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A Comparison Between Skew-logistic and Skew-normal Distributions

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Abstract Skew distributions are reasonable models for describing claims in property-liability insurance. We consider two well-known datasets from actuarial science and fit skew-normal and skew-logistic distributions to these dataset. We find that the skew-logistic distribution is reasonably competitive compared to skew-normal in the literature when describing insurance data. The value at risk and tail value at risk are estimated for the dataset under consideration. Also, we compare the skew distributions via Kolmogorov-Smirnov goodness-of-fit test, log-likelihood criteria and AIC.

Keywords Skew-logistic distribution; skew-normal distribution; value at risk; tail value at risk; log-likelihood criteria; AIC; Kolmogorov-Smirnov goodness-of-fit test.

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1 Introduction

Fitting an adequate distribution to real data sets is a relevant problem and not an easy task in actuarial literature, mainly due to the nature of the data, which shows several features to be accounted for. Eling [1] showed that the skew-normal and the skew-Student t distributions are reasonably competitive compared to some models when describing insurance data. Bolance *et al.* [2] provided strong empirical evidence in favor of the use of the skew-normal, and log-skew-normal distributions to model bivariate claims data from the Spanish motor insurance industry. Ahn *et al.* [3] used the log-phase-type distribution as a parametric alternative in fitting heavy tailed data. In the study of Burnecki *et al.* [4] usually claims distributions showed the presence of small, medium and large size claims, characteristics that are hardly compatible with the choice of fitting a single parametric analytical distribution.

In this paper, we compare skew-normal and skew-logistic distributions as reasonably good models for describing insurance claims. We consider two dataset widely used in literature and fit the skew-normal and skew-logistic distributions to these data. We find that the skew-logistic distribution is compared to skew-normal for two datasets. For this, the value at risk and tail value at risk are estimated for the dataset under consideration and two distributions are compared via Kolmogorov-Smirnov goodness-of-fit test, log-likelihood criteria and AIC.

2 Risk measures

Risk measures and their properties have been widely studied in the literature (see [5–7] and references therein). Most of those contributions and applications in risk management usually assume a parametric distribution for the loss random variable.

The value at risk, or VaR risk measure was actually used by management long before it was reinvented for investment banking. In actuarial contexts it is known as the quantile risk measure or quantile premium principle. VaR is always specified with a given confidence level γ. In broad terms, the γ-VaR represents the loss that, with probability γ will not be exceeded. Since that may not define a unique value, for example if there is a probability mass around the value, we define the γ-VaR more specifically, for $0 \leq \gamma \leq 1$, as

$$\mathrm{VaR}_\gamma(X) = \inf\{x,\ F_X(x) \geq \gamma\} = F_X^{-1}(\gamma), \tag{1}$$

where X is a random variable with probability density function (pdf) f_X, and cumulative distribution function (cdf) F_X. The value at risk is widely used in applications [8].

The quantile risk measure assesses the worst case loss, where worst case is defined as the event with a $1-\gamma$ probability. One problem with the quantile risk measure is that it does not take into consideration what the loss will be if that $1-\gamma$ worst case event actually occurs. The loss distribution above the quantile does not affect the risk measure. The conditional tail expectation (or CTE) was chosen to address some of the problems with the quantile risk measure. It was proposed more or less simultaneously by several research groups, so it has a number of names, including tail value at risk (or Tail-VaR), tail conditional expectation (or TCE) and expected shortfall [9]. Like the quantile risk measure, the CTE is defined using some confidence level γ, $0 \leq \gamma \leq 1$. In words, the CTE is the expected loss given that the loss falls in the worst $(1-\gamma)$ part of the loss distribution. The worst $(1-\gamma)$ part of the loss distribution is the part above the γ-quantile, Q_γ. If Q_γ falls in a continuous part of the loss distribution (that is, not in a probability mass) then we can interpret the CTE at confidence level γ, given the γ-quantile risk measure Q_γ, as

$$\mathrm{CTE}_\gamma(X) = \mathbb{E}(X|X > Q_\gamma). \tag{2}$$

3 Skew Distribution

Skewed distributions have played an important role in the statistical literature since the pioneering work of Azzalini [10]. He has provided a methodology to introduce skewness in a normal distribution. Since then a number of papers appeared in this area. He showed if $f(.)$ is any symmetric density function defined on $(-\infty, +\infty)$ and $F(.)$ is its distribution function, then for any $\alpha \in (-\infty, +\infty)$,

$$2f(x)F(\alpha x)\mathbb{I}_{(-\infty,+\infty)}(x), \tag{3}$$

is a proper density function and it is skewed if $\alpha \neq 0$. This property has been studied extensively in the literature to study skew-t and skew-Cauchy distributions [11].

