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Approximating Common Fixed Points for Four Asymptotically Quasi-nonexpansive Mappings in CAT(0) Spaces

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Abstract The aim of this paper to study a three-step iterative algorithm for four asymptotically quasi-nonexpansive mappings in the framework of **CAT(0)** spaces (for definition, see page 2). Also we establish some strong convergence theorems for above said scheme and mappings by using semi-compactness and condition (GA) (see, definition 2) which is more general than condition (A) (see, definition 1). Our results improve and extend the corresponding results from the previous work.

Keywords Asymptotically quasi-nonexpansive mapping; three-step iteration scheme; common fixed point; strong convergence; CAT(0) space.

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1 Introduction and Preliminaries

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [1]), $\mathbb{R}$-trees (see [2]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [4]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [1].

Fixed point theory in a CAT(0) space was first studied by Kirk (see [5, 6]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since, then the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g. [7–19] and references therein). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see,e.g., [1]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) a *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denoted by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [1]).

2 CAT(0) Space

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x,y) \quad \leq \quad d_{\mathbb{R}^2}(\overline{x},\overline{y}). \tag{1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [20]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \quad \leq \quad \frac{1}{2}\, d^2(x,y_1) + \frac{1}{2}\, d^2(x,y_2) - \frac{1}{4}\, d^2(y_1,y_2). \tag{2}$$

The inequality (2) is the (CN) inequality of Bruhat and Tits [21]. The above inequality has been extended in [10] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1-\alpha)y) \quad \leq \quad & \alpha d^2(z,x) + (1-\alpha) d^2(z,y) \\ & -\alpha(1-\alpha) d^2(x,y) \end{aligned} \tag{3}$$

for any $\alpha \in [0,1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [1, page 163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0,1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x,y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \quad \leq \quad \alpha d(z,x) + (1-\alpha) d(z,y), \tag{4}$$

for any $z \in X$ and $[x,y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$.

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x,y] \subset C$.

Let T be a self map on a nonempty subset C of X. Denote the set of fixed points of T by $F(T) = \{x \in C : T(x) = x\}$. We say that T is said to be:

(1) asymptotically nonexpansive if there exists a sequence $\{r_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$d(T^n x, T^n y) \quad \leq \quad (1+r_n) d(x,y), \tag{5}$$

for all $x, y \in C$ and $n \geq 1$;

(2) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$d(T^n x, p) \quad \leq \quad (1+r_n) d(x,p), \tag{6}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$;

(3) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \quad \leq \quad L\ d(x,y), \tag{7}$$

for all $x, y \in C$ and $n \geq 1$;

(4) semi-compact if for any bounded sequence $\{x_n\}$ in C with $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there is a convergent subsequence of $\{x_n\}$.

Remark 1 From the above definitions, it is clear that the classes of quasi-nonexpansive mappings and asymptotically nonexpansive mappings include nonexpansive mappings when their fixed point sets are nonempty, whereas the class of asymptotically quasi-nonexpansive mapping is larger than that of quasi-nonexpansive mappings and asymptotically nonexpansive mappings when their fixed point sets are nonempty. The reverse of these implications may not be true as the following examples show:

Example 1 Let $E = [-\pi, \pi]$ and let T be defined by

$$Tx = x\, cosx$$

for each $x \in E$. Clearly $F(T) = \{0\}$. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Tx - z| = |Tx - 0| = |x||cos\, x| \leq |x| = |x - z|,$$

and hence T is an asymptotically quasi-nonexpansive mapping with constant sequences $\{k_n\} = \{1\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{\pi}{2}$ and $y = \pi$, then

$$|Tx - Ty| = \left|\frac{\pi}{2}\, cos\, \frac{\pi}{2} - \pi\, cos\, \pi\right| = \pi,$$

whereas

$$|x - y| = \left|\frac{\pi}{2} - \pi\right| = \frac{\pi}{2}.$$

Example 2 Let $E = \mathbb{R}$ and let T be defined by

$$T(x) = \begin{cases} \frac{x}{2}\, cos\, \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \neq 0$ and $Tx = x$, then $x = \frac{x}{2}\, cos\, \frac{1}{x}$. Thus $2 = cos\, \frac{1}{x}$. This is not hold. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Tx - z| = |Tx - 0| = \left|\frac{x}{2}\right|\left|cos\, \frac{1}{x}\right| \leq \frac{|x|}{2} < |x| = |x - z|,$$

