorphic solutions; iterated p-order; iterated exponent of convergence of the sequence of distinct zeros.

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1 Introduction and Statement of Results

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions [1, 2]. Throughout this paper, the term "meromorphic" will mean meromorphic in the whole complex plane. For the definition of the iterated order of a meromorphic function, we use the same definition as in [3–5]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r = \exp\left(\exp_p r\right),\ p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right),\ p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r,\ \log_0 r := r,\ \log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1 [4] Let f be a meromorphic function. Then the iterated p-order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} \quad (p \geqslant 1 \text{ is an integer}),$$

where $T(r,f)$ is the Nevanlinna characteristic function of f. For $p = 1$, this notation is called order and for $p = 2$ hyper-order.

Definition 2 [4] The finiteness degree of the order of a meromorphic function f is defined by

$$i(f) = \begin{cases} 0, & \text{for } f \text{ rational}, \\ \min\{p \in \mathbb{N} : \rho_p(f) < +\infty\}, & \text{for } f \text{ transcendental for which} \\ & \text{some } p \in \mathbb{N} \text{ with } \rho_p(f) < +\infty \text{ exists}, \\ +\infty, & \text{for } f \text{ with } \rho_p(f) = +\infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

Definition 3 [4] The iterated convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_p(f) = \limsup_{r\to+\infty} \frac{\log_p N(r, 1/f)}{\log r} \ \ (p \geqslant 1 \text{ is an integer}),$$

where $N\left(r, \frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z : |z| \leqslant r\}$. Similarly, the iterated convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}(r, 1/f)}{\log r} \ \ (p \geqslant 1 \text{ is an integer}),$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z : |z| \leqslant r\}$.

Definition 4 [4] The finiteness degree of the iterated convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$i_\lambda(f) = \begin{cases} 0, & \text{if } n(r, 1/f) = O(\log r), \\ \min\{p \in \mathbb{N} : \lambda_p(f) < \infty\}, & \text{if } \lambda_p(f) < \infty \text{ for some } p \in \mathbb{N}, \\ \infty, & \text{if } \lambda_p(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

Remark 1 Similarly, we can define the finiteness degree $i_{\overline{\lambda}}(f)$ of $\overline{\lambda}_p(f)$.

Definition 5 [6] Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of fixed points of $f(z)$ is defined by

$$\tau_p(f) = \lambda_p(f - z) = \limsup_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{f-z}\right)}{\log r}, \quad (p \geqslant 1 \text{ is an integer}).$$

Similarly, the iterated exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}_p(f) = \overline{\lambda}_p(f - z) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}, \quad (p \geqslant 1 \text{ is an integer}).$$

For $p = 1$, this notation is called the exponent of convergence of the sequence of distinct fixed points and for $p = 2$ the hyper-exponent of convergence of the sequence of distinct fixed points [7]. Thus $\overline{\tau}_p(f) = \overline{\lambda}_p(f - z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

For $k \geqslant 2$, we consider the linear differential equation

$$f^{(k)} + A(z) f = 0, \tag{1}$$

where $A(z)$ is a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$.

Many important results have been obtained on the fixed points of general transcendental meromorphic functions [8]. In the year 2000, Chen [9] considered fixed points of solutions of second-order linear differential equations and obtained precise estimation of the number of fixed points of solutions. After him, Wang and Lü [10] investigated the fixed points and hyper-order of solutions of second order linear differential equations with meromorphic coefficients and their derivatives, and obtained the following result.

Theorem 1 *[10] Suppose that $A(z)$ is a transcendental meromorphic function satisfying $\delta(\infty, A) = \liminf_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, $\rho(A) = \rho < +\infty$. Then every meromorphic solution $f(z) \not\equiv 0$ of the equation*

$$f'' + A(z) f = 0,$$

satisfies that f and f', f'' all have infinitely many fixed points and

$$\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \rho(f) = +\infty,$$

$$\overline{\tau}_2(f) = \overline{\tau}_2(f') = \overline{\tau}_2(f'') = \rho_2(f) = \rho.$$

Theorem 1 has been generalized to higher order differential equations by Liu Ming-Sheng and Zhang Xiao-Mei [7] as follows.

