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Implicit 7-stage Tenth Order Runge-Kutta Methods Based on Gauss-Kronrod-Lobatto Quadrature Formula

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Abstract In this paper, four new implicit Runge-Kutta methods which based on 7-point Gauss-Kronrod-Lobatto quadrature formula were developed. The resulting implicit methods were 7-stage tenth order Gauss-Kronrod-Lobatto III (GKLM(7,10)-III), 7-stage tenth order Gauss-Kronrod-Lobatto IIIA (GKLM(7,10)-IIIA), 7-stage tenth order Gauss-Kronrod-Lobatto IIIB (GKLM(7,10)-IIIB) and 7-stage tenth order Gauss-Kronrod-Lobatto IIIC (GKLM(7,10)-IIIC). Each of these methods required 7 function of evaluations at each integration step and gave accuracy of order 10. Theoretical analyses showed that the stage order for GKLM(7,10)-III, GKLM(7,10)-IIIA, GKLM(7,10)-IIIB and GKLM(7,10)-IIIC are 6, 7, 3 and 4, respectively. GKLM(7,10)-IIIC possessed the strongest stability condition i.e. L-stability, followed by GKLM(7,10)-IIIA and GKLM(7,10)-IIIB which both possessed A-stability, and lastly GKLM(7,10)-III having finite region of absolute stability. Numerical experiments compared the accuracy of these four implicit methods and the classical 5-stage tenth order Gauss-Legendre method in solving some test problems. Numerical results revealed that, GKLM(7,10)-IIIA was the most accurate method in solving a scalar stiff problem. All the proposed methods were found to have comparable accuracy and more accurate than the 5-stage tenth order Gauss-Legendre method in solving a two-dimensional stiff problem. Last but not least, all the proposed methods were implemented to solve two real-world problems i.e. the Van der Pol oscillator and the Brusselator. The numerical solutions which generated by the proposed methods were found to be comparable to the numerical solutions found in the existing literature.

Keywords Initial value problem; Gauss-Kronrod-Lobatto quadrature formula; Gauss-Kronrod-Lobatto III; Gauss-Kronrod-Lobatto IIIA; Gauss-Kronrod-Lobatto IIIB; Gauss-Kronrod-Lobatto IIIC; Van der Pol oscillator; Brusselator.

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1 Introduction

The numerical solution of first order initial value problem of the form

$$\begin{gathered} y' = f\left(x, y\right), \ y\left(a\right) = \eta, \\ y, f\left(x, y\right) \in \mathbb{R}, \ x \in [a, b] \subset \mathbb{R}, \end{gathered} \tag{1}$$

using one-step Runge-Kutta method is always a popular practice in science and engineering. The general form of Runge-Kutta method is given by

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i f\left(x_n + c_i h, Y_i\right), \tag{2}$$

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f\left(x_n + c_j h, Y_j\right),\ i = 1, \ldots, s. \tag{3}$$

From formula (2), the numbers $b_1, b_2, \ldots, b_s$ and $c_1, c_2, \ldots, c_s$ are independent of the function f, and they are called the quadrature weights and nodes, respectively. The function Y_i is the stage value and also the approximations to $y\left(x_n + c_j h\right)$ computed by formula (3) on the interval $[x_n, x_n + c_j h]$.

If $a_{ij} = 0,\ j \geq i,\ i = 1(1)s$, then Y_i is said to be defined explicitly so that formuale (2) and (3) form an explicit Runge-Kutta method. In most cases, explicit Runge-Kutta method is preferable because it allows explicit stage-by-stage implementation which is very easy to program using computer. However, numerical analysts also aware that the computational costs involving function evaluations increases rapidly as higher order requirements are imposed, [1]. Another disadvantage of explicit Runge-Kutta method is that it has relatively small interval of absolute stability renders them unsuitable for stiff initial value problems, [2]. In view of this, we are thus taking interest in implicit Runge-Kutta methods.

Implicit Runge-Kutta method is defined by the formulae (2) and (3) where Y_i are defined by a set of s implicit equations. In an implicit Runge-Kutta method, the explicit stage-by-stage implementation scheme enjoyed by explicit Runge-Kutta method is no longer available and needs to be replaced by an iterative computation, [3]. Other than this computational difficulty, implicit Runge-Kutta method is an appealing method where higher accuracy can be obtained with fewer function evaluations, and it has relatively bigger interval of absolute stability. For excellent surveys and various perspectives of implicit Runge-Kutta method, see, for examples, Hall and Watt [1], Fatunla [2], Butcher [3–5], Dekker and Verwer [6], Jain [7], Lambert [8], Hairer *et al.* [9, 10] and Iserles [11].

According to Butcher [3, 4], Dekker and Verwer [6], Jain [7], Lambert [8], Hairer *et al.* [9], Iserles [11] and many others, there are three special classes of implicit Runge-Kutta methods which based on three different Gauss-Legendre type quadrature formulae, namely Gauss-Legendre quadrature formulae, Gauss-Radau quadrature formulae and Gauss-Lobatto quadrature formulae. It has been shown in the literatures that Gauss-Legendre type implicit Runge-Kutta methods have high order of accuracy and highly stable, [3]. These particular types of implicit Runge-Kutta methods also hinted the possibility to search for implicit Runge-Kutta methods that are based on other types of quadrature formulae. In conjunction to this, Teh and Yaacob [12] had developed two 5-stage eighth order Gauss-Kronrod methods that are based on 5-point Gauss-Kronrod quadrature formula. After that, four implicit Runge-Kutta methods that are based on Gauss-Kronrod-Radau quadrature formuale were reported in Teh and Yaacob [13].

