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Compact Schemes for the Simulations of Wave Propagations

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Abstract In this paper, a general frame work for the development of compact schemes in particular for time harmonic wave equation is presented. The salient features this frame work offers are (a) exact values of numerical solutions at the nodes of the spatial grid irrespective of one or higher dimensions are obtained; (b) compact schemes preserves same stencil structure as that of the standard finite difference and finite element schemes; (c) requirement of fine mesh size to enjoy desired level of accuracy is removed which is real trouble in the case of high wave numbers.

Keywords Time harmonic wave equation; High wave number; Compact schemes; Wave propagations.

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1 Introduction

In present era of science and technology human race is enjoying diverse range of devices such as mobile phones, microwave oven, radars, ultrasound scans etc. Therefore, importance of sound understanding of waves and its propagation is evident from given examples. Mathematically complex nature of this phenomenon resulted into untiring efforts of many physicists, engineers and mathematicians to find analytical solutions which are not mostly possible. Therefore, a greater interest and investment of numerical analysts for efficient and reliable numerical solutions for such problems is natural and we find tremendous amount of work done by many [1–8] in this regard. It is widely known in the literature of wave propagations [2–4, 9] that using discretization schemes such as finite difference and finite elements, numerical dispersion and numerical dissipation are not avoidable. These issues are real hard in the case of higher dimensions and demand the development of state of the art numerical schemes. However, developed schemes work well for low order wave numbers but fails completely in the case of high order wave numbers. We take up this challenge of developing schemes which are less dispersive and dissipative even for high wave numbers and perform optimal in higher dimensions and are compact at the same time.

2 General framework for the development of compact numerical schemes

In order to motivate the idea of developing compact numerical schemes for high wave numbers, we consider one dimensional Helmholtz equation [4], given by

$$u''(x) + k^2u(x) = 0 \text{ for } x \in \mathbb{R} \tag{1}$$

where $k \in \mathbb{C}$ is the wave number. In following sections, we present framework for both finite difference and finite element schemes.

2.1 Framework for compact finite difference scheme

For the development of compact finite difference scheme, we replace second order derivative present in equation (1) by standard second order central finite difference approximations at the node $x_j = jh$ of the uniformly spaced grid $h\mathbb{Z}$ with h as the mesh size, given by

$$u_j'' = \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} + \frac{h^2}{12} u_j^4(x) + \cdots,$$

and we obtain

$$u_{j-1} + ((kh)^2 - 2)u_j + u_{j+1} = 0. \tag{2}$$

Inserting, a non-trivial solutions of the form $u_j = e^{ij\tilde{k}h}$ into equation (2) with $\tilde{k}$ as the discrete wave number and writing above as a series in kh, we get

$$\tilde{k}h - kh = +\frac{(kh)^3}{24} + \cdots. \tag{3}$$

Equation (3) has already been obtained by many [1,3,5]. Interestingly, it is evident from above expression that to enjoy dispersion free propagations for all wave range of wave numbers, one requires $\tilde{k} = k$, which is possible only when $kh \to 0$. Therefore, for finite difference schemes, we make use *Bloch* wave property [10] given by

$$u_{j+n} = e^{ikhn} u_j \quad \forall n \in \mathbb{Z}. \tag{4}$$

Using property given in equation (4), we get compact scheme in the case of finite differences given by

$$u_{j-1} - 2\cos(kh)u_j + u_{j+1} = 0 \tag{5}$$

which leads back to standard finite difference scheme on Taylor series expansion of the middle node coefficients $-2\cos(kh)$. Now, inserting a non-trivial solutions of the form $u_j = e^{ij\tilde{k}h}$ into equation (5) and performing simplifications give $\tilde{k} = k$.

2.2 Framework for compact finite element scheme

For finite element setting, we start with the variational formulation of equation (1), given by: Find $u \in H^1(\mathbb{R})$ such that

$$B(u, v) = (u', v') - k^2 (u, v) = 0 \tag{6}$$

holds for all $v \in H^1(\mathbb{R})$ where $(\cdot, \cdot)$ denotes the L^2-inner product on $\mathbb{R}$ and $H^1(\mathbb{R})$ is the usual Sobolev space [10]. Let $V_h \subset H^1(\mathbb{R})$ denote the set of continuous piecewise linear polynomials relative to the grid $\mathcal{G}_h = \{nh, n = 0, \pm1, \pm2 \ldots\}$. We now seek an approximate solution $u_h \in V_h$

$$u_h(x) = \sum_{i \in \mathbb{Z}} U_i \theta_i(x), \quad x \in \mathbb{R}$$

where U_i are unknowns to be determined that satisfy

$$\sum_{i \in \mathbb{Z}} \left((\theta_i', \theta_j') - k^2 (\theta_i, \theta_j) \right) = 0 \text{ for all } j \in \mathbb{Z}$$

	$k=10$		$k=10000$		$\lvert\tilde{k}-k\rvert$
	N	$\tilde{k}$	N	$\tilde{k}$	
F.E. Scheme	1450	9.99998	$6*10^7$	9999.99998	0.00001
F.D. Scheme	1450	10.00001	$6*10^7$	10000.00001	0.00001