Theorem 2 *[7] Suppose that $k \geqslant 2$ and $A(z)$ is a transcendental meromorphic function satisfying $\delta(\infty, A) = \delta > 0$, $\rho(A) = \rho < +\infty$. Then every meromorphic solution $f(z) \not\equiv 0$ of (1), satisfies that f and $f', f'', \cdots, f^{(k)}$ all have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \cdots = \overline{\tau}\left(f^{(k)}\right) = \rho(f) = +\infty,$$

$$\overline{\tau}_2(f) = \overline{\tau}_2(f') = \overline{\tau}_2(f'') = \cdots = \overline{\tau}_2\left(f^{(k)}\right) = \rho_2(f) = \rho.$$

In the year 2008, Belaïdi [11] obtained an extension of Theorem 2, by studying the relation between solutions and their derivatives of the differential equation (1) and meromorphic functions of finite iterated $p-$order and proved the following result.

Theorem 3 *[11] Let $k \geqslant 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:*
(i) all poles of f are of uniformly bounded multiplicity or that
(ii) $\delta(\infty, f) > 0$. If $\varphi(z) \not\equiv 0$ is a meromorphic function with finite iterated $p-$order $\rho_p(\varphi) < +\infty$, then every meromorphic solution $f(z) \not\equiv 0$ of (1) satisfies

$$\overline{\lambda}_p(f - \varphi) = \overline{\lambda}_p(f' - \varphi) = \cdots = \overline{\lambda}_p\left(f^{(k)} - \varphi\right) = \rho_p(f) = +\infty, \tag{2}$$

$$\overline{\lambda}_{p+1}(f - \varphi) = \overline{\lambda}_{p+1}(f' - \varphi) = \cdots = \overline{\lambda}_{p+1}\left(f^{(k)} - \varphi\right) = \rho_{p+1}(f) = \rho_p(A) = \rho. \tag{3}$$

The main purpose of this paper is to study the relation between solutions and their derivatives of the differential equation (1) and meromorphic functions of infinite iterated $p-$order. We obtain an extension of Theorem 3. In fact, we prove the following results.

Theorem 4 *Let $k \geqslant 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:*
(i) all poles of f are of uniformly bounded multiplicity or that
(ii) $\delta(\infty, f) > 0$.

If $\varphi(z) \not\equiv 0$ is a meromorphic function with $\rho_{p+1}(\varphi) < \rho_p(A)$, then every meromorphic solution $f(z) \not\equiv 0$ of (1) satisfies $i_{\overline{\lambda}}\left(f^{(j)} - \varphi\right) = i_{\lambda}\left(f^{(j)} - \varphi\right) = i(f) = p + 1$, $j \in \{0, 1, \cdots, k\}$ and (3).

Remark 2 For $\rho_p(\varphi) < \infty$ in Theorem 4, we obtain Theorem 3.

Setting $p = 1$ and $\varphi(z) = z$ in Theorem 4, we obtain the following corollary.

Corollary 1 Let $k \geqslant 2$ and $A(z)$ be a transcendental meromorphic function of finite order $\rho(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either: (i) all poles of f are of uniformly bounded multiplicity or that
(ii) $\delta(\infty, f) > 0$.

Then every meromorphic solution $f(z) \not\equiv 0$ *of* (1) *satisfies that* f *and* $f', f'', \cdots, f^{(k)}$ all have infinitely many fixed points and

$$\overline{\tau}\left(f^{(j)}\right) = \tau\left(f^{(j)}\right) = \rho(f) = +\infty,\ j \in \{0, 1, \cdots, k\},$$

$$\overline{\tau}_2\left(f^{(j)}\right) = \tau_2\left(f^{(j)}\right) = \rho_2(f) = \rho,\ j \in \{0, 1, \cdots, k\}.$$

When $A(z)$ is a transcendental entire function, we get the following results.

