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On the Asymptotic Stability and Contraction of Solutions of An Evolution Equation

[1]**Maduabuchi Orakwelu,**[2]**Michael Egwurube,** [3]**Usman Aminu Bawa,**[4] **Chidinma Nwosu**
[1]Department of Mathematics and Statistics, Caritas University
P.M.B. 01784, Enugu State, Nigeria
[2,3]Department of Mathematics and Computer Sciences, Federal University of Technology Yola
P.M.B. 2076, Adamawa State, Nigeria
[4]Department of Mathematics, Michael Okpara University of Agriculture Umudike
P.M.B. 7267, Umuahia, Abia State, Nigeria
e-mail: [1]maduabuchi@gmail.com, [2]mikegwurube@yahoo.com,
[3]aminuusmanbawa@gmail.com,[4]calculus20032002@yahoo.co.uk

Abstract An Evolution Equation on a Banach Space with m-accretive operator is considered. The solution of this Evolution Equation is shown to exist, asymptotically stable and contract.

Keywords Evolution Equation; Asymptotic stability; Contraction; Banach Space; Existence.

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1 Introduction

An Evolution Equation is an equation that can be interpreted as the differential law of the development in time of a system [1]. The term does not have an exact definition, and its meaning depends not only on the equation itself but also on the formation of the problem for which it is used. Typical of an Evolution Equation is the possibility of constructing the solution from a prescribed initial condition that can be interpreted as the description of the initial state of a system [1].

Merenkov [2] considered an Evolution Equation in which the stability was investigated using Lyapunov's functional. Egwurube and Garba [3] considered a quasi-linear hyperbolic differential equation.

$$\begin{aligned} u_t + f(u)_x &= 0, && 0 < x < 1 \\ u(0,x) &= u_0(x), && 0 < x < 1 \\ u(t,0) &= 0, && t > 0 \end{aligned} \tag{1}$$

which was transformed into an initial value problem

$$\frac{du}{dt} + Au(t) = 0 \ , \quad u(0) = u_0 \tag{2}$$

defined on a Banach space $L^1[0,1]$ with $D(A) = C[0,1]$ and proved that the operator A is m-accretive and that it does admit a solution. Egwurube [4] also considered the same Evolution problem and gave condition for the asymptotic stability of its solution in $C[0,1]$. Igobi *et al*. [5] investigated the existence and uniqueness, and asymptotic stability analysis of solution of a retarded equation model of HIV/AIDS transmission. Egwurube *et al*. [6] showed the existence and asymptotic stability of solutions of the same Evolution Equation.

Abstract differential equation involves regarding the solution of a differential equation as an element of some function space in X that depends on the parameter t [1]. We shall consider the solution of this evolution problem, prove its existence, asymptotic stability and also show contraction of the solution in $L^1[0,1]$.

Definition 1 [7] Let P be a positive real number. A function defined on $[0,1]$ is said to belong to the space $L^P[0,1]$ if $\int_0^1 |f|^P < \infty$. For a function $f \in L^P$, we define $\|f\| = \|f\|_P = \left\{\int_0^1 |f|^P\right\}^{1/P}$.

2 Main Result

Let u^* be the steady state solution of (2) and assume $x = u - u^*$ so that

$$\frac{dx}{dt} + a\,x(t) = h(x)\,,\;\; x(0) = x_0 \tag{3}$$

where a is a bounded linear operator and $h(x)$ represents the non-linear term.
The solution of (3) is

$$|x(t)| \leq e^{\alpha t}|x_0|\left[1 + \int_{\infty}^{t} e^{\alpha\tau}\right]$$

$$X(\tau+\gamma) = \begin{cases} 0, & \forall\tau < \gamma \\ 1, & \forall\tau > \gamma \end{cases}$$

then

$$|x(t)| \leq e^{\alpha t}\left|x_0\left[1 + \int_0^t e^{\alpha\tau}d\tau\right]\right|. \tag{4}$$

Theorem 1 *Suppose $x(t)$ is a measureable function on $[0,1]$ and $P = 1$, then*

$$\int_0^1 |x(t)|^P \leq \frac{|x_0|}{\alpha}\left[e^{\alpha} - 1\right]\left[1 - \frac{1}{\alpha}\left[1 - e^{-\alpha\gamma}\right]\right] < \infty\,.$$

Proof On integrating (4) with respect to t we obtain

$$\begin{aligned}
\int_0^1 |x(t)|\,dt &\leq \int_0^1 \left|e^{\alpha t}|x_0|\left[1 + \int_0^{\gamma} e^{-\alpha\tau}d\tau\right]\right| dt \\
&\leq \int_0^1 e^{\alpha t}|x_0|\,dt + |x_0|\int_0^1\int_0^{\gamma} e^{-\alpha\tau}e^{\alpha\tau}d\tau\,dt \\
&\leq \frac{|x_0|}{\alpha}\left[e^{\alpha} - 1\right] + \frac{|x_0|}{\alpha}\left[1 - e^{-\alpha\gamma}\right]\int_0^1 e^{\alpha t}dt \\
&\leq \frac{|x_0|}{\alpha}\left[e^{\alpha} - 1\right]\left[1 - \frac{1}{\alpha}\left[1 - e^{-\alpha\tau}\right]\right] = M \text{ (say)}
\end{aligned}$$

But $M < \infty. \Rightarrow\; x(t) \in L^1[0,1]$, since $\int_0^1 |x(t)|\,dt < \infty$.