In this paper, we have considered the Gauss-Kronrod-Lobatto quadrature formula to construct four Kronrod type implicit Runge-Kutta methods. These new methods will serve as counterparts of the classical Lobatto III methods developed by Butcher and Ehle. This paper is organized as follows. Section 2 presents the basic information of Gauss-Kronrod-Lobatto quadrature formula. Section 3 is divided into four sub-sections, with each subsection presents the development of a new Kronrod-Lobatto type Runge-Kutta method. Numerical comparisons among these new Runge-Kutta methods and the classical 5-stage tenth order Gauss-Legendre method are presented in Section 4. Lastly, some conclusions are given in Section 5.

2 Gauss-Kronrod-Lobatto Quadrature Formula

A n-point Gauss-Lobatto quadrature formula for the integral

$$\Im(f) = \int_a^b f(x)\, dx, \tag{4}$$

is a formula of the form

$$G_n(f) = \sum_{k=1}^{n} w_k f(x_k), \tag{5}$$

with the nodes $a = x_1 < x_2 < x_3 < \cdots < x_n = b$ and positive weights w_k are chosen so that

$$G_n(f) = \Im(f)\,,\ \forall f \in \mathrm{P}_{2n-3}, \tag{6}$$

where P_{2n-3} denotes the set of polynomials of degree at most $2n-3$, [14,15]. The associated Gauss-Kronrod-Lobatto quadrature formula is given by

$$K_{2n-1}(f) = \sum_{k=1}^{n} \widehat{w}_k f(\widehat{x}_k) + \sum_{k=1}^{n-1} \widetilde{w}_k f(\widetilde{x}_k), \tag{7}$$

where $\{\widehat{x}_k = x_k\}$, $k = 1\,(1)\,n$ are precisely the one used in (5), while all the other $3n-2$ parameters $\{\widehat{w}_k\}$, $\{\widetilde{w}_k\}$ and $\{\widetilde{x}_k\}$ are chosen in such a way that

$$K_{2n-1}(f) = \Im(f)\,,\ \forall f \in \mathrm{P}_{3n-2}, \tag{8}$$

where P_{3n-2} denotes the set of polynomials of degree at most $3n-2$, [16]. According to Calvetti *et al.* [16], the nodes in the Gauss-Kronrod-Lobatto quadrature formula are ordered so that the following interlacing property is satisfied:

$$a = \widehat{x}_1 < \widetilde{x}_1 < \widehat{x}_2 < \widetilde{x}_2 < \widehat{x}_3 < \widetilde{x}_3 < \cdots < \widehat{x}_{n-1} < \widetilde{x}_{n-1} < \widehat{x}_n = b.$$

3 7-stage Implicit Runge-Kutta Methods based on 7-point Gauss-Kronrod-Lobatto Quadrature Formula

In this section, we have developed four implicit Runge-Kutta methods which based on 7-point Gauss-Kronrod-Lobatto quadrature formula for the numerical solution of (1). A 7-point Gauss-Kronrod-Lobatto quadrature formula is a 7-point formula which consists of four fixed nodes from the 4-point Gauss-Lobatto quadrature formula, and 3 additional points.

The first step is to obtain the quadrature nodes with respect to a 4-point Gauss-Lobatto quadrature formula and suppose that $f(x)$ is a polynomial of degree 5 given by

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5. \tag{9}$$

On substituting (4), (5) and (9) into (6) with $a = 0$, $b = 1$, then we obtained the following result:

$$\begin{aligned}
&\int_0^1 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right) dx \\
&= \sum_{k=1}^{4} w_k f(x_k) \\
&= w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + w_4 f(x_4) \\
&= w_1 \left(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + a_4 x_1^4 + a_5 x_1^5\right) \\
&\quad + w_2 \left(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3 + a_4 x_2^4 + a_5 x_2^5\right) \\
&\quad + w_3 \left(a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^3 + a_4 x_3^4 + a_5 x_3^5\right) \\
&\quad + w_4 \left(a_0 + a_1 x_4 + a_2 x_4^2 + a_3 x_4^3 + a_4 x_4^4 + a_5 x_4^5\right).
\end{aligned} \tag{10}$$

The integration of integral of (10) yields the following result:

$$\int_0^1 \left(a_0 + a_1 x + a_2 x^2 + a_4 x^4 + a_5 x^5\right) dx = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} + \frac{a_4}{5} + \frac{a_5}{6}. \tag{11}$$

