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Linear Fuzzy Delay Differential System: Analysis on Stability of Steady State

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Abstract In this paper, a first order linear fuzzy delay differential system is proposed by using parametric form of alpha-cut representation of symmetric triangular fuzzy number. The steady state and linear stability of the system are also determined. The steady state is stable in absence of delay by Routh-Hurwitz criteria. For increasing delay we determine the positive real roots of the polynomial and developing the conditions. Several examples are considered to illustrate the stability of the steady state for the proposed system.

Keywords Delay Differential Equations; Fuzzy Differential Equations; Fuzzy Matrices.

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1 Introduction

The dynamics of many control systems explained by time-delay differential equations. The delay may appear because of physical properties of equipment used in the system, signal transmission or measurement of system variables. The class of differential equations with delay include a large number of differential equations. Differential equations with delay play an important role in an increasing number of system models in biology, engineering, physics and other sciences. Delay differential equations covered extensive literature dealing with functional equations and their applications. We refer to the book [1], and references therein. In such cases, modelling the real life problems usually involves vagueness or variations in some parameters. In this respect the development in fuzzy theory has contributed to modelling uncertain systems, which leads to the consideration of fuzzy differential equations.

The concept of fuzzy set and the corresponding fuzzy set-theoretic operation was introduced by [2] used the membership function. In such cases, modeling the real life problems usually includes vagueness or uncertainty in some of the parameters. On the other hand, the development has contributed to various problems of the theory and applications of fuzzy systems, in particular to the theory of differential equations with uncertainty. The use of fuzzy differential equations has been the best way to model dynamical systems under possibility uncertainty. Moreover in view of the development of calculus for fuzzy functions, the investigation of fuzzy delay differential equations has been initiated by many researchers such as [3,4] and etc. [5] proposed the first order linear fuzzy time-delay dynamical systems by using complex number representation of the $\alpha-$cut sets and obtained the solution by using a Runge-Kutta method.

Although, the concepts of the steady states refer to the absence of changes in a system, in some cases, studying the stability of the steady state solutions become an important subject since, by examining what happens in a steady state, we can better understand the behavior of a system.

The aim of this paper, is to implement parametric form of $\alpha-$ cut for linear fuzzy delay differential system and to determine the stability of steady state in the absence of delay and including delay.

The organization of this paper is as follows. In Section 2 , the basic definitions of fuzzy number, steady states and characteristic equation are briefly presented. In Section 3 linear fuzzy delay initial value problem is explained and existences of critical delays were introduced. In Section 4 several examples are illustrated. Finally, Section 5 presents the concluding remarks.

2 Preliminaries

Definition 1 *A fuzzy number is a function such as* $u : R \rightarrow [0,1]$ *satisfying the following properties [2]:*

(i) *u is normal, i.e* $\exists x_0 \in R$ *with* $u(x_0) = 1$.

(ii) *u is a convex fuzzy set i.e* $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\} \forall x, y \in R, \lambda \in [0,1]$.

(iii) *u is upper semi-continuous on R.*

(iv) $\overline{\{x \in R : u(x) > 0\}}$ *is compact where* $\bar{A}$ *denotes the closure of A.*

Definition 2 *An* $\alpha-$ *cut,* u_α*, is a crisp set which contains all the elements of the universal set X that have a membership function at least to the degree of* α *and can be expressed as* $u_\alpha = \{x \in X : \mu_u(x) \geq \alpha\}$ *[5].*

And the set $u'_\alpha = \{x \in X : \mu_u(x) > \alpha\}$ *is called the strong* $\alpha-$ *cut.*

Definition 3 *A fuzzy number u is completely determined by any pair* $u = (\underline{u}, \overline{u})$ *of functions* $\underline{u}(\alpha), \overline{u}(\alpha) : [0,1] \rightarrow R$ *satisfying the three conditions [5]:*

(i) $\underline{u}(\alpha)$, $\overline{u}(\alpha)$ *is a bounded, monotonic, (nondecreasing,nonincreasing) left- continuous function for all* $\alpha \in (0,1]$ *and right-continuous for* $\alpha = 0$.