3.1 Skew-normal Distribution

The normal distribution is the most popular distribution used for modeling in economics and finance. The insurance risks have skewed distributions, which is why in many cases the normal distribution is not an appropriate model for insurance risks or losses (see [12] and [13]). Besides skewness, some insurance risks also exhibit extreme tails [14].

The skew-normal distribution as well as other distributions from the skew-elliptical class might be promising alternatives to the normal distribution since they preserve advantages of the normal distribution with the additional benefit of flexibility with regards to skewness and kurtosis.

A random variable Z has a skew-normal (SN) distribution with parameter α, denoted by $SN(0,1,\alpha)$ and can be written as $SN(\alpha)$, if its density is given by

$$f(z,\alpha) = 2\phi(z)\Phi(\alpha z)\mathbb{I}_{(-\infty,+\infty)}(z), \tag{4}$$

where Φ and ϕ are the standard normal cdf and the standard normalpdf, respectively, and z and α are real numbers. Some basic properties of the $SN(\alpha)$ distribution given in [10] are:
1) $SN(0) = N(0,1)$,
2) if $Z \sim SN(\alpha)$, then $-Z \sim SN(-\alpha)$,
3) as $\alpha \to \pm\infty$, then $SN(\alpha)$ tends to the half-normal distribution, i.e., the distribution of $\pm|X|$, when $X \sim N(0,1)$,
4) if $Z \sim SN(\alpha)$, then $Z^2 \sim \chi_1^2$,
5) the moment generating function $M_Z(t)$ of the r.v. Z is

$$M_Z(t) = \mathbb{E}[e^{tZ}] = 2\exp\left(\frac{t^2}{2}\right)\Phi(\delta t),$$

where $\delta = \frac{\alpha}{\sqrt{1+\alpha^2}}$ and thus

$$\begin{aligned} \mathbb{E}(Z) &= \sqrt{\frac{2}{\pi}}\delta, \\ \mathbb{V}ar(Z) &= 1 - 2\frac{\delta^2}{\pi}. \end{aligned}$$

Also the measure of skewness and kurtosis are

$$\begin{aligned} S(Z) &= \frac{4-\pi}{2} sign(\alpha)\left(\frac{\alpha^2}{\pi/2 + (\pi/2-1)\alpha^2}\right)^{3/2}, \\ K(Z) &= 2(\pi-3)\left(\frac{\alpha^2}{\pi/2 + (\pi/2-1)\alpha^2}\right)^2. \end{aligned}$$

In practice it is useful to consider random variable under an affine transformation $Y = \xi + \sigma Z$, where $\xi \in \mathbb{R}$ and $\sigma > 0$. If $Z \sim SN(\alpha)$, then the density of Y is

$$f_{SN}(y;\xi,\sigma,\alpha) = \frac{2}{\sigma}\phi\left(\frac{y-\xi}{\sigma}\right)\Phi\left(\alpha\frac{y-\xi}{\sigma}\right)\mathbb{I}_{(-\infty,+\infty)}(y) \tag{5}$$

with location parameter ξ, scale parameter σ and shape parameter α. We denote this by $Y \sim SN(\xi,\sigma,\alpha)$. Figure 1 illustrates skew-normal distribution for three different values of shape parameter.