Theorem 5 *Assume $A(z)$ is a transcendental entire function of finite iterated order $\rho = \rho_p(A) > 0$. If $\varphi(z) \not\equiv 0$ is an entire function with $\rho_{p+1}(\varphi) < \rho_p(A)$, then every solution $f(z) \not\equiv 0$ of* (1) *satisfies* $i_{\overline{\lambda}}(f^{(j)} - \varphi) = i_\lambda(f^{(j)} - \varphi) = i(f) = p + 1,\ j \in \{0, 1, \cdots, k\}$ *and* (3).

Setting $p = 1$ and $\varphi(z) = z$ in Theorem 5, we obtain the following corollary.

Corollary 2 Let $k \geqslant 2$ and $A(z)$ be a transcendental entire function of finite order $\rho(A) = \rho > 0$. Then every solution $f(z) \not\equiv 0$ *of* (1) *satisfies that* f *and* $f', f'', \cdots, f^{(k)}$ all have infinitely many fixed points and

$$\overline{\tau}\left(f^{(j)}\right) = \tau\left(f^{(j)}\right) = \rho(f) = +\infty,\ j \in \{0, 1, \cdots, k\},$$

$$\overline{\tau}_2\left(f^{(j)}\right) = \tau_2\left(f^{(j)}\right) = \rho_2(f) = \rho,\ j \in \{0, 1, \cdots, k\}.$$

The following corollary may to study the relation between the solutions of two differential equations and entire functions.

Corollary 3 Let $k, n \geqslant 2$ and $A(z), B(z)$ be transcendental entire functions of finite iterated order $\rho_p(A) = \rho_A > \rho_p(B) = \rho_B > 0$ such that $\rho_A \neq \rho_B$. Let f_1 be a solution of the equation

$$f^{(k)} + A(z) f = 0 \tag{4}$$

and f_2 be a solution of the equation

$$f^{(n)} + B(z) f = 0. \tag{5}$$

If $\varphi(z) \not\equiv 0$ is an entire function with $\rho_{p+1}(\varphi) < \rho_A$ and $\rho_{p+1}(\varphi) \neq \rho_B$, then f_1 and f_2 satisfy

$$\overline{\lambda}_{p+1}\left(f_1^{(i)} - f_2^{(j)} - \varphi\right) = \lambda_{p+1}\left(f_1^{(i)} - f_2^{(j)} - \varphi\right) = \rho_{p+1}\left(f_1^{(i)} - f_2^{(j)} - \varphi\right) = \rho_A,$$

$0 \leqslant i \leqslant k,\ j \in \mathbb{N}$.

Theorem 6 *Let $k \geqslant 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:*

1. *all poles of f are of uniformly bounded multiplicity or that*
2. *$\delta(\infty, f) > 0$.*

Let $\varphi(z) \not\equiv 0$ be a meromorphic function with $\rho_{p+1}(\varphi) < \rho_p(A)$ and $\varphi'(z) - \varphi(z) \not\equiv 0$. If $f \not\equiv 0$ is a meromorphic solution of equation (1), then the differential polynomial $g = f^{(k-1)} + f^{(k-2)} + \cdots + f' + f$ satisfies $i_{\overline{\lambda}}(g-\varphi) = i_\lambda(g-\varphi) = i(f) = p+1$ and

$$\overline{\lambda}_{p+1}(g-\varphi) = \lambda_{p+1}(g-\varphi) = \rho_{p+1}(f) = \rho_p(A) = \rho.$$

In 1994, Chen [12] studied the oscillation of solutions of nonhomogeneous linear differential equations and gave a very good general theorem : If the order of solution of nonhomogeneous linear differential equation is dominate the order of the coefficients and the second member, then the exponent of convergence and the order of solution are equal.

The natural question which is arises: What about the oscillation of derivatives of solution of nonhomogeneous differential equations?

The second main purpose of this paper is to give an answer to the above question for the nonhomogeneous differential equation

$$f^{(k)} + A(z) f = F, \tag{6}$$

where A, F are meromorphic functions. In fact, we prove the following result.