Table 1: Analysis of the number of elements for a dispersion error of $|\tilde{k}-k| = 10^{-4}$.

and has the following matrix form

$$\left(\mathbf{K}-k^2\mathbf{M}\right)\mathbf{U}=0$$

with

$$\mathbf{K}_{ij}=\left(\theta_i',\theta_j'\right) \quad \text{and} \quad \mathbf{M}_{ij}=\left(\theta_i,\theta_j\right) \ \text{ for } i,j\in\mathbb{Z}. \tag{7}$$

Moreover, for a single physical element $(0,h)$, the *stifness* and *mass* matrices given in equation (7) takes the form

$$\mathbf{K}=\frac{2}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \text{ and } \mathbf{M}=\frac{h}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}. \tag{8}$$

The assembling of matrices given in equation (8) lead to a system and picking up the j-th row, we get

$$\left(1+\frac{(kh)^2}{6}(1-\tau)\right)\left(U_{j-1}+U_{j+1}\right)-2\left(1-\frac{(kh)^2}{6}(2+\tau)\right)U_j=0. \tag{9}$$

Substituting, again a plane wave solution of the form $U_j=e^{ij\tilde{k}h}$ in equation (9) with $\tilde{k}$ as the discrete wave number, we obtain the following expression

$$\tilde{k}-k=-\frac{k^3h^2}{24}+\cdots$$

which is widely known in the literature of finite elements [1-6]. It is clear from Table 1 that the discrete wave number $\tilde{k}$ is overestimated in the case of finite difference scheme where as underestimated for finite element scheme which results into phase lag and phase lead a major cause of numerical dispersion. Now, to have dispersion free propagation, one requires that both exact wave and wave obtained using numerical approximations propagate with the exact wave number i.e, $\tilde{k}=k$, and for case of finite element scheme, we propose following modification to bilinear form

$$B^{\alpha}(u,v)=(u',v')-\alpha k^2\,(u,v)=0$$

with matrix form given by

$$\left(\mathbf{K}-k^2\mathbf{M}^{\alpha}\right)\mathbf{U}=0 \text{ with } \mathbf{M}^{\alpha}=\alpha\mathbf{M}.$$

Now we want to find a value of the unknown α such that the modified bilinear form provides the exact solution at the nodes of the spatial grid irrespective of low or high wave numbers. This means that $\tilde{k}=k$ and the value of α is given by

$$\alpha=\frac{6(1-\cos kh)}{k^2h^2\,(2+\cos kh)}=1+\frac{(kh)^2}{12}+\cdots$$

Scheme	$kh = 0.01$	$kh = 1$	$kh = 100$	$kh = 10000$
Compact F.D.	$3.1 * 10^{-14}$	$1.3 * 10^{-14}$	$9.1 * 10^{-13}$	$1.1 * 10^{-10}$
Standard F.D.	$1.0 * 10^{-6}$	2.8621	$9.9 * 10^{-1}$	$9.9 * 10^{-1}$
Compact F.E.	$1.9 * 10^{-13}$	$1.5 * 10^{-14}$	$9.1 * 10^{-13}$	$1.1 * 10^{-10}$
Standard F.E.	$1.0 * 10^{-6}$	2.7210	1.1305	$9.9 * 10^{-1}$

Table 2: Comparison of ℓ_∞ errors for both standard and compact schemes with fixed $h = 10^{-2}$ and varying wave numbers in the case of Dirichlet boundary conditions (10).

which is consistent with standard scheme as $\alpha \to 0$.

3 Results and discussion

In order to present the superiority of compat schemes over standard schemes, we solve (1) on $\Omega = (0, 1) \subset \mathbb{R}$ with Dirichlet boundary conditions applied at both ends given by

$$u(0) = 1 \quad \text{and} \quad u(1) = e^{ik}. \tag{10}$$

Numerical error is measured using the discrete ℓ_∞ norm, defined by $\ell_\infty = \max_j |u_j - u(x_j)|, j = 0, 1, 2, \ldots, N$ with $u(x_j)$ representing the analytical solution and u_j the computed numerical solution. Moreover, N denotes the number of elements in a uniformly spaced grid.

In Table 2, dispersion errors with fixed $h = 10^{-2}$ are given. It is evident that in the case of standard schemes, dispersion error is less when $kh \ll 0$ and gets worse for all $kh > 1$. This means that dispersion error depends upon the non-dimensional wave number kh where as for compact schemes, Table 2 is representing entirely different picture reflecting that dispersion error stays low irrespective of low $kh = 10^{-2}$ and high $kh = 10^4$.

4 Conclusion

The proposed general frame work for the development of compact schemes in particular for time harmonic wave equation offers following advantages over standard schemes (a) dispersion and dissipation less numerical solutions at the nodes of the spatial grid for one and higher dimensions are obtained; (b) Tri-diagonal banded matrices are obtained even for compact schemes.

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