(ii) *For all* $\alpha \in (0,1]$ *we have:* $\underline{u}(\alpha) \leq \overline{u}(\alpha)$.

For every $u = (\underline{u}, \overline{u}), v = (\underline{v}, \overline{v})$ *and* $k > 0$, $(\underline{u+v})(\alpha) = \underline{u}(\alpha) + \underline{v}(\alpha)$

$(\overline{u+v})(\alpha) = \overline{u}(\alpha) + \overline{v}(\alpha)$

$(\underline{ku})(\alpha) = k\underline{u}(\alpha), (\overline{ku})(\alpha)) = k\overline{u}(\alpha)$

Fuzzy sets are always mapping a universal set into $[0,1]$. Conversely, every function $\mu : X \longrightarrow [0,1]$ is considered as a fuzzy set ([2]). We defined a set $F_1 = \{x \in \Re | x \text{ is about } a_2\}$ with triangular membership function as below.

Definition 4

$$\mu_{F_1}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & x \in [a_1, a_2) \\ 1 & x = a_2 \\ \frac{-x+a_3}{a_3-a_2} & x \in (a_2, a_3] \\ 0 & otherwise \end{cases}$$

So the Fuzzy set F can be written as any ordinary function $F = \{(x, \mu_F(x)) : x \in X\}$ [2].

Definition 5 *The function $f : \Re \to E$ is called a fuzzy function if for an arbitrary fixed $\hat{t} \in \Re$ and $\epsilon > 0$ there is a $\delta > 0$ such that $|t - \hat{t}| < \delta \to d[f(t), f(\hat{t})] < \epsilon$, then f is said to be continuous. Note that d is the metric space.*

Definition 6 *Let $u, v \in E$. If there exists $w \in E$ such that $u = v - w$ then w is called the H-difference of u, v and it is denoted by $u - v$ [5].*

Definition 7 *A function $f : (a, b) \to E$ is called $H-$ differentiable at $\hat{t} \in (a, b)$ if, for $h > 0$ sufficiently small, there exist the $H-$differences $f(\hat{t} + h) - f(\hat{t})$, $f(\hat{t}) - f(\hat{t} - h)$, and an element $f'(\hat{t}) \in E$ such that [5]:*

$$\lim_{h \to 0^+} d(\frac{f(\hat{t} + h) - f(\hat{t})}{h}, f'(\hat{t})) = \lim_{h \to 0^+} d(\frac{f(\hat{t}) - f(\hat{t} - h)}{h}, f'(\hat{t})) = 0$$

Then $f'(\hat{t})$ is called the fuzzy derivative of f at $\hat{t}$.

Definition 8 *Consider the steady state solution $x(t) = x^*$ where $x \in R^n$ as a solution of the nonlinear difference system [6],*

$$f(x_1^*, x_2^*, ..., x_n^*) = 0.$$

The linear part of the solution is given by the variational equation

$$x'(t) = \boldsymbol{A}_0 x(t) + \sum_{j=1}^{n} \boldsymbol{A}_j x(t - \tau_j)$$

where $A_j(t) = \frac{\partial f}{\partial x_j} |_{x^*(t), x^*(t-\tau_1)\ , \cdots, x^*(t-\tau_n)}$ $j = 1, ...n$

In the case of a steady state solution, the matrices $\boldsymbol{A}_j(t) = \boldsymbol{A}_j$ are constants. If we substitute the solution $x(t) = x^* e^{-\lambda t}$ into a variational equation, we may define the matrix $\boldsymbol{B}$ as

$$\boldsymbol{B}(x^*, \lambda) = \lambda I - \boldsymbol{A}_0 - \sum_{j=1}^{n} \boldsymbol{A}_j e^{-\lambda \tau_j}$$