Theorem 7 *Let $k \geqslant 2$ and $A(z), F(z)$ be transcendental meromorphic functions of finite iterated order such that $\frac{F(z)}{A(z)}$ is not a polynomial of degree less than k. If f is a meromorphic solution of equation (6) with $\rho_p(f) = \rho > \max\{\rho_p(F), \rho_p(A)\}$, then f satisfies*

$$\overline{\lambda}_p\left(f^{(j)}\right) = \lambda_p\left(f^{(j)}\right) = \rho_p(f), \ j \in \{0, 1, \cdots, k\}.$$

Remark 3 In Theorem 7, if we do not have the condition $\frac{F(z)}{A(z)}$ is not a polynomial of degree less than k, then the conclusions of Theorem 7 can not hold. For example the equation

$$f'' - \left(e^{2z} + e^z\right) f = -z\left(e^{2z} + e^z\right),$$

has a solution $f(z) = e^{e^z} + z$ with $\rho_2(f) = 1$ and $\overline{\lambda}_2(f'') = \lambda_2(f'') = 0$, here we have $\frac{F(z)}{A(z)} = z$.

2 Auxiliary Lemmas

The following lemmas will be used in the proofs of our theorems.

Lemma 1 [11] Let $k \geqslant 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$. Suppose, moreover, that either:

1. all poles of f are of uniformly bounded multiplicity or that

2. $\delta(\infty, f) > 0$.

Then every meromorphic solution $f(z) \not\equiv 0$ *of* (1) *satisfies* $\rho_p(f) = +\infty$ *and* $\rho_{p+1}(f) = \rho_p(A) = \rho$.

Lemma 2 [11] Let $p \geq 1$ be an integer and let $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ be finite iterated $p-$order meromorphic functions. If f is a meromorphic solution with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho < +\infty$ of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F, \tag{7}$$

then $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p+1$, $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$.

Lemma 3 [13] Let $p \geqslant 1$ be an integer, and let $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ be meromorphic functions. If f is a meromorphic solution of equation (7) such that

1. $\max\{i(F), i(A_j) \ \ (j = 0, ..., k-1)\} < i(f) = p$ or
2. $\max\{\rho_p(F), \rho_p(A_j) \ \ (j = 0, ..., k-1)\} < \rho_p(f) < +\infty$,

then $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p$ *and* $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f)$.

Lemma 4 (see Remark 1.3 of [4]) If f is a meromorphic function with $i(f) = p$, then $\rho_p(f') = \rho_p(f)$.

The following lemma is a corollary of Theorem 2.3 in [4].

Lemma 5 Assume $A(z)$ is an entire function with $i(A) = p$, and assume $1 \leqslant p < +\infty$. Then, for all non-trivial solutions f of (1), we have

$$\rho_p(f) = \infty \ \textit{and} \ \rho_{p+1}(f) = \rho_p(A).$$

3 Proof of The Theorems and Corollary

Proof **[Theorem 4]** Assume that $f \not\equiv 0$ is a meromorphic solution of equation (1). Then by Lemma 1, we have $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_p(A) = \rho$. Set $w_j = f^{(j)} - \varphi$, $j \in \{0, 1, \cdots, k\}$. Then by $\rho_{p+1}(\varphi) < \rho_{p+1}(f) = \rho_p(A)$, Lemma 4 and Lemma 1 we get $\rho_{p+1}(w_j) = \rho_p(A) = \rho$. We can rewrite (1) as

$$f + \frac{f^{(k)}}{A} = 0. \tag{8}$$

By differentiating both sides of (8), j times we get

$$f^{(j)} + \left(\frac{f^{(k)}}{A}\right)^{(j)} = 0. \tag{9}$$

Thus, $w_j = f^{(j)} - \varphi$ is a solution of the equation

$$w_j + \left(\frac{w_j^{(k-j)}}{A}\right)^{(j)} = -\left[\varphi + \left(\frac{\varphi^{(k-j)}}{A}\right)^{(j)}\right] = B. \tag{10}$$