On substituting the result in (11) and the preassigned Gauss-Lobatto quadrature nodes, $x_1 = 0$ and $x_4 = 1$ into (10) and rearrange in terms of a_i for $i = 0(1)5$, we obtained the following equation

$$\begin{aligned}
&(w_1 + w_2 + w_3 + w_4)\, a_0 + (w_1(0) + w_2 x_2 + w_3 x_3 + w_4(1))\, a_1 \\
&+ \left(w_1(0)^2 + w_2 x_2^2 + w_3 x_3^2 + w_4(1)^2\right) a_2 + \left(w_1(0)^3 + w_2 x_2^3 + w_3 x_3^3 + w_4(1)^3\right) a_3 \\
&+ \left(w_1(0)^4 + w_2 x_2^4 + w_3 x_3^4 + w_4(1)^4\right) a_4 + \left(w_1(0)^5 + w_2 x_2^5 + w_3 x_3^5 + w_4(1)^5\right) a_5 \\
&= a_0 + \tfrac{a_1}{2} + \tfrac{a_2}{3} + \tfrac{a_3}{4} + \tfrac{a_4}{5} + \tfrac{a_5}{6}.
\end{aligned} \tag{12}$$

On matching the coefficients of a_0, a_1, a_2, a_3, a_4 and a_5 in (12), we arrived to a system of 6 equations. On solving these 6 equations simultaneously using *MATHEMATICA 5.0*, we obtained the following weights and quadrature nodes of a 4-point Gauss-Lobatto quadrature formula as shown below:

$$\left\{ w_1 = \frac{1}{12}, w_2 = \frac{5}{12}, w_3 = \frac{5}{12}, w_4 = \frac{1}{12}, x_1 = 0, x_2 = \frac{5-\sqrt{5}}{10}, x_3 = \frac{5+\sqrt{5}}{10}, x_4 = 1 \right\}. \tag{13}$$

The weights of a 4-point Gauss-Lobatto quadrature formula will not be reused when constructing a 7-point Gauss-Kronrod-Lobatto quadrature formula, therefore, only the quadrature nodes are of importance. Now, the second step is to derive the 7-point Gauss-Kronrod-Lobatto quadrature formula and suppose that $f(x)$ is a polynomial of degree 9 given by

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9. \tag{14}$$

On substituting (4), (7) and (14) into (8) with $a = 0$, $b = 1$, then we obtained the following result:

$$
\begin{aligned}
&\int_0^1 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9\right) dx \\
&= \sum_{k=1}^{4} \widehat{w}_k f\left(\widehat{x}_k\right) + \sum_{k=1}^{3} \widetilde{w}_k f\left(\widetilde{x}_k\right) \\
&= \widehat{w}_1 f\left(\widehat{x}_1\right) + \widetilde{w}_1 f\left(\widetilde{x}_1\right) + \widehat{w}_2 f\left(\widehat{x}_2\right) + \widetilde{w}_2 f\left(\widetilde{x}_2\right) + \widehat{w}_3 f\left(\widehat{x}_3\right) + \widetilde{w}_3 f\left(\widetilde{x}_3\right) + \widehat{w}_4 f\left(\widehat{x}_4\right) \\
&= \widehat{w}_1 \left(a_0 + a_1 \widehat{x}_1 + a_2 \widehat{x}_1^2 + a_3 \widehat{x}_1^3 + a_4 \widehat{x}_1^4 + a_5 \widehat{x}_1^5 + a_6 \widehat{x}_1^6 + a_7 \widehat{x}_1^7 + a_8 \widehat{x}_1^8 + a_9 \widehat{x}_1^9\right) \\
&+ \widetilde{w}_1 \left(a_0 + a_1 \widetilde{x}_1 + a_2 \widetilde{x}_1^2 + a_3 \widetilde{x}_1^3 + a_4 \widetilde{x}_1^4 + a_5 \widetilde{x}_1^5 + a_6 \widetilde{x}_1^6 + a_7 \widetilde{x}_1^7 + a_8 \widetilde{x}_1^8 + a_9 \widetilde{x}_1^9\right) \\
&+ \widehat{w}_2 \left(a_0 + a_1 \widehat{x}_2 + a_2 \widehat{x}_2^2 + a_3 \widehat{x}_2^3 + a_4 \widehat{x}_2^4 + a_5 \widehat{x}_2^5 + a_6 \widehat{x}_2^6 + a_7 \widehat{x}_2^7 + a_8 \widehat{x}_2^8 + a_9 \widehat{x}_2^9\right) \\
&+ \widetilde{w}_2 \left(a_0 + a_1 \widetilde{x}_2 + a_2 \widetilde{x}_2^2 + a_3 \widetilde{x}_2^3 + a_4 \widetilde{x}_2^4 + a_5 \widetilde{x}_2^5 + a_6 \widetilde{x}_2^6 + a_7 \widetilde{x}_2^7 + a_8 \widetilde{x}_2^8 + a_9 \widetilde{x}_2^9\right) \\
&+ \widehat{w}_3 \left(a_0 + a_1 \widehat{x}_3 + a_2 \widehat{x}_3^2 + a_3 \widehat{x}_3^3 + a_4 \widehat{x}_3^4 + a_5 \widehat{x}_3^5 + a_6 \widehat{x}_3^6 + a_7 \widehat{x}_3^7 + a_8 \widehat{x}_3^8 + a_9 \widehat{x}_3^9\right) \\
&+ \widetilde{w}_3 \left(a_0 + a_1 \widetilde{x}_3 + a_2 \widetilde{x}_3^2 + a_3 \widetilde{x}_3^3 + a_4 \widetilde{x}_3^4 + a_5 \widetilde{x}_3^5 + a_6 \widetilde{x}_3^6 + a_7 \widetilde{x}_3^7 + a_8 \widetilde{x}_3^8 + a_9 \widetilde{x}_3^9\right) \\
&+ \widehat{w}_4 \left(a_0 + a_1 \widehat{x}_4 + a_2 \widehat{x}_4^2 + a_3 \widehat{x}_4^3 + a_4 \widehat{x}_4^4 + a_5 \widehat{x}_4^5 + a_6 \widehat{x}_4^6 + a_7 \widehat{x}_4^7 + a_8 \widehat{x}_4^8 + a_9 \widehat{x}_4^9\right).
\end{aligned}
\tag{15}
$$