So, the characteristic equation can be obtained by solving the determinant of this matrix $\boldsymbol{B}$ and let the equation be equal to zero. that is ,

$$det(\boldsymbol{B}) = 0 \tag{1}$$

3 Linear Fuzzy Delay Initial Value Problem

Consider the first order linear fuzzy delay initial value differential equation :

$$\begin{aligned} &\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau) && t \in I = [t_0, T] \\ &\mathbf{x}(t) = \mathbf{x}_0 && t \in [t_0 - \tau, t_0], \end{aligned} \tag{2}$$

where $\mathbf{x}(t)$ and $\mathbf{x}(t-\tau) \in \{\mathbf{x}(t) : \tau < t\}$ are n-dimensional fuzzy functions of t, every element of matrices $\mathbf{A} = [a_{ij}]_{n\times n}$, $a_{ij} \in F(\Re)$ and $\mathbf{B} = [b_{ij}]_{n\times n}$, $b_{ij} \in F(\Re)$ are assumed to be fuzzy numbers where $F(\Re)$ represents the fuzzy sets on $\Re$. The function $\dot{\mathbf{x}}(t)$ is a fuzzy derivatives of $\mathbf{x}(t)$ at $t \in I$ and $\mathbf{x}_0$ is a fuzzy number and delay parameter τ is a known positive number.

The α-level sets of $\mathbf{x}(t)$ and $\mathbf{x}(t-\tau)$ for $t \in [t_0, T]$ are given as follows:

$$\begin{aligned} {x_\alpha}^k(t) &= [{\underline{x}_\alpha}^k, {\overline{x}_\alpha}^k], \\ {x_\alpha}^k(t-\tau) &= [{\underline{x}_\alpha}^k(t-\tau), {\overline{x}_\alpha}^k(t-\tau)], \quad (k = 1, 2, ..., n) \end{aligned} \tag{3}$$

Let $\mathbf{x}_\alpha(t) = [\underline{\mathbf{x}}_\alpha(t), \overline{\mathbf{x}}_\alpha(t)]$ be the solution of the fuzzy delay systems (2) in which the elements of matrices $\mathbf{A}$ and $\mathbf{B}$ are fuzzy numbers then (2) can be written with $\mathbf{x}_\alpha(t) = [\underline{\mathbf{x}}_\alpha(t), \overline{\mathbf{x}}_\alpha(t)]$ as its solution as expressed below:

$$\begin{aligned} \dot{\underline{\mathbf{x}}}_\alpha(t) =& \mathbf{A}_\alpha \underline{\mathbf{x}}_\alpha(t) + \mathbf{B}_\alpha \underline{\mathbf{x}}_\alpha(t-\tau), \\ \dot{\overline{\mathbf{x}}}_\alpha(t) =& \mathbf{A}_\alpha \overline{\mathbf{x}}_\alpha(t) + \mathbf{B}_\alpha \overline{\mathbf{x}}_\alpha(t-\tau), \qquad 0 \le \alpha \le 1 \\ \underline{\mathbf{x}}_\alpha(t) =& \underline{\mathbf{x}}_{\alpha 0}, \qquad\qquad t \in [t_0 - \tau, t_0] \\ \overline{\mathbf{x}}_\alpha(t) =& \overline{\mathbf{x}}_{\alpha 0}. \end{aligned} \tag{4}$$

Suppose $(a_{ij})_\alpha = [(a_{ij})^-_\alpha, (a_{ij})^+_\alpha]$, $\mathbf{A}_\alpha = [\mathbf{A}^-_\alpha, \mathbf{A}^+_\alpha]$ where $\mathbf{A}^-_\alpha = [(a_{ij})^-_\alpha]_{n\times n}$ $\mathbf{A}^+_\alpha = [(a_{ij})^+_\alpha]_{n\times n}$ and $(b_{ij})_\alpha = [(b_{ij})^-_\alpha, (b_{ij})^+_\alpha]$, $\mathbf{B}_\alpha = [\mathbf{B}^-_\alpha, \mathbf{B}^+_\alpha]$ where $\mathbf{B}^-_\alpha = [(b_{ij})^-_\alpha]_{n\times n}$, $\mathbf{B}^+_\alpha = [(b_{ij})^+_\alpha]_{n\times n}$.