Now, we prove that $B \not\equiv 0$. Suppose that $B = 0$. If φ is a polynomial, then $\rho_p(\varphi) = \rho_p\left(\left(\frac{\varphi^{(k-j)}}{A}\right)^{(j)}\right) = \rho_p(A) = 0$ and this is a contradiction. Hence $B \not\equiv 0$. If φ is not a polynomial, then we have $\varphi + \left(\frac{\varphi^{(k-j)}}{A}\right)^{(j)} = 0$. By differentiation this expression $(k-j)$ times we get

$$\left(\frac{\varphi^{(k-j)}}{A}\right)^{(k)} + A\left(\frac{\varphi^{(k-j)}}{A}\right) = 0. \tag{11}$$

Thus $\frac{\varphi^{(k-j)}}{A}$ is a nontrivial solution of equation (1) with $\rho_{p+1}\left(\frac{\varphi^{(k-j)}}{A}\right) < \rho_p(A)$, this is a contradiction by Lemma 1. Hence $B \not\equiv 0$. By (10) we can write

$$w_j^{(k)} + \Phi_{k-1} w_j^{(k-1)} + \cdots + \Phi_0 w_j = AB, \tag{12}$$

where Φ_i $(i = 0, 1, \cdots, k)$ are meromorphic functions with $\rho_p(\Phi_i) < \infty$ $(i = 0, 1, \cdots, k)$. By $AB \not\equiv 0$ and $\rho_p(AB) < +\infty$ with Lemma 2 we have $i_{\overline{\lambda}}(w_j) = i_\lambda(w_j) = i(w_j) = p+1$ and $\overline{\lambda}_{p+1}(w_j) = \lambda_{p+1}(w_j) = \rho_{p+1}(w_j) = \rho_p(A)$. Thus $i_{\overline{\lambda}}\left(f^{(j)} - \varphi\right) = i_\lambda\left(f^{(j)} - \varphi\right) = i(f) = p+1$ and $\overline{\lambda}_{p+1}\left(f^{(j)} - \varphi\right) = \lambda_{p+1}\left(f^{(j)} - \varphi\right) = \rho_{p+1}(f) = \rho_p(A)$ $(j \in \{0, 1, \cdots, k\})$. □

Proof [**Theorem 5**] Assume that $f \not\equiv 0$ is a solution of equation (1). Then by Lemma 5, we have $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_p(A) = \rho$. By using similar arguments as in the proof of Theorem 4, we obtain Theorem 5. □

Proof [**Corollary 3**] Assume that f_1 and f_2 are solutions of equations (4) and (5). Set $\psi = f_2^{(j)} + \varphi$ $(j \in \mathbb{N})$. Then by $\rho_{p+1}(\varphi) < \rho_A$, $\rho_{p+1}(f_2) = \rho_B < \rho_A$ and $\rho_{p+1}(\varphi) \neq \rho_B$ we have $\rho_{p+1}(\psi) < \rho_A$. By application of Theorem 5 for the solution f_1 of (4), we get

$$\overline{\lambda}_{p+1}\left(f_1^{(i)} - \psi\right) = \lambda_{p+1}\left(f_1^{(i)} - \psi\right) = \rho_{p+1}\left(f_1^{(i)} - f_2^{(j)} - \varphi\right) = \rho_{p+1}(f_1) = \rho_A,$$