The integration of integral of (15) yields the following result:

$$
\begin{aligned}
&\int_0^1 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9\right) dx \\
&= a_0 + \tfrac{a_1}{2} + \tfrac{a_2}{3} + \tfrac{a_3}{4} + \tfrac{a_4}{5} + \tfrac{a_5}{6} + \tfrac{a_6}{7} + \tfrac{a_7}{8} + \tfrac{a_8}{9} + \tfrac{a_9}{10}.
\end{aligned}
\tag{16}
$$

On substituting the result in (16) and

$$
\left\{ \widehat{x}_1 = x_1 = 0, \widehat{x}_2 = x_2 = \frac{5-\sqrt{5}}{10}, \widehat{x}_3 = x_3 = \frac{5+\sqrt{5}}{10}, \widehat{x}_4 = x_4 = 1 \right\}
$$

into (15) and rearrange in terms of a_i for $i = 0(1)9$, we obtained the following equation

$$
\begin{aligned}
&\left(\widehat{w}_1 + \widetilde{w}_1 + \widehat{w}_2 + \widetilde{w}_2 + \widehat{w}_3 + \widetilde{w}_3 + \widehat{w}_4\right) a_0 \\
&+ \left(\widehat{w}_1 (0) + \widetilde{w}_1 \widetilde{x}_1 + \widehat{w}_2 \left(\tfrac{5-\sqrt{5}}{10}\right) + \widetilde{w}_2 \widetilde{x}_2 + \widehat{w}_3 \left(\tfrac{5+\sqrt{5}}{10}\right) + \widetilde{w}_3 \widetilde{x}_3 + \widehat{w}_4 (1)\right) a_1 \\
&+ \left(\widehat{w}_1 (0)^2 + \widetilde{w}_1 \widetilde{x}_1^2 + \widehat{w}_2 \left(\tfrac{5-\sqrt{5}}{10}\right)^2 + \widetilde{w}_2 \widetilde{x}_2^2 + \widehat{w}_3 \left(\tfrac{5+\sqrt{5}}{10}\right)^2 + \widetilde{w}_3 \widetilde{x}_3^2 + \widehat{w}_4 (1)^2\right) a_2 \\
&+ \left(\widehat{w}_1 (0)^3 + \widetilde{w}_1 \widetilde{x}_1^3 + \widehat{w}_2 \left(\tfrac{5-\sqrt{5}}{10}\right)^3 + \widetilde{w}_2 \widetilde{x}_2^3 + \widehat{w}_3 \left(\tfrac{5+\sqrt{5}}{10}\right)^3 + \widetilde{w}_3 \widetilde{x}_3^3 + \widehat{w}_4 (1)^3\right) a_3 \\
&+ \left(\widehat{w}_1 (0)^4 + \widetilde{w}_1 \widetilde{x}_1^4 + \widehat{w}_2 \left(\tfrac{5-\sqrt{5}}{10}\right)^4 + \widetilde{w}_2 \widetilde{x}_2^4 + \widehat{w}_3 \left(\tfrac{5+\sqrt{5}}{10}\right)^4 + \widetilde{w}_3 \widetilde{x}_3^3 + \widehat{w}_4 (1)^4\right) a_4 \\
&+ \left(\widehat{w}_1 (0)^5 + \widetilde{w}_1 \widetilde{x}_1^3 + \widehat{w}_2 \left(\tfrac{5-\sqrt{5}}{10}\right)^5 + \widetilde{w}_2 \widetilde{x}_2^5 + \widehat{w}_3 \left(\tfrac{5+\sqrt{5}}{10}\right)^5 + \widetilde{w}_3 \widetilde{x}_3^3 + \widehat{w}_4 (1)^5\right) a_5 \\
&+ \left(\widehat{w}_1 (0)^6 + \widetilde{w}_1 \widetilde{x}_1^6 + \widehat{w}_2 \left(\tfrac{5-\sqrt{5}}{10}\right)^6 + \widetilde{w}_2 \widetilde{x}_2^6 + \widehat{w}_3 \left(\tfrac{5+\sqrt{5}}{10}\right)^6 + \widetilde{w}_3 \widetilde{x}_3^6 + \widehat{w}_4 (1)^6\right) a_6 \\
&+ \left(\widehat{w}_1 (0)^7 + \widetilde{w}_1 \widetilde{x}_1^7 + \widehat{w}_2 \left(\tfrac{5-\sqrt{5}}{10}\right)^7 + \widetilde{w}_2 \widetilde{x}_2^7 + \widehat{w}_3 \left(\tfrac{5+\sqrt{5}}{10}\right)^7 + \widetilde{w}_3 \widetilde{x}_3^7 + \widehat{w}_4 (1)^7\right) a_7 \\
&+ \left(\widehat{w}_1 (0)^8 + \widetilde{w}_1 \widetilde{x}_1^8 + \widehat{w}_2 \left(\tfrac{5-\sqrt{5}}{10}\right)^8 + \widetilde{w}_2 \widetilde{x}_2^8 + \widehat{w}_3 \left(\tfrac{5+\sqrt{5}}{10}\right)^8 + \widetilde{w}_3 \widetilde{x}_3^8 + \widehat{w}_4 (1)^8\right) a_8 \\
&+ \left(\widehat{w}_1 (0)^9 + \widetilde{w}_1 \widetilde{x}_1^9 + \widehat{w}_2 \left(\tfrac{5-\sqrt{5}}{10}\right)^9 + \widetilde{w}_2 \widetilde{x}_2^3 + \widehat{w}_3 \left(\tfrac{5+\sqrt{5}}{10}\right)^9 + \widetilde{w}_3 \widetilde{x}_3^9 + \widehat{w}_4 (1)^9\right) a_9 \\
&= a_0 + \tfrac{a_1}{2} + \tfrac{a_2}{3} + \tfrac{a_3}{4} + \tfrac{a_4}{5} + \tfrac{a_5}{6} + \tfrac{a_6}{7} + \tfrac{a_7}{8} + \tfrac{a_8}{9} + \tfrac{a_9}{10}.
\end{aligned}
\tag{17}
$$