Then we introduce the following definition:

Definition 9 *If $\boldsymbol{A}(\mu, \alpha) = [a_{ij}(\mu, \alpha)]_{n\times n} = (1-\mu){\boldsymbol{A}_\alpha}^- + \mu {\boldsymbol{A}_\alpha}^+$, $\boldsymbol{B}(\mu, \alpha) = [b_{ij}(\mu, \alpha)]_{n\times n} = (1-\mu){\boldsymbol{B}_\alpha}^- + \mu {\boldsymbol{B}_\alpha}^+$, for $\mu \in [0, 1]$. Then the solution of (4) is $(\underline{\boldsymbol{x}}_\alpha(t), \overline{\boldsymbol{x}}_\alpha(t))$, if $(\underline{\boldsymbol{x}}_\alpha(t), \overline{\boldsymbol{x}}_\alpha(t))$ is also a solution of the following systems:*

$$\begin{aligned} \dot{\underline{\boldsymbol{x}}}_\alpha(t) &= \bigcup_{\mu=0}^{1} \boldsymbol{C}(\mu, \alpha) \underline{\boldsymbol{x}}_\alpha(t) + \bigcup_{\mu=0}^{1} \boldsymbol{D}(\mu, \alpha) \underline{\boldsymbol{x}}_\alpha(t-\tau), \\ \dot{\overline{\boldsymbol{x}}}_\alpha(t) &= \bigcup_{\mu=0}^{1} \boldsymbol{C}(\mu, \alpha) \overline{\boldsymbol{x}}_\alpha(t) + \bigcup_{\mu=0}^{1} \boldsymbol{D}(\mu, \alpha) \overline{\boldsymbol{x}}_\alpha(t-\tau), \\ \underline{\boldsymbol{x}}_\alpha(t) &= \underline{\boldsymbol{x}}_{\alpha 0} \qquad t \in [t_0 - \tau, t_0], \qquad 0 \le \alpha \le 1 \\ \overline{\boldsymbol{x}}_\alpha(t) &= \overline{\boldsymbol{x}}_{\alpha 0}, \end{aligned} \tag{5}$$

where the elements of the matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ are determined from $\boldsymbol{A}(\mu, \alpha)$ and $\boldsymbol{B}(\mu, \alpha)$ as follows:

$$c_{ij} = \begin{cases} e a_{ij}(\mu, \alpha), & a_{ij} \ge 0 \\ g a_{ij}(\mu, \alpha), & a_{ij} < 0 \end{cases}$$

and

$$d_{ij} = \begin{cases} eb_{ij}(\mu, \alpha), & b_{ij} \geq 0 \\ gb_{ij}(\mu, \alpha). & b_{ij} < 0 \end{cases}$$

e is the identity operation and g corresponds to negative value in $\Re$
$\forall z, w \in \Re,$
$e : (z, w) \to (z, w),$
$g : (z, w) \to (w, z).$

3.1 Existence of Critical Delays

Consider the characteristic equation of the delay differential equation at a steady state will have the form:

$$P(\lambda, \tau) = P_1(\lambda) + P_2(\lambda)e^{-\lambda\tau} \tag{6}$$

where τ is the length of discrete delay, and P_1,P_2 are polynomials. We rewrite (6) as follows: $\sum_{j=0}^{N} a_j\lambda^j + e^{-\lambda\tau} \sum_{j=0}^{M} b_j\lambda^j = 0$

Consider that the steady state is stable at $\tau = 0$. Then for absence of delay all the roots of the polynomial have negative real part. At increasing τ these roots change. We will focus in any critical values of τ at which the characteristic root transfer from having negative to having positive real parts. If this happens, there must be a boundary point, a critical value of τ in order that the characteristic equation has a pure imaginary root [7]. To determine whether or not such a τ exists, by reducing (6) to a polynomial case and looking for particular types of roots, thus determining whether a bifurcation can occur as a result of the introduction of delay (see [7]).