$0 \leqslant i \leqslant k,\ j \in \mathbb{N}$. □

Proof [**Theorem 6**] Assume that $f \not\equiv 0$ is a meromorphic solution of equation (1). Then by Lemma 1, we have $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho_p(A) = \rho$. Set $g = f^{(k)} + f^{(k-1)} + \cdots + f'$. First, we prove that $\rho_p(g) = \rho_p(f) = +\infty$ and $\rho_{p+1}(g) = \rho_{p+1}(f) = \rho_p(A)$. We have $g = f^{(k-1)} + f^{(k-2)} + \cdots + f' + f$ then $g' = f^{(k)} + f^{(k-1)} + \cdots + f'$ and

$$g - g' = f - f^{(k)}. \tag{13}$$

Substituting $f^{(k)} = -Af$ into (13) we get

$$f = \frac{g - g'}{1 + A}. \tag{14}$$

If we suppose that $\rho_p(g) < \infty$ then by (14), we obtain that $\rho_p(f) < \infty$. This is a contradiction. Hence $\rho_p(g) = \rho_p(f) = +\infty$. By $g = f^{(k-1)} + f^{(k-2)} + \cdots + f' + f$ we have by Lemma 4, $\rho_{p+1}(g) \leqslant \rho_{p+1}(f)$ and by (14) we have $\rho_{p+1}(f) \leqslant \rho_{p+1}(g)$ then $\rho_{p+1}(g) = \rho_{p+1}(f)$. Now, we prove that $\overline{\lambda}_{p+1}(g-\varphi) = \lambda_{p+1}(g-\varphi) = \rho_{p+1}(f) = \rho_p(A)$. Suppose that $f \not\equiv 0$ is a meromorphic solution of equation (1). Set $w = g - \varphi$, then we have $\rho_{p+1}(w) = \rho_{p+1}(g) = \rho_{p+1}(f) = \rho_p(A)$. In order to prove $\overline{\lambda}_{p+1}(g-\varphi) = \lambda_{p+1}(g-\varphi) = \rho_{p+1}(f) = \rho_p(A)$, we need to prove $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(f) = \rho_p(A)$. By $g = w + \varphi$, we get from (14)

$$f = \frac{w - w'}{1+A} + \frac{\varphi - \varphi'}{1+A}. \tag{15}$$

Substituting (15) into equation (1), we obtain

$$\left(\frac{w-w'}{1+A}\right)^{(k)} + A\left(\frac{w-w'}{1+A}\right) = -\left[\left(\frac{\varphi-\varphi'}{1+A}\right)^{(k)} + A\left(\frac{\varphi-\varphi'}{1+A}\right)\right]$$

which implies that

$$w^{(k+1)} + \Phi_k w^{(k)} + \cdots + \Phi_0 w = (A+1)\left(\left(\frac{\varphi-\varphi'}{1+A}\right)^{(k)} + A\left(\frac{\varphi-\varphi'}{1+A}\right)\right) = F, \tag{16}$$

where Φ_j $(j = 0, 1, \cdots, k)$ are meromorphic functions with $\rho_p(\Phi_j) < \infty$ $(j = 0, 1, \cdots, k)$. Since $\varphi - \varphi' \not\equiv 0$ and $\rho_{p+1}\left(\frac{\varphi-\varphi'}{1+A}\right) < \rho_p(A)$, then by Lemma 1 it follows that $F \not\equiv 0$. By Lemma 2, we obtain $i_{\overline{\lambda}}(w) = i_\lambda(w) = i(w) = p+1$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_p(A)$, i.e., $i_{\overline{\lambda}}(g-\varphi) = i_\lambda(g-\varphi) = i(g-\varphi) = i(f) = p+1$ and $\overline{\lambda}_{p+1}(g-\varphi) = \lambda_{p+1}(g-\varphi) = \rho_{p+1}(f) = \rho_p(A)$. □

Proof **[Theorem 7]** Assume that f is a meromorphic solution of equation (6). Set $w_j = f^{(j)}$, $j \in \{0, 1, \cdots, k\}$. Then, by Lemma 4, we have $\rho_p(w_j) = \rho_p(f)$. We can rewrite (6) as

$$f + \frac{f^{(k)}}{A} = \frac{F}{A}. \tag{17}$$

By differentiating both sides of (17), j times we get

$$f^{(j)} + \left(\frac{f^{(k)}}{A}\right)^{(j)} = \left(\frac{F}{A}\right)^{(j)}, \ j \in \{1, 2, \cdots, k\}. \tag{18}$$