On matching the coefficients of a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 and a_9, we arrived to a system of 10 equations. On solving these 10 equations simultaneously using *MATHEMATICA 5.0*, we obtained the following weights and quadrature nodes of a 7-point Gauss-Kronrod-Lobatto quadrature formula as shown below:

$$\begin{Bmatrix} \widehat{w}_1 = \frac{11}{420}, \widetilde{w}_1 = \frac{36}{245}, \widehat{w}_2 = \frac{125}{588}, \widetilde{w}_2 = \frac{8}{35}, \widehat{w}_3 = \frac{125}{588}, \widetilde{w}_3 = \frac{36}{245}, \widehat{w}_4 = \frac{11}{420}, \\ \widehat{x}_1 = 0, \widetilde{x}_1 = \frac{3-\sqrt{6}}{6}, \widehat{x}_2 = \frac{5-\sqrt{5}}{10}, \widetilde{x}_2 = \frac{1}{2}, \widehat{x}_3 = \frac{5+\sqrt{5}}{10}, \widetilde{x}_3 = \frac{3+\sqrt{6}}{6}, \widehat{x}_4 = 1 \end{Bmatrix}. \tag{18}$$

The direct substitution of (18) in the sense of the weights and abscissas of an implicit Runge-Kutta method is

$$\begin{Bmatrix} b_1 = \frac{11}{420}, b_2 = \frac{36}{245}, b_3 = \frac{125}{588}, b_4 = \frac{8}{35}, b_5 = \frac{125}{588}, b_6 = \frac{36}{245}, b_7 = \frac{11}{420}, \\ c_1 = 0, c_2 = \frac{3-\sqrt{6}}{6}, c_3 = \frac{5-\sqrt{5}}{10}, c_4 = \frac{1}{2}, c_5 = \frac{5+\sqrt{5}}{10} c_6 = \frac{3+\sqrt{6}}{6}, c_7 = 1 \end{Bmatrix}. \tag{19}$$

Before moving on with our developments, we would like to mention about the technique to derive the order conditions for a Runge-Kutta method. The technique for deriving the order conditions is to match the expansion of the solution generated by the Runge-Kutta method with the Taylor expansion of the exact solution, [8]. However, as the order being sought increases, the number of conditions rises rapidly and becomes unmanageable, [3]. Therefore, the following definitions and results on the simplified order conditions which relates the parameters a_{ij}, c_i and b_i of a Runge-Kutta method will be found useful, [3, 9]:

$$B\left(p\right): \quad \sum_{i=1}^{s} b_i c_i^{k-1} = \frac{1}{k}, \ k = 1, \ldots, p, \tag{20}$$

$$C\left(\eta\right): \quad \sum_{j=1}^{s} a_{ij} c_j^{k-1} = \frac{c_i^k}{k}, \ i = 1, \ldots, s, \ k = 1, \ldots, \eta, \tag{21}$$

$$D\left(\zeta\right): \quad \sum_{i=1}^{s} b_i c_i^{k-1} a_{ij} = \frac{b_j}{k}\left(1 - c_j^k\right), \ j = 1, \ldots, s, \ k = 1, \ldots, \zeta. \tag{22}$$

We note that the simplifying assumptions shown in (20)–(22) are very useful in developing our new implicit methods and also facilitate some of our discussions later.