and hence T is an asymptotically quasi-nonexpansive mapping with constant sequences $\{k_n\} = \{1\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{2}{3\pi}$ and $y = \frac{1}{\pi}$, then

$$|Tx - Ty| = \left|\frac{1}{3\pi}\, cos\, \frac{3\pi}{2} - \frac{1}{2\pi}\, cos\, \pi\right| = \frac{1}{2\pi},$$

whereas

$$|x - y| = \left|\frac{2}{3\pi} - \frac{1}{\pi}\right| = \frac{1}{3\pi}.$$

In 2002, Xu and Noor [22] introduced a three-step iterative scheme as follows:

$$\begin{cases} x_0 \in C, \\ \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\ \quad\quad y_n = (1 - \beta_n)x_n + \beta_n T^n z_n, \\ \quad\quad z_n = (1 - \gamma_n)x_n + \gamma_n T^n x_n, \quad n \geq 0 \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$.

Recently, Y. Niwongsa and B. Panyanak [23] studied the Noor iteration scheme in CAT(0) spaces and they proved some $\triangle$ and strong convergence theorems for asymptotically nonexpansive mappings which extend and improve some recent results from the literature.

The aim of this paper is to study a three-step iterative scheme for four asymptotically quasi-nonexpansive mapping in the setting of $CAT(0)$ spaces. Also we establish some strong convergence theorems for said scheme and mappings. Our results extend the corresponding results of [23] and many others.

We need the following useful lemma to prove our convergence results.

Lemma 1 (See [24]) Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three sequences of nonnegative real numbers satisfying the following conditions:

$$p_{n+1} \le (1+q_n)p_n + r_n, \quad n \ge 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty.$$

Then

(1) $\lim_{n\to\infty} p_n$ exists.

(2) In addition, if $\liminf_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

3 Strong Convergence Theorems in CAT(0) Space

We establish some convergence results of a three-step iteration scheme to a common fixed point for four asymptotically quasi-nonexpansive self mappings in the framework of CAT(0) spaces.

Theorem 1 *Let X be a complete $CAT(0)$ space and let C be a nonempty closed convex subset of X. Let R, S, T, $U\colon C \to C$ be four asymptotically quasi-nonexpansive mappings with sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $F = F(R)\cap F(S)\cap F(T)\cap F(U)$ is closed. Let $\{x_n\}$ be the three-step iteration process defined as : For a given $x_1 \in C$, define*

$$\begin{cases} z_n = \gamma_n R^n x_n \oplus (1-\gamma_n)U^n x_n, \\ y_n = \beta_n T^n z_n \oplus (1-\beta_n)R^n x_n, \\ x_{n+1} = \alpha_n S^n y_n \oplus (1-\alpha_n)R^n x_n, \quad n \ge 1, \end{cases} \tag{8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in [0,1]. Then $\{x_n\}$ converges strongly to a common fixed point p of the mappings R, S, T and U if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x,F) = \inf_{p\in F}\{d(x,p)\}$.

Proof The necessity is obvious and so it is omitted. Now, we prove the sufficiency. For any $p \in F$, from (8), we have

$$\begin{aligned} d(z_n, p) &= d(\gamma_n R^n x_n \oplus (1-\gamma_n) U^n x_n, p) \\ &\leq \gamma_n d(R^n x_n, p) + (1-\gamma_n) d(U^n x_n, p) \\ &\leq \gamma_n k_n d(x_n, p) + (1-\gamma_n) k_n d(x_n, p) \\ &\leq k_n d(x_n, p) \end{aligned} \tag{9}$$

again from (8) and using (9), we have

$$\begin{aligned} d(y_n, p) &= d(\beta_n T^n z_n \oplus (1-\beta_n) R^n x_n, p) \\ &\leq \beta_n d(T^n z_n, p) + (1-\beta_n) d(R^n x_n, p) \\ &\leq \beta_n k_n d(z_n, p) + (1-\beta_n) k_n d(x_n, p) \\ &\leq \beta_n k_n [k_n d(x_n, p)] + (1-\beta_n) k_n^2 d(x_n, p) \\ &\leq k_n^2 d(x_n, p) \end{aligned} \tag{10}$$