Thus, $w_j = f^{(j)}$ is a solution of the equation

$$w_j + \left(\frac{w_j^{(k-j)}}{A}\right)^{(j)} = \left(\frac{F}{A}\right)^{(j)}. \tag{19}$$

By (19) we can write

$$w_j^{(k)} + \Phi_{k-1} w_j^{(k-1)} + \cdots + \Phi_0 w_j = A\left(\frac{F}{A}\right)^{(j)},$$

where $\Phi_i\ (i = 0, 1, \cdots, k)$ are meromorphic functions with $\rho_p(\Phi_i) < \rho_p(w_j)\ (i = 0, 1, \cdots, k-1)$. By the condition $\frac{F}{A}$ is not a polynomial of degree less than k, we have $\left(\frac{F}{A}\right)^{(j)} \not\equiv 0$, $j \in \{0, 1, \cdots, k\}$. By $A\left(\frac{F}{A}\right)^{(j)} \not\equiv 0$ and $\rho_p\left(A\left(\frac{F}{A}\right)^{(j)}\right) < \rho_p(w_j)$ with Lemma 3 we have $\overline{\lambda}_p(w_j) = \lambda_p(w_j) = \rho_p(w_j) = \rho_p(f)$. Thus $\overline{\lambda}_p\left(f^{(j)}\right) = \lambda_p\left(f^{(j)}\right) = \rho_p(f)$, $j \in \{0, 1, \cdots, k\}$. □

4 Conclusion

In this paper, we have investigated the interaction between the solutions and the coefficients of some higher order homogeneous and nonhomogeneous linear differential equations with meromorphic coefficients and we obtained some properties of the growth and the zeros of the solutions and their derivatives.

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References

[1] Hayman, W. K. *Meromorphic Functions*. Oxford: Clarendon Press. 1964.

[2] Yang, C. C. and Yi, H. X. *Uniqueness Theory of Meromorphic Functions. Mathematics and Its Applications*. Vol. 557. Dordrecht: Kluwer Academic Publishers Group. 2003.

[3] Bernal, L. G. On growth $k-$order of solutions of a complex homogeneous linear differential equation. *Proceedings of the American Mathematical Society.* 1987. 101(2): 317–322.

[4] Kinnunen, L. Linear differential equations with solutions of finite iterated order. *Southeast Asian Bull. Math.* 1998. 22(4): 385–405.

[5] Laine, I. *Nevanlinna Theory and Complex Differential Equations.* Vol 15. Berlin: Walter de Gruyter. 1993.

[6] Laine, I. and Rieppo, J. Differential polynomials generated by linear differential equations. *Complex Var. Theory Appl.* 2004. 49(12): 897–911.

[7] Liu, M. S. and Zhang, X. M. Fixed points of meromorphic solutions of higher order linear differential equations. *Ann. Acad. Sci. Fenn. Math.* 2006. 31(1): 191–211.

[8] Zhang, Q. T. and Yang, C. C. *The Fixed Points and Resolution Theory of Meromorphic Functions.* Beijing: Beijing University Press. 1988 (in Chinese).

[9] Chen, Z. X. The fixed points and hyper order of solutions of second order complex differential equations. *Acta Math. Sci. Ser. A Chin. Ed.* 2000. 20(3): 425–432. (in Chinese).

[10] Wang, J. and Lü, W. R. The fixed points and hyper-order of solutions of second order linear differential equations with meromorphic coefficients. *Acta Math. Appl. Sin.* 2004. 27(1): 72–80. (in Chinese).

[11] Belaïdi, B. Oscillation of fixed points of solutions of some linear differential equations. *Acta Math. Univ. Comenian. (N.S.)* 2008. 77(2): 263–269.

[12] Chen, Z. X. Zeros of meromorphic solutions of higher order linear differential equations. *Analysis.* 1994. 14(4): 425–438.

[13] Cao, T. B., Chen, Z. X., Zheng, X. M. and Tu, J. On the iterated order of meromorphic solutions of higher order linear differential equations. *Ann. Differential Equations.* 2005. 21(2): 111–122.