3.1 7-stage Tenth Order Gauss-Kronrod-Lobatto III Method

In order to complete the development of the 7-stage tenth order Gauss-Kronrod-Lobatto III method, the choice of a_{ij}, $i, j = 1(1)7$ is to satisfy all the 42 order conditions of

$$C\left(6\right): \quad \sum_{j=1}^{7} a_{ij} c_j^{k-1} = \frac{c_i^k}{k}, \ i = 1, \ldots, 7, \ k = 1, \ldots, 6. \tag{23}$$

On substituting the abscissas in (19) into (23), assuming $a_{17} = a_{27} = a_{37} = a_{47} = a_{57} = a_{67} = a_{77} = 0$ and solve these 42 equations simultaneously using *MATHEMATICA 5.0*, yield the solution of the parameters a_{ij}, $i, j = 1(1)7$ as shown below:

$$
\begin{aligned}
&\{a_{11} = 0, a_{12} = 0, a_{13} = 0, a_{14} = 0, a_{15} = 0, a_{16} = 0, a_{17} = 0, a_{21} = \tfrac{31}{864}, \\
&a_{22} = \tfrac{123+2\sqrt{6}}{2016}, a_{23} = \tfrac{5\left(299-5\sqrt{5}-128\sqrt{6}\right)}{12096}, a_{24} = \tfrac{41-16\sqrt{6}}{432}, a_{25} = \tfrac{5\left(299+5\sqrt{5}-128\sqrt{6}\right)}{12096}, \\
&a_{26} = \tfrac{123-50\sqrt{6}}{2016}, a_{27} = 0, a_{31} = \tfrac{1}{50}, a_{32} = \tfrac{3\left(95+\sqrt{5}+40\sqrt{6}\right)}{3500}, a_{33} = \tfrac{50+\sqrt{5}}{525}, \\
&a_{34} = \tfrac{95-47\sqrt{5}}{750}, a_{35} = \tfrac{2}{21} - \tfrac{43}{210\sqrt{5}}, a_{36} = \tfrac{3\left(95+\sqrt{5}-40\sqrt{6}\right)}{3500} a_{37} = 0, a_{41} = \tfrac{1}{32}, \\
&a_{42} = \tfrac{3\left(5+2\sqrt{6}\right)}{224}, a_{43} = \tfrac{5\left(31+15\sqrt{5}\right)}{1344}, a_{44} = \tfrac{5}{48}, a_{45} = \tfrac{5\left(31-15\sqrt{5}\right)}{1344}, a_{46} = \tfrac{3\left(5-2\sqrt{6}\right)}{224}, \\
&a_{47} = 0, a_{51} = \tfrac{1}{50}, a_{52} = \tfrac{3\left(95-\sqrt{5}+40\sqrt{6}\right)}{3500}, a_{53} = \tfrac{2}{21} + \tfrac{43}{210\sqrt{5}}, a_{54} = \tfrac{95+47\sqrt{5}}{750}, \\
&a_{55} = \tfrac{50-\sqrt{5}}{525}, a_{56} = \tfrac{3\left(95-\sqrt{5}-40\sqrt{6}\right)}{3500}, a_{57} = 0, a_{61} = \tfrac{31}{864}, a_{62} = \tfrac{123+50\sqrt{6}}{2016}, \\
&a_{63} = \tfrac{5\left(299-5\sqrt{5}+128\sqrt{6}\right)}{12096}, a_{64} = \tfrac{41+16\sqrt{6}}{432}, a_{65} = \tfrac{5\left(299+5\sqrt{5}+128\sqrt{6}\right)}{12096}, a_{66} = \tfrac{123-2\sqrt{6}}{2016}, \\
&a_{67} = 0, a_{71} = 0, a_{72} = \tfrac{3}{14}, a_{73} = \tfrac{5}{42}, a_{74} = \tfrac{1}{3}, a_{75} = \tfrac{5}{42}, a_{76} = \tfrac{3}{14}, a_{77} = 0\}.
\end{aligned}
\tag{24}
$$

On substituting the values in (19) and (24) with $s = 7$ into (2) and (3), we obtained the 7-stage tenth order Gauss-Kronrod-Lobatto III method, or in brief as GKLM(7,10)-III. GKLM(7,10)-III has proved to possess tenth order of accuracy because the values in (19) satisfy all the order conditions in

$$B\,(10): \quad \sum_{i=1}^{7} b_i c_i^{k-1} = \frac{1}{k}, \; k = 1, \ldots, 10.$$

In addition, the values in (19) and (24) also satisfy the order conditions in

$$D\,(4): \quad \sum_{i=1}^{7} b_i c_i^{k-1} a_{ij} = \frac{b_j}{k}\left(1 - c_j^k\right), \; j = 1, \ldots, 7, \; k = 1, \ldots, 4.$$

Since GKLM(7,10)-III satisfies $C(6)$, then we can claim that GKLM(7,10)-III has stage order 6.