3.2 Skew-logistic Distribution

Using the same basic principle of [10], the skewness can be easily introduced to the logistic distribution. It has location, scale and skewness parameters. The probability density

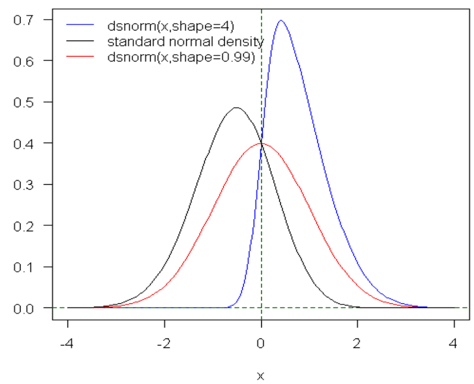

Figure 1: Skew-normal Distribution for Three Different Values of Shape Parameter

function of the skew logistic distribution can have different shapes with both positive and negative skewness depending on the skewness parameter (see Figure 2).

Although the probability density function of the skew logistic distribution is unimodal and log-concave, but the distribution function, failure rate function and the different moments cannot be obtained in explicit forms. Moreover, even when the location and scale parameters are known, the maximum likelihood estimator of the skewness parameter may not always exist [11]. Due to this problem, it becomes difficult to use this distribution for data analysis purposes. The logistic distribution [15] has been used in many different fields, for detailed description of the various properties and applications [16]. The standard logistic distribution has the pdf and the cdf specified by

$$f(x) = \frac{e^{-\frac{x-\eta}{\beta}}}{\beta\left(1+e^{-\frac{x-\eta}{\beta}}\right)^2}\mathbb{I}_{(-\infty,+\infty)}(x), \qquad \eta \in \mathbb{R}, \beta > 0$$

and

$$F(x) = \frac{1}{1+e^{-\frac{x-\eta}{\beta}}},$$

respectively. A random variable X is said to have skew-logistic distribution if its pdf is

$$f_{SL}(x;\eta,\beta,\alpha) = \frac{2e^{-\frac{x-\eta}{\beta}}}{\beta\left(1+e^{-\frac{x-\eta}{\beta}}\right)^2\left(1+e^{-\alpha\frac{x-\eta}{\beta}}\right)}, \tag{6}$$

where $\alpha \in (-\infty, +\infty)$. Such X is said to follow a skew-logistic distribution with skewness parameter α. We denote this by $X \sim SL(\eta, \beta, \alpha)$. Therefore $SL(0, 1, \alpha)$ can be written as $SL(\alpha)$. From Figure 2 it is clear that $SL(\alpha)$ is positively skewed when α is positive. It takes similar shapes on the negative side for $\alpha < 0$. Therefore, $SL(\alpha)$ can take positive and negative skewness. As α goes to $\pm\infty$, it converges to the half logistic distribution. Comparing with the shapes of the skew normal density function, it is clear that $SL(\alpha)$ produces heavy tailed skewed distribution than the skew normal ones. For large values of α, the tail behaviors of the different members of the $SL(\alpha)$ family are very similar. It is clear from Figure 2 that the tail behaviors of the different family members of $SL(\alpha)$ are the same for large values of $|\alpha|$. Some of the properties which are true for skew normal distribution are also true for skew logistic distribution.

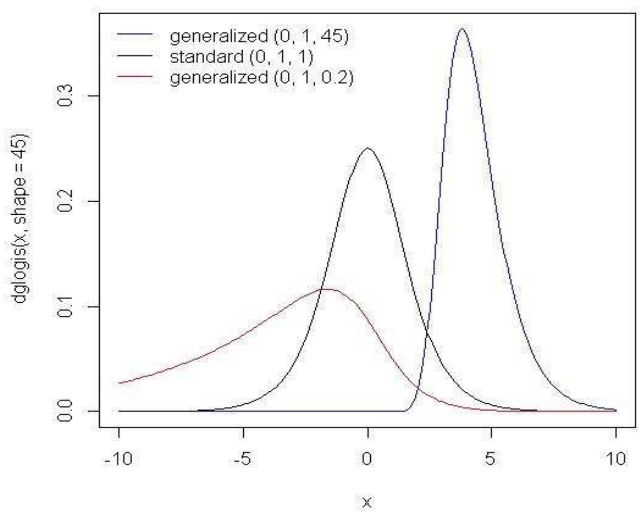

Figure 2: Skew-logistic Distribution for Three Different Values of Shape Parameter

4 Data

In the following we consider two well-known datasets:

The US indemnity losses- The US indemnity losses used in Frees and Valdez [17]. The dataset consists of 1500 general liability claims, giving for each the indemnity payment denoted in the data as “loss” and the allocated loss adjustment expense denoted in the data as “alae”, both in USD. The latter is the additional expense associated with settlement of the claim (e.g., claims investigation expenses and legal fees). We focus here on the pure loss data and do not consider the expenses, but results taking these expenses into consideration are available upon request. The dataset can be found in the R packages copula and evd, and have been used in works of [18] and [19].