again using (8) and (10), we obtain

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n S^n y_n \oplus (1-\alpha_n) R^n x_n, p) \\ &\leq \alpha_n d(S^n y_n, p) + (1-\alpha_n) d(R^n x_n, p) \\ &\leq \alpha_n k_n d(y_n, p) + (1-\alpha_n) k_n d(x_n, p) \\ &\leq \alpha_n k_n [k_n^2 d(x_n, p)] + (1-\alpha_n) k_n^3 d(x_n, p) \\ &\leq k_n^3 d(x_n, p) \\ &= [1+\theta_n] d(x_n, p) \end{aligned} \tag{11}$$

where $\theta_n = (k_n^3 - 1) = (k_n - 1)(k_n^2 + k_n + 1)$, since by hypothesis, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, it follows that $\sum_{n=1}^{\infty} \theta_n < \infty$, from Lemma 1, we know that $\lim_{n\to\infty} d(x_n, p)$ exists. Also from (11), we have

$$d(x_{n+1}, F) \quad \leq \quad (1+\theta_n) d(x_n, F), \tag{12}$$

since $\sum_{n=1}^{\infty} \theta_n < \infty$, from Lemma 1, we know that $\lim_{n\to\infty} d(x_n, F)$ exists.

Now, we prove that $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

If $x_n \to p \in F$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. Since $\lim_{n\to\infty} d(x_n, F)$ exists, by hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$, we conclude that $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence.

Since $1 + x \leq e^x$ for $x \geq 0$, from (11), we obtain

$$d(x_{n+1}, p) \quad \leq \quad \{e^{\theta_n}\} d(x_n, p). \tag{13}$$

Hence for any positive integers m, n and from (13) with $\sum_{n=1}^{\infty} \theta_n < \infty$, we have

$$\begin{aligned} d(x_{n+m}, p) &\leq \Big\{e^{\theta_{n+m-1}}\Big\} d(x_{n+m-1}, p) \\ &\leq \Big\{e^{\theta_{n+m-1}}\Big\}\Big[e^{\theta_{n+m-2}} d(x_{n+m-2}, p)\Big] \\ &\leq \Big\{e^{(\theta_{n+m-1}+\theta_{n+m-2})}\Big\} d(x_{n+m-2}, p) \\ &\leq \dots \\ &\leq \dots \\ &\leq \Big\{e^{\sum_{k=n}^{n+m-1} \theta_k}\Big\} d(x_n, p). \end{aligned} \tag{14}$$

Let $K = e^{\sum_{k=n}^{n+m-1} \theta_k}$. Then $0 < K < \infty$ and

$$d(x_{n+m}, p) \leq K d(x_n, p), \tag{15}$$

for the natural numbers m, n and $p \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, therefore for any $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \varepsilon/2K$ for all $n \geq n_0$. So, we can find $p^* \in F$ such that $d(x_{n_0}, p^*) < \varepsilon/2K$. Hence, for all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\leq K d(x_{n_0}, p^*) + K d(x_{n_0}, p^*) \\ &= 2K d(x_{n_0}, p^*) \\ &< 2K.\frac{\varepsilon}{2K} = \varepsilon. \end{aligned} \tag{16}$$

This proves that $\{x_n\}$ is a Cauchy sequence. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = z$. Since C is closed, therefore $z \in C$. Next, we show that $z \in F$. Now, the following two inequalities:

$$\begin{aligned} d(z, p) &\leq d(z, x_n) + d(x_n, p) \quad \forall p \in F, \;\; n \geq 1, \\ d(z, x_n) &\leq d(z, p) + d(x_n, p) \quad \forall p \in F, \;\; n \geq 1 \end{aligned} \tag{17}$$

give that

$$-d(z, x_n) \leq d(z, F) - d(x_n, F) \leq d(z, x_n), \;\; n \geq 1. \tag{18}$$

That is,

$$|d(z, F) - d(x_n, F)| \leq d(z, x_n), \;\; n \geq 1. \tag{19}$$

As $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, F) = 0$, we conclude that $z \in F$. This completes the proof. □

We deduce some results from Theorem 1 as follows.