The stability function of a Runge-Kutta method can be easily obtained by using the following formula, [8]

$$R\,(z) = \frac{\det\left[\mathbf{I} - z\mathbf{A} + \mathbf{e}\mathbf{b}^{\mathrm{T}}\right]}{\det\left[\mathbf{I} - z\mathbf{A}\right]}, \tag{25}$$

where in the case of a 7-stage Runge-Kutta method, $\mathbf{I}$ is a 7×7 identity matrix, $\mathbf{A}$ is a matrix containing the elements a_{ij} for $i, j = 1(1)7$, $\mathbf{e} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}$, $\mathbf{b}$ is a row vector containing the elements b_i for $i = 1(1)7$. On substituting the values from (19) and (24) into (25), the stability function for GKLM(7,10)-III is given by

$$R\,(z)_{\mathrm{GKLM(7,10)-III}} = \frac{36288000+21168000z+5785920z^2+970200z^3+109200z^4+8400z^5+420z^6+11z^7}{36288000-15120000z+2761920z^2-279720z^3+15960z^4-420z^5}. \tag{26}$$

A Runge-Kutta method is said to be absolute stable if $|R\,(z)| \leq 1$ holds. The region S of the complex z-plane for which $|R\,(z)| \leq 1$ holds is the region of absolute stability of the Runge-Kutta method. Figure 1 is the plot of the stability function (26). The shaded region in Figure 1 is the region of absolute stability of GKLM(7,10)-III. We have observed that the region of absolute stability of GKLM(7,10)-III is a bounded region in the left-half complex plane, which suggest that GKLM(7,10)-III is not A-stable.

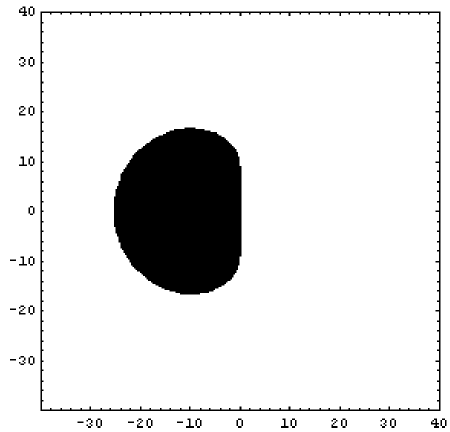

Figure 1: Stability Region of GKLM(7,10)-III

3.2 7-stage Tenth Order Gauss-Kronrod-Lobatto IIIC Method

As for the 7-stage tenth order Gauss-Kronrod-Lobatto IIIC method, the choice of a_{ij}, $i, j = 1(1)7$ is to satisfy all the 42 order conditions of

$$D(6): \quad \sum_{i=1}^{7} b_i c_i^{k-1} a_{ij} = \frac{b_j}{k}\left(1 - c_j^k\right), \; j = 1, \ldots, 7, \; k = 1, \ldots, 6. \tag{27}$$

On substituting the weights and abscissas in (19) into (27), assuming $a_{71} = a_{77} = \frac{11}{420}$, $a_{72} = a_{76} = \frac{36}{245}$, $a_{73} = a_{75} = \frac{125}{588}$, $a_{74} = \frac{8}{35}$ and solve these 42 equations simultaneously using *MATHEMATICA 5.0*, yield the solution of the parameters a_{ij}, $i, j = 1(1)7$ as shown below:

$$\begin{aligned}
&\Big\{a_{11} = \tfrac{11}{420}, a_{12} = -\tfrac{293}{5390}, a_{13} = \tfrac{325}{6468}, a_{14} = -\tfrac{17}{385}, a_{15} = \tfrac{325}{6468}, a_{16} = -\tfrac{293}{5390},\\
&a_{17} = \tfrac{11}{420}, a_{21} = \tfrac{11}{420}, a_{22} = \tfrac{6063-70\sqrt{6}}{70560}, a_{23} = \tfrac{5\left(535-7\sqrt{5}-280\sqrt{6}\right)}{28224}, a_{24} = \tfrac{209-70\sqrt{6}}{1680},\\
&a_{25} = \tfrac{5\left(535+7\sqrt{5}-280\sqrt{6}\right)}{28224}, a_{26} = \tfrac{6063-1750\sqrt{6}}{70560}, a_{27} = -\tfrac{121}{10080}, a_{31} = \tfrac{11}{420},\\
&a_{32} = \tfrac{1507+35\sqrt{5}+896\sqrt{6}}{24500}, a_{33} = \tfrac{23}{196} - \tfrac{1}{105\sqrt{5}}, a_{34} = \tfrac{3\left(61-35\sqrt{5}\right)}{1750}, a_{35} = \tfrac{23}{196} - \tfrac{43}{210\sqrt{5}},\\
&a_{36} = \tfrac{1507+35\sqrt{5}-896\sqrt{6}}{24500}, a_{37} = \tfrac{121}{10500}, a_{41} = \tfrac{11}{420}, a_{42} = \tfrac{2021}{23520} + \tfrac{1}{7\sqrt{6}},\\
&a_{43} = \tfrac{5\left(535+329\sqrt{5}\right)}{28224}, a_{44} = \tfrac{209}{1680}, a_{45} = \tfrac{5\left(535-329\sqrt{5}\right)}{28224}, a_{46} = \tfrac{2021}{23520} - \tfrac{1}{7\sqrt{6}},\\
&a_{47} = -\tfrac{121}{10080}, a_{51} = \tfrac{11}{420}, a_{52} = \tfrac{1507-35\sqrt{5}+896\sqrt{6}}{24500}, a_{53} = \tfrac{23}{196} + \tfrac{43}{210\sqrt{5}},\\
&a_{54} = \tfrac{3\left(61+35\sqrt{5}\right)}{1750}, a_{55} = \tfrac{23}{196} + \tfrac{1}{105\sqrt{5}}, a_{56} = \tfrac{1507-35\sqrt{5}-896\sqrt{6}}{24500}, a_{57} = \tfrac{121}{10500},\\
&a_{61} = \tfrac{11}{420}, a_{62} = \tfrac{6063+1750\sqrt{6}}{70560}, a_{63} = \tfrac{5\left(535-7\sqrt{5}+280\sqrt{6}\right)}{28224}, a_{64} = \tfrac{209+70\sqrt{6}}{1680},\\
&a_{65} = \tfrac{5\left(535+7\sqrt{5}+280\sqrt{6}\right)}{28224}, a_{66} = \tfrac{6063+70\sqrt{6}}{70560}, a_{67} = -\tfrac{121}{10080}, a_{71} = \tfrac{11}{420}, a_{72} = \tfrac{36}{245},\\
&a_{73} = \tfrac{125}{588}, a_{74} = \tfrac{8}{35}, a_{75} = \tfrac{125}{588}, a_{76} = \tfrac{36}{245}, a_{77} = \tfrac{11}{420}\Big\}.
\end{aligned} \tag{28}$$

On substituting the values in (19) and (28) with $s = 7$ into (2) and (3), we obtained the 7-stage tenth order Gauss-Kronrod-Lobatto IIIC method, or in brief as GKLM(7,10)-IIIC.

GKLM(7,10)-IIIC has proved to possess tenth order of accuracy because the values in (19) satisfy all the order conditions in $B(10)$. In addition, the values in (19) and (28) also satisfy the order conditions in

$$C\left(4\right): \quad \sum_{j=1}^{7} a_{ij} c_j^{k-1} = \frac{c_i^k}{k},\ i = 1,\ldots,7,\ k = 1,\ldots,4.$$

Since GKLM(7,10)-IIIC satisfies $C(4)$, then we can claim that GKLM(7,10)-IIIC has stage order 4.

The stability function for GKLM(7,10)-IIIC can be easily obtained by substituting the values in (19) and (28) into (25). Upon these substitutions, the stability function for GKLM(7,10)-IIIC is given by

$$R\left(z\right)_{\mathrm{GKLM(7,10)-IIIC}} = \frac{36288000+15120000z+2761920z^2+279720z^3+15960z^4+420z^5}{36288000-21168000z+5785920z^2-970200z^3+109200z^4-8400z^5+420z^6-11z^7}. \tag{29}$$

Figure 2 is the plot of the stability function (29). The shaded region in Figure 2 is the region of absolute stability of GKLM(7,10)-IIIC. We have observed that the region of

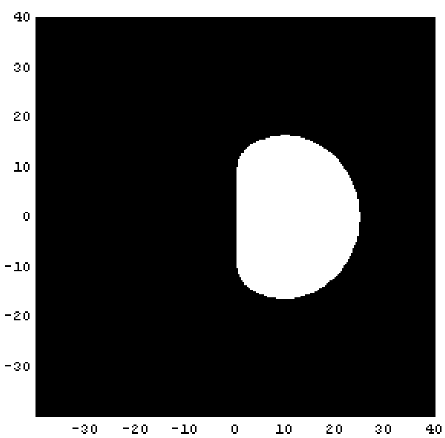

Figure 2: Stability Region of GKLM(7,10)-IIIC

absolute stability of GKLM(7,10)-IIIC contains the whole left-half complex plane, which suggest that GKLM(7,10)-IIIC is A-stable. In addition, (29) also satisfies the condition: $\left|R\left(z\right)_{\mathrm{GKLM(7,10)-IIIC}}\right| \to 0$ as $\mathrm{Re}\left(z\right) \to -\infty$. Therefore, GKLM(7,10)-IIIC is L-stable.

3.3 7-stage Tenth Order Gauss-Kronrod-Lobatto IIIA Method

As for the third implicit Runge-Kutta method, we choose the a_{ij}, $i, j = 1(1)7$ to satisfy all the 49 order conditions of

$$C\left(7\right): \quad \sum_{j=1}^{7} a_{ij} c_j^{k-1} = \frac{c_i^k}{k},\ i = 1,\ldots,7,\ k = 1,\ldots,7. \tag{30}$$

On substituting the abscissas in (19) into (30) and solve these 49 equations simultaneously using *MATHEMATICA 5.0*, yield the solution of the parameters a_{ij}, $i, j = 1(1)7$ as shown below:

$$
\begin{aligned}
&\Big\{a_{11}=0, a_{12}=0, a_{13}=0, a_{14}=0, a_{15}=0, a_{16}=0, a_{17}=0, a_{21}=\tfrac{1877+96\sqrt{6}}{60480},\\
&a_{22}=\tfrac{2592-109\sqrt{6}}{35280}, a_{23}=\frac{25\left(360-\sqrt{5\left(30769+448\sqrt{30}\right)}\right)}{84672}, a_{24}=\tfrac{108-41\