We start by looking for a purely imaginary root $i\mu$, $\mu \in \Re$ of (6),
$P_1(i\mu) + P_2(i\mu)e^{-i\mu\tau} = 0$

We write the polynomial as real and imaginary parts, and the exponential in trigonometric functions as follows:

$$R_1(\mu) + iQ_1(\mu) + (R_2(\mu) + iQ_2(\mu))(cos(\mu\tau) - isin(\mu\tau)) = 0. \tag{7}$$

From the original polynomial coefficients of (6), the new polynomials are
$R_1(\mu) = \sum_j (-1)^{j+1} a_{2j}\mu^{2j}$,
$Q_1(\mu) = \sum_j (-1)^j a_{2j+1}\mu^{2j+1}$,
$R_2(\mu) = \sum_j (-1)^{j+1} b_{2j}\mu^{2j}$,
$Q_2(\mu) = \sum_j (-1)^j b_{2j+1}\mu^{2j+1}$.

We see that since $i\mu$ is purely imaginary, R_1 and R_2 are even polynomials of μ but Q_1 and Q_2 are odd polynomials.

In order for (7) to arise, both real and imaginary parts should be 0, we get the equations
$R_1(\mu) + R_2(\mu)cos(\mu\tau) + Q_2(\mu)sin(\mu\tau) = 0$
$Q_1(\mu) - R_2sin(\mu\tau) + Q_2(\mu)cos(\mu\tau) = 0$

Rearranging, we obtain

$$\begin{aligned} -R_1(\mu) &= R_2(\mu)cos(\mu\tau) + Q_2(\mu)sin(\mu\tau) \\ Q_1(\mu) &= R_2(\mu)sin(\mu\tau) - Q_2(\mu)cos(\mu\tau). \end{aligned} \tag{8}$$

Squaring each each equations and summing the results yields

$$R_1(\mu)^2 + Q_1(\mu)^2 = R_2(\mu)^2 + Q_2(\mu)^2 \tag{9}$$

Consider a new variable $\gamma = \mu^2 \in \Re$. Then (9) can be written in terms of γ as

$$S(\gamma) = 0, \tag{10}$$

where S is a polynomial. We are only interested in $\mu \in \Re$, and thus if all of the real roots of S are negative, we can conclude that there is no solution $\mu*$ of (8). Conversely, if there is a positive real root $\gamma*$ to S, there is a delay τ corresponding to $\mu* = \pm\sqrt{\gamma*}$ which solves equation (8).

4 Examples

To show the properties of this new definition, two examples will be given in this section. Example 1 Consider the delay differential equation:

$$\dot{x}(t) = x(t) - 2x(t-\tau) \tag{11}$$

We fuzzify the system (11) by using symmetric triangular fuzzy number and Definition 9 as follows:

$$\begin{aligned} \widetilde{1} &= [1-(1-\alpha)\sigma_1, 1+(1-\alpha)\sigma_1], \\ \widetilde{2} &= [2-(1-\alpha)\sigma_2, 2+(1-\alpha)\sigma_2] \end{aligned} \tag{12}$$