The Danish fire losses- The Danish fire losses was analyzed in [20]. These data represents the Danish fire losses in million Danish Krones and were collected by a Danish reinsurance company. The dataset contains individual losses above 1 million Danish Krones, a total of 2167 individual losses, covering the period from January 3, 1980 to December 31,

1990. The data is adjusted for inflation to reflect 1985 values. The dataset can be found in the R packages fEcofin and fExtremes.

Table 1 presents descriptive statistics for the dataset. In addition to the number of observations, indicators for the first four moments (mean, standard deviation, skewness, excess kurtosis), and minimum and maximum values, we also present the 99% quantile and the mean loss, if the loss is above 99%. The 99% quantile is the value at risk (at 99% confidence level) and the mean loss exceeding the 99% quantile is the tail value at risk. The descriptive statistics show the skewness and kurtosis for the data. Figure 3 presents histogram and normal Q-Q plot for the dataset considered. Both histograms reveal a very typical feature of insurance claims data: a large number of small losses and a lower number of very large losses. The absolute values for the indemnity losses presented in the left histogram are higher than the values presented in the right histogram, which is simply due to scaling (TUSD on the left, million Danish Krones on the right).

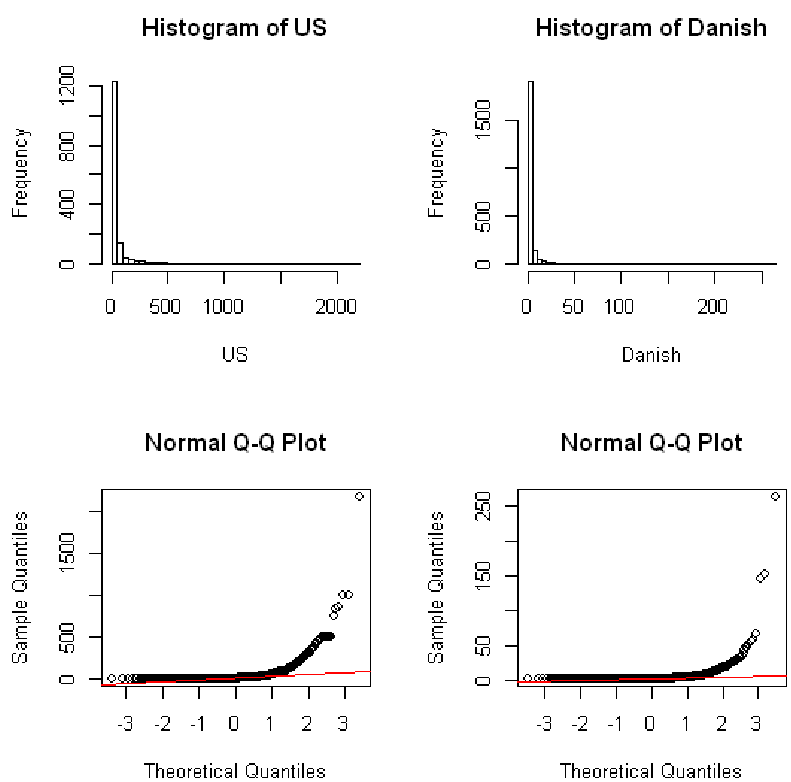

Figure 3: US Indemnity Losses (Left) and Danish Fire Losses (Right)

Table 1: Descriptive Statistics for Data

	US indemnity losses	Danish fire losses
No.of observations	1500	2167
$\mathbb{E}(X)$	41.21	3.39
St.Dev(X)	10.27	8.51
Skewness(X)	9.15	18.74
Kurtosis	141.98	482.20
Minimum	0.01	1.00
Maximum	2173.60	263.25
99%Quantile(value at risk)	475.06	26.04
$\mathbb{E}(X\|X \geq$ value at risk)	739.62	58.59