Corollary 1 Let X be a complete CAT(0) space and let C be a nonempty closed convex subset of X. Let R, S, T, $U\colon C \to C$ be four asymptotically quasi-nonexpansive mappings with sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$. Suppose that $F = F(R) \cap F(S) \cap F(T) \cap F(U)$ is closed. Let $\{x_n\}$ be the three-step iteration process defined as by (8). Then $\{x_n\}$ converges strongly to a common fixed point p of the mappings R, S, T and U if and only if there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to $p \in F$.

Corollary 2 Let X be a Banach space, and let C be a nonempty closed convex subset of X. Let R, S, T, $U\colon C \to C$ be four asymptotically quasi-nonexpansive mappings with sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$. Suppose that $F = F(R)\cap F(S)\cap F(T)\cap F(U)$ is closed. Let $\{x_n\}$ be the three-step iteration process defined as by (8). Then $\{x_n\}$ converges strongly to a common fixed point p of the mappings R, S, T and U if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p\in F}\{d(x, p)\}$.

Proof The proof of Corollary 2 immediately follows by taking $\lambda x \oplus (1-\lambda)y = \lambda x + (1-\lambda)y$ in Corollary 1. This completes the proof. □

Lemma 2 Let X be a complete CAT(0) space and let C be a nonempty closed convex subset of X. Let R, S, T, $U\colon C \to C$ be four uniformly 1-Lipschitzian asymptotically quasi-nonexpansive mappings with sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$. Let $\{x_n\}$ be the three-step iteration process defined as by (8). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be the real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If $F = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$,

$$d(x, Sy) \quad \leq \quad d(Rx, Sy),\ \forall\, x,\, y \in C \tag{20}$$

and

$$d(x, Rx) \quad \leq \quad d(Ux, Rx),\ \forall\, x\, \in C. \tag{21}$$

Then

$$\begin{aligned}\lim_{n\to\infty} d(Rx_n, x_n) &= \lim_{n\to\infty} d(Sx_n, x_n) = \lim_{n\to\infty} d(Tx_n, x_n) \\ &= \lim_{n\to\infty} d(Ux_n, x_n) = 0.\end{aligned}$$

Proof Let $p \in F$. Then, by Theorem 1, we have $\lim_{n\to\infty} d(x_n, p)$ exists. Let $\lim_{n\to\infty} d(x_n, p)$ $= a$. If $a = 0$, then by the continuity of T the conclusion follows. Now suppose $a > 0$. Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $\{x_n\}\,\{y_n\}$, $\{z_n\} \subset B_R(p)$ for all $n \geq 1$. Using (3) and (8), we have

$$\begin{aligned} d^2(z_n, p) &= d^2(\gamma_n R^n x_n \oplus (1-\gamma_n)U^n x_n, p) \\ &\leq \gamma_n d^2(R^n x_n, p) + (1-\gamma_n) d^2(U^n x_n, p) \\ &\quad -\gamma_n(1-\gamma_n) d(R^n x_n, U^n x_n) \\ &\leq \gamma_n k_n^2 d^2(x_n, p) + (1-\gamma_n) k_n^2 d^2(x_n, p) \\ &\quad -\gamma_n(1-\gamma_n) d(R^n x_n, U^n x_n) \\ &\leq k_n^2 d^2(x_n, p) - \gamma_n(1-\gamma_n) d(R^n x_n, U^n x_n) \end{aligned} \tag{22}$$