Let

$$\begin{aligned} a_1 =& (1-\mu)(1-(1-\alpha)\sigma_1) + \mu(1+(1-\alpha)\sigma_1), \\ a_2 =& (1-\mu)(2-(1-\alpha)\sigma_2) + \mu(2+(1-\alpha)\sigma_2), \end{aligned} \tag{13}$$

where $0 \le \mu \le 1$. By using the Equation (5), (11) can be written as:

$$\begin{bmatrix} \dot{\underline{x}}_\alpha(t) \\ \dot{\overline{x}}_\alpha(t) \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} \underline{x}_\alpha \\ \overline{x}_\alpha \end{bmatrix} + \begin{bmatrix} 0 & -a_2 \\ -a_2 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_{\alpha t} \\ \overline{x}_{\alpha t} \end{bmatrix}, \tag{14}$$

where $x_{\alpha t} = x_\alpha(t-\tau)$. The system is a fuzzy delay differential equation. This system has a trivial steady state. The characteristic equation is $\lambda^2 - (a_1+a_2)\lambda + a_1a_2 - a_2^2 e^{-2\lambda\tau} = 0$. If $\tau = 0$ then we obtain $\lambda = \frac{1}{2}((a_1+a_2) \mp \sqrt{(a_1+a_2)^2 - 4(a_1a_2 - a_2^2)}$. By Routh-Hurwitz criteria the roots have negative real parts if and only if $(a_1+a_2) < 0$, and $(a_1a_2 - a_2^2) > 0$. The steady state is stable under such conditions. As τ increases, the roots change. If $\lambda = i\mu$, $\mu > 0$ is a root of the characteristic equation then we have $S(\mu) = \mu^4 + \mu^2(a_1^2 + a_2^2 + 4a_1a_2) - a_2^4 = 0$ then we obtain

$$\mu^2 = \frac{1}{2}(-(a_1^2 + a_2^2 + 4a_1a_2) \mp \sqrt{(a_1^2 + a_2^2 + 4a_1a_2)^2 + 4a_2^4}). \tag{15}$$

From equation (15), μ is a postive real root if
$(a_1^2 + a_2^2 + 4a_1a_2) < \sqrt{(a_1^2 + a_2^2 + 4a_1a_2)^2 + 4a_2^4}$.

Then the steady state $(0,0)$ is unstable under the condition $(a_1^2 + a_2^2 + 4a_1a_2) < \sqrt{(a_1^2 + a_2^2 + 4a_1a_2)^2 + 4a_2^4}$.

Example 2 Consider the linear delay differential equations:

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -2y + x(t-\tau) \end{aligned} \tag{16}$$

In the same way as for example 1 we fuzzify the system by using (12) and (13) to obtain:

$$\begin{bmatrix} \underline{\dot{x}}_\alpha(t) \\ \dot{\overline{x}}_\alpha(t) \\ \underline{\dot{y}}_\alpha(t) \\ \dot{\overline{y}}_\alpha(t) \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & -a_2 \\ 0 & 0 & -a_2 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_\alpha \\ \overline{x}_\alpha \\ \underline{y}_\alpha \\ \overline{y}_\alpha \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_{\alpha t} \\ \overline{x}_{\alpha t} \\ \underline{y}_{\alpha t} \\ \overline{y}_{\alpha t} \end{bmatrix}. \tag{17}$$

The system (17) is a linear fuzzy delay differential equations. And the system has $(\underline{x}*, \overline{x}*, \underline{y}*, \overline{y}*) = (0,0,0,0)$ steady state with the characteristic equation $(a_1 - \lambda)^2(\lambda^2 - a_2^2) = 0$.

We obtain $\lambda = a_1$ and $\lambda = \mp a_2$. So, λ can be a positive or a negative fuzzy numbers. Thus, the steady state is unstable for all values of τ.

5 Conclusions

We have presented a new representation which is parametric representation of the α-level sets of the fuzzy linear delay system. We successfully studied the first order linear delay differential systems with fuzzy matrices. The linear stability of steady states and the occurrence of characteristic roots crossing the imaginary axis from left to right as the result of changing the stable steady state to unstable are given in two examples.

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