5 Results

In this section, we estimate the parameters of the skew-normal and skew-logistic distributions and analyze their properties for the empirical dataset introduced in Section 4 based on maximum likelihood estimation. A comparison of the distributions is made based on the Akaikes information and log-likelihood criteria. Eling [1] showed that both the skew-normal and skew-logistic are competitive compared to some distributions in widespread use. We calculate value at risk and tail value at risk using the estimated parameters and compare the estimation results with the empirical values for value at risk and tail value at risk. All tests presented in this section were conducted with the R packages sn and glogis for skew-normal and skew-logistic distributions, respectively.

Table 2 presents the estimated parameters for the skew-normal and skew-logistic distributions. The model's skewness values thus confirm the right skew of the empirical data, but the skewness values the model can take are less extreme. This might be seen as a limitation of the skew-normal model compared to other skewed distributions.

Table 3 presents a model comparison based on the log-likelihood criteria and AIC. Considering AIC and log-likelihood criteria, we conclude that the skew-logistic distribution is better in comparison with the skew-normal distribution for fitting to the dataset. We recall that we can compare the results with the transformation kernel approach described in [2]. Also, Table 4 reveals that the skew-logistic is better than skew-normal for describing the two datasets. Overall, the test results are thus highly correlated with the AIC results and confirm the ability of the skew-logistic distribution to describe insurance claims for the data at hand.

Finally, in Table 5 we use the model results to derive estimators for value at risk and tail value at risk and compare them with the empirical data. In Table 5, only values for a confidence level of 99% are presented.

6 Conclusion

The aim of this work is to fit two standard dataset of insurance claims to two skewed distributions used in finance literature. The motivation for conducting this study is to

Table 2: Estimated Parameters for the Skew-normal and Skew-logistic Distributions

Model		US indemnity losses	Danish fire
skew-normal	Location	-0.18729	0.9721663
	Scale	110.6819405	8.8584110
	Shape	1533.7683374	1533.7683374
Skew-logistic	Location	-216.93770	-16.844338
	Scale	31.17305	1.738632
	Shape	1781.45321	50301.946727

Table 3: Log-likelihood and AIC for Distributions

Model	R package	Log-likelihood		AIC	
		US indemnity	Danish fire	US indemnity	Danish fire
Skew-normal	sn	−8149.49	−6295.59	16304.98	12597.18
Skew-logistic	glogis	-7854	-5120	15714	10246

Table 4: Kolmogorov-Smirnov Goodness-of-fit for Distributions

Model	US indemnity	Danish fire
Skew-normal	0.5159	0.5798
Skew-logistic	0.2272	0.2212
Critical value	0.0351	0.0292

Table 5: Value at Risk and Tail Value at Risk at 99% Confidence Level (original data)

	Model	Value at risk	Tail value at risk
US indemnity data	Skew-normal	285.31	320.32
	Skew-logistic	159.86	191.04
	Empirical	475.05	739.61
Danish fire data	Skew-normal	23.78	26.62
	Skew-logistic	9.98	11.70
	Empirical	26.04	58.59

discover whether these models are also appropriate for describing insurance claims data. Claims data in non-life insurance are very skewed and exhibit high kurtosis. For this reason, the skew-normal and skew-logistic might be promising candidates for both theoretical and empirical work in actuarial science.

For both distributions, the value at risk and tail value at risk do not perform very well when the original data are considered; the estimators derived using the theoretical distributions are in general much lower than the empirical values. The results for value at risk and tail value at risk look better when the log data are considered; the risk estimators derived using the theoretical distributions are very close to the empirical values (see Table 6). We see that the VaR and Tail-Var for skew-logistic distribution are closee to the empirical values for two datasets.

Table 6: Value at Risk and Tail Value at Risk at 99% Confidence Level (log data)

	Model	Value at risk	Tail value at risk
log of US indemnity data	Skew-normal	10.75	11.27
	Skew-logistic	10.76	11.22
	Empirical	10.77	11.10
log of Danish fire data	Skew-normal	2.90	3.26
	Skew-logistic	2.91	3.35
	Empirical	3.26	3.82

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