Now equation (22) implies that

$$d^2(z_n, p) \quad \leq \quad k_n^2 d^2(x_n, p). \tag{23}$$

Again using (3), (8) and (23), we obtain that

$$
\begin{aligned}
d^2(y_n, p) &= d^2(\beta_n T^n z_n \oplus (1-\beta_n) R^n x_n, p) \\
&\le \beta_n d^2(T^n z_n, p) + (1-\beta_n) d^2(R^n x_n, p) \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, R^n x_n) \\
&\le \beta_n k_n^2 d^2(z_n, p) + (1-\beta_n) k_n^2 d^2(x_n, p) \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, R^n x_n) \\
&\le \beta_n k_n^2 [k_n^2 d^2(x_n, p)] + (1-\beta_n) k_n^4 d^2(x_n, p) \\
&\quad -\beta_n(1-\beta_n) d^2(T^n z_n, R^n x_n) \\
&\le k_n^4 d^2(x_n, p) - \beta_n(1-\beta_n) d^2(T^n z_n, R^n x_n). \qquad (24)
\end{aligned}
$$

Now equation (23) implies that

$$
d^2(y_n, p) \le k_n^4 d^2(x_n, p). \qquad (25)
$$

Now using (3), (8) and (25), we obtain that

$$
\begin{aligned}
d^2(x_{n+1}, p) &= d^2(\alpha_n S^n y_n \oplus (1-\alpha_n) R^n x_n, p) \\
&\le \alpha_n d^2(S^n y_n, p) + (1-\alpha_n) d^2(R^n x_n, p) \\
&\quad -\alpha_n(1-\alpha_n) d^2(S^n y_n, R^n x_n) \\
&\le \alpha_n k_n^2 d^2(y_n, p) + (1-\alpha_n) k_n^2 d^2(x_n, p) \\
&\quad -\alpha_n(1-\alpha_n) d^2(S^n y_n, R^n x_n) \\
&\le \alpha_n k_n^2 [k_n^4 d^2(x_n, p)] + (1-\alpha_n) k_n^6 d^2(x_n, p) \\
&\quad -\alpha_n(1-\alpha_n) d^2(S^n y_n, R^n x_n) \\
&\le k_n^6 d^2(x_n, p) - \alpha_n(1-\alpha_n) d^2(S^n y_n, R^n x_n) \\
&= [1 + (k_n^6 - 1)] d^2(x_n, p) - \alpha_n(1-\alpha_n) d^2(S^n y_n, R^n x_n) \\
&= [1 + (k_n - 1)(k_n^5 + k_n^4 + k_n^3 + k_n^2 + k_n + 1)] d^2(x_n, p) \\
&\quad -\alpha_n(1-\alpha_n) d^2(S^n y_n, R^n x_n) \\
&= [1 + t_n] d^2(x_n, p) - \alpha_n(1-\alpha_n) d^2(S^n y_n, R^n x_n) \qquad (26)
\end{aligned}
$$

where $t_n = (k_n - 1)(k_n^5 + k_n^4 + k_n^3 + k_n^2 + k_n + 1)$. Since by hypothesis of the theorem $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, it follows that $\sum_{n=1}^{\infty} t_n < \infty$. Observe that $\alpha_n(1-\alpha_n) \ge \delta^2$ and $\sum_{n=1}^{\infty} t_n < \infty$. For $m \ge 1$, (26) implies that

$$
\begin{aligned}
\sum_{n=1}^{m} d^2(S^n y_n, R^n x_n) &\le \frac{1}{\delta^2}\Big[d^2(x_1, p) - d^2(x_{m+1}, p) + \sum_{n=1}^{m} t_n d^2(x_n, p)\Big] \\
&\le \frac{1}{\delta^2}\Big[d^2(x_1, p) + R^2 \sum_{n=1}^{m} t_n\Big]. \qquad (27)
\end{aligned}
$$

When $m \to \infty$, we have $\sum_{n=1}^{\infty} d^2(S^n y_n, R^n x_n) < \infty$, since $\sum_{n=1}^{\infty} t_n < \infty$ and $d(x_n, p) \le R$, $\forall\, n$.

Hence

$$
\lim_{n\to\infty} d(S^n y_n, R^n x_n) = 0. \qquad (28)
$$

Using (20) and (28), it follows that

$$\begin{aligned} d(R^n x_n, x_n) &\leq d(R^n x_n, S^n y_n) + d(S^n y_n, x_n) \\ &\leq 2d(R^n x_n, S^n y_n) \to 0 \text{ as } n \to \infty, \end{aligned} \tag{29}$$

and hence

$$\begin{aligned} d(S^n y_n, x_n) &\leq d(S^n y_n, R^n x_n) + d(R^n x_n, x_n) \\ &\quad \to 0 \text{ as } n \to \infty. \end{aligned} \tag{30}$$

Now, we have

$$\begin{aligned} d(x_n, p) &\leq d(x_n, S^n y_n) + d(S^n y_n, p) \\ &\leq d(x_n, S^n y_n) + k_n d(y_n, p), \end{aligned} \tag{31}$$

from which we deduce that $a \leq \liminf_{n\to\infty} d(y_n, p)$. On the other hand, taking $\limsup_{n\to\infty}$ on both the sides of (10), we have

$$\limsup_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} k_n^2 d(x_n, p) = a,$$

which implies that

$$\lim_{n\to\infty} d(y_n, p) = a. \tag{32}$$

Again consider equation (24), we have

$$\begin{aligned} d^2(y_n, p) &\leq k_n^4 d^2(x_n, p) - \beta_n(1-\beta_n) d^2(T^n z_n, R^n x_n) \\ &= [1 + (k_n^4 - 1)] d^2(x_n, p) - \beta_n(1-\beta_n) d^2(T^n z_n, R^n x_n) \\ &= [1 + (k_n - 1)(k_n^3 + k_n^2 + k_n + 1)] d^2(x_n, p) \\ &\quad -\beta_n(1-\beta_n) d^2(T^n z_n, R^n x_n) \\ &= (1 + m_n) d^2(x_n, p) - \beta_n(1-\beta_n) d^2(T^n z_n, R^n x_n) \end{aligned} \tag{33}$$

where $m_n = (k_n^4 - 1) = (k_n - 1)(k_n^3 + k_n^2 + k_n + 1)$, since by assumption of the theorem $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, it follows that $\sum_{n=1}^{\infty} m_n < \infty$. For $m \geq 1$, (33) implies that

$$\begin{aligned} \sum_{n=1}^{m} d^2(T^n z_n, R^n x_n) &\leq \frac{1}{\delta^2}\Big[\sum_{n=1}^{m} m_n d^2(x_n, p)\Big] \\ &\leq \frac{R^2}{\delta^2} \sum_{n=1}^{m} m_n. \end{aligned} \tag{34}$$

When $m \to \infty$, we have $\sum_{n=1}^{\infty} d^2(T^n z_n, R^n x_n) < \infty$, since $\sum_{n=1}^{\infty} m_n < \infty$ and $d(x_n, p) \leq R$, $\forall\, n$.

Hence

$$\lim_{n\to\infty} d(T^n z_n, R^n x_n) = 0, \tag{35}$$

consequently,

$$d(T^n z_n, x_n) \leq d(T^n z_n, R^n x_n) + d(R^n x_n, x_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{36}$$

Again note that

$$\limsup_{n\to\infty} d(U^n x_n, p) \leq \limsup_{n\to\infty} k_n d(x_n, p) = a, \tag{37}$$

and

$$\limsup_{n\to\infty} d(R^n x_n, p) \leq \limsup_{n\to\infty} k_n d(x_n, p) = a. \tag{38}$$

Also,

$$\begin{aligned} d(x_n, p) &\leq d(x_n, T^n z_n) + d(T^n z_n, p) \\ &\leq d(x_n, T^n z_n) + k_n d(z_n, p). \end{aligned} \tag{39}$$

Using (36) in above inequality, we obtain

$$a = \lim_{n\to\infty} d(x_n, p) \leq \liminf_{n\to\infty} d(z_n, p). \tag{40}$$

On the other hand, taking $\limsup_{n\to\infty}$ on both the sides of (9), we have

$$\limsup_{n\to\infty} d(z_n, p) \leq \limsup_{n\to\infty} k_n d(x_n, p) = a.$$

This together with (40) gives

$$\lim_{n\to\infty} d(z_n, p) = a. \tag{41}$$

Again consider equation (22), we have

$$\begin{aligned} d^2(z_n, p) &\leq k_n^2 d^2(x_n, p) - \gamma_n(1-\gamma_n) d^2(R^n x_n, U^n x_n) \\ &= [1 + (k_n^2 - 1)] d^2(x_n, p) - \gamma_n(1-\gamma_n) d^2(R^n x_n, U^n x_n) \\ &= [1 + (k_n - 1)(k_n + 1)] d^2(x_n, p) \\ &\quad -\gamma_n(1-\gamma_n) d^2(R^n x_n, U^n x_n) \\ &\leq (1 + u_n) d^2(x_n, p) - \gamma_n(1-\gamma_n) d^2(R^n x_n, U^n x_n) \end{aligned} \tag{42}$$

where $u_n = (k_n - 1)(k_n + 1)$, since by assumption of the theorem $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, it follows that $\sum_{n=1}^{\infty} u_n < \infty$. For $m \geq 1$, (42) implies that

$$\begin{aligned} \sum_{n=1}^{m} d^2(R^n x_n, U^n x_n) &\leq \frac{1}{\delta^2}\Big[\sum_{n=1}^{m} u_n d^2(x_n, p)\Big] \\ &\leq \frac{R^2}{\delta^2} \sum_{n=1}^{m} u_n. \end{aligned} \tag{43}$$

When $m \to \infty$, we have $\sum_{n=1}^{\infty} d^2(R^n x_n, U^n x_n) < \infty$, since $\sum_{n=1}^{\infty} u_n < \infty$ and $d(x_n, p) \leq R$, $\forall\, n$.

Hence

$$\lim_{n\to\infty} d(R^n x_n, U^n x_n) = 0. \tag{44}$$

Using (21) and (44), it follows that

$$\begin{aligned} d(U^n x_n, x_n) &\leq d(U^n x_n, R^n x_n) + d(R^n x_n, x_n) \\ &\leq 2d(U^n x_n, R^n x_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{45}$$

Consequently, we have

$$\begin{aligned} d(x_n, T^n x_n) &\leq d(x_n, T^n z_n) + d(T^n z_n, T^n x_n) \\ &\leq d(x_n, T^n z_n) + k_n d(z_n, x_n) \\ &\leq d(x_n, T^n z_n) + k_n[(1-\gamma_n)d(U^n x_n, x_n) \\ &\quad +\gamma_n d(R^n x_n, x_n)] \\ &= d(x_n, T^n z_n) + k_n(1-\gamma_n)d(U^n x_n, x_n) \\ &\quad +k_n\gamma_n d(R^n x_n, x_n). \end{aligned} \tag{46}$$

Using (29), (36) and (45) in (46), we have

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0. \tag{47}$$

Thus,

$$\begin{aligned} d(x_n, S^n x_n) &\leq d(x_n, S^n y_n) + d(S^n y_n, S^n x_n) \\ &\leq d(x_n, S^n y_n) + k_n d(y_n, x_n) \\ &\leq d(x_n, S^n y_n) + k_n[(1-\beta_n)d(R^n x_n, x_n) \\ &\quad +\beta_n d(T^n z_n, x_n)] \\ &= d(x_n, S^n y_n) + k_n(1-\beta_n)d(R^n x_n, x_n) \\ &\quad +k_n\beta_n d(T^n z_n, x_n). \end{aligned} \tag{48}$$

Using (29), (30) and (36) in (48), we have

$$\lim_{n\to\infty} d(x_n, S^n x_n) = 0. \tag{49}$$

Again note that

$$d(x_{n+1}, x_n) \leq (1-\alpha_n)d(R^n x_n, x_n) + \alpha_n d(S^n y_n, x_n)$$

using (29) and (30), we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0, \tag{50}$$

and

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad +d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n). \end{aligned}$$

Since T is uniformly 1-Lipschitzian, we obtain that

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(x_{n+1}, x_n) + d(T^n x_n, x_n). \end{aligned} \tag{51}$$

Using (47) and (50),

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{52}$$

Similarly, we can prove that

$$\lim_{n\to\infty} d(x_n, Rx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Ux_n) = 0. \tag{53}$$

This completes the proof. □

Theorem 2 *Let X be a complete $CAT(0)$ space and let C be a nonempty closed convex subset of X. Let R, S, T, $U\colon C \to C$ be four uniformly 1-Lipschitzian asymptotically quasi-nonexpansive mappings with sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the three-step iteration process defined as by (8). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be the real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$ and R, S, U satisfy the conditions (20) and (21). Suppose one of the mappings in $\{R, S, T, U\}$ is semi-compact. If $F = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U.*

Proof Suppose R is semi-compact. By Lemma 2, we have $\lim_{n\to\infty} d(x_n, Rx_n) = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = x_1 \in C$. Now, Lemma 2 guarantees that $\lim_{n_j\to\infty} d(x_{n_j}, Rx_{n_j}) = 0$, $\lim_{n_j\to\infty} d(x_{n_j}, Sx_{n_j}) = 0$, $\lim_{n_j\to\infty} d(x_{n_j}, Tx_{n_j}) = 0$, $\lim_{n_j\to\infty} d(x_{n_j}, Ux_{n_j}) = 0$ and so $d(x_1, Rx_1) = 0$, $d(x_1, Sx_1) = 0$, $d(x_1, Tx_1) = 0$, $d(x_1, Ux_1) = 0$. This implies that $x_1 \in F = F(R) \cap F(S) \cap F(T) \cap F(U)$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, it follows, as in the proof of Theorem 1, that $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U. This completes the proof. □

For further analysis, we need the following concept.

Senter and Dotson [18] introduced the concept of Condition (A) as follows.

Definition 1 (See [18]) A mapping $T\colon C \to C$ is said to satisfy Condition (A) if there exists a non-decreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$, for all $x \in C$. It is to be noted that Condition (A) is weaker than compactness of the domain C.

Now, we generalize the above definition for four mappings.

Definition 2 Four mappings R, S, T and $U\colon C \to C$ where C is a nonempty subset of a metric space (X, d) with at least one common fixed point is said to satisfy **Condition** (GA) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $a_1 d(x, Rx) + a_2 d(x, Sx) + a_3 d(x, Tx) + a_4 d(x, Ux) \geq f(d(x, F))$ for all $x \in C$, where $d(x, F) = \inf\{d(x, p) : p \in F = F(R) \cap F(S) \cap F(T) \cap F(U)\}$ and a_1, a_2, a_3, a_4 are four nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 = 1$.

Remark 2 Condition (GA) reduces to condition (A) [18] when $R = S = T = U$.

Theorem 3 *Let X be a complete $CAT(0)$ space and let C be a nonempty closed convex subset of X. Let R, S, T, $U: C \to C$ be four uniformly 1-Lipschitzian asymptotically quasi-nonexpansive mappings with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and satisfying condition (GA). Let $\{x_n\}$ be the three-step iteration process defined as by (8). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be the real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$ and R, S, U satisfy the conditions (20) and (21). If $F = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U.*

Proof We proved in Lemma 2 that

$$\begin{aligned} \lim_{n\to\infty} d(x_n, Rx_n) &= \lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Tx_n) \\ &= \lim_{n\to\infty} d(x_n, Ux_n) = 0. \end{aligned} \tag{54}$$

From the condition (GA) and (54), we have

$$\begin{aligned} \lim_{n\to\infty} f(d(x_n, F)) &\leq \lim_{n\to\infty} [a_1\, d(x_n, Rx_n) + a_2\, d(x_n, Sx_n) \\ &\quad + a_3\, d(x_n, Tx_n) + a_4\, d(x_n, Ux_n)] \\ &\leq a_1 \lim_{n\to\infty} d(x_n, Rx_n) + a_2 \lim_{n\to\infty} d(x_n, Sx_n) \\ &\quad + a_3 \lim_{n\to\infty} d(x_n, Tx_n) + a_4 \lim_{n\to\infty} d(x_n, Ux_n) = 0. \end{aligned}$$

Hence

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since $f: [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

It follows, as in the proof of Theorem 1, that $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U. This completes the proof. □

4 Conclusion

The class of asymptotically quasi-nonexpansive mappings is more general than the class of nonexpansive, quasi-nonexpansive and asymptotically nonexpansive mappings and the modified Noor iteration scheme is more general than Noor iteration scheme. Thus the results presented in this paper are good improvement and generalization of corresponding results of Xu and Noor [22], Niwongsa and Panyanak [23] and many others from the existing literature.

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