In obtaining asymptotic stability behavior of the solution of this quasi-linear hyperbolic differential equation, the interval of the space ranges from 0 to t (where $t \leq \infty$). So that

$$\|x(s)\|_{L^P[0,t]} = \left\{ \int_0^t |x(s)|^P \right\}^{\frac{1}{P}}$$

but $P = 1$, therefore

$$\begin{aligned}
\|x(s)\|_{L^1[0,t]} &= \int_0^t e^{\alpha s} |x_0| \left[1 + \frac{1 - e^{-\alpha\gamma}}{\alpha} \right] ds \\
\|x(s)\|_{L^1[0,t]} &= \frac{|x_0|}{\alpha} \left[1 + \frac{1 - e^{-\alpha\gamma}}{\alpha} \right] \left| e^{\alpha t} - 1 \right| \\
\lim_{t\to\infty} \|x(s)\|_{L^1[0,t]} &= \frac{|x_0|}{\alpha} \left[1 + \frac{1 - e^{-\alpha\gamma}}{\alpha} \right] = M \,(\text{where } M = \text{constant})
\end{aligned}$$

Lemma 1 Let α, m, m_i and $M \in R$. Then

(i) $m_i \leq m$

(ii) $\frac{M}{m} = +ve$

(iii) $\frac{M}{m_i} = +ve.$

Theorem 2 *Suppose the solution* $x(t) \in L^P[0,1]$ *and* $S(t)$ *a strongly continuous semi-group, then,* $\|S(t)\, x(t)\|_{L^P[0,1]} \leq \|x(t)\|_{L^P[0,1]}$ *for* $P = 1$.

Proof For a function $f \in L^P$, then $\|f\|_P = \left\{ \int_0^1 |f|^P \right\}^{\frac{1}{P}}$.
Therefore $\|x(t)\|_{L^P[0,1]} = \left\{ \int_0^1 |x(t)|^P \right\}^{\frac{1}{P}}$, then $\|S(t)\, x(t)\|_{L^P[0,1]} = \left\{ \int_0^1 |S(t)\, x(t)|^P \right\}^{\frac{1}{p}}$.

Let $S(t)$ the strongly continuous semi-group be e^{at}, then,

$$\left\| e^{at} x(t) \right\|_{L^P[0,1]} = \left\{ \int_0^1 \left| e^{at} x(t) \right|^P dt \right\}^{\frac{1}{P}}.$$

For $P = 1$

$$\begin{aligned}
\left\| e^{\alpha t} x(t) \right\|_{L^1[0,1]} &= \int_0^1 \left| e^{\alpha t} x(t) \right| dt \\
&= \int_0^1 e^{\alpha t} e^{at} |x_0| \left[1 + \frac{1 - e^{-\alpha\gamma}}{\alpha} \right] dt = \int_0^1 e^{(\alpha + a)t} |x_0| \left[1 + \frac{1 - e^{-\alpha\gamma}}{\alpha} \right]
\end{aligned}$$

Let $M = |x_0| \left[1 + \frac{1 - e^{-\alpha\gamma}}{\alpha} \right]$ and $m = (a + \alpha)$, then,

$$\left\| e^{\alpha t} x(t) \right\|_{L^P[0,1]} = \int_0^1 e^{mt} M \, dt = \frac{M}{m} \left[e^m - 1 \right] \tag{5}$$

Also,

$$\|x(t)\|_{L^P[0,1]} = \int_0^1 |x(t)|\, dt = \int_0^1 e^{\alpha t} |x_0| \left[1 + \frac{1 - e^{-\alpha\gamma}}{\alpha}\right] dt.$$

Let $M = |x_0| \left[1 + \frac{1-e^{-\alpha\gamma}}{\alpha}\right]$ and $m_i = \alpha$, then,

$$\begin{aligned} \|x(t)\|_{L^P}[0,1] &= \int_0^1 e^{m_i t} M\, dt \\ &= \frac{M}{m_i}\left[e^{m_i - 1}\right] \end{aligned} \tag{6}$$

On comparing (5) and (6), it is easy to see that $\|s(t)\,x(t)\|_{L^P[0,1]} \leq \|x(t)\|_{L^P[0,1]}$. Hence a contraction.

3 Conclusion

Thus, the solution of the Evolution Equation exists, is asymptotically stable and contracts in $L^1[0,1]$. Having shown this it is hoped that the existence, asymptotic stability and contraction of solutions can perhaps be shown for other types of related Evolution Equations in the same space or other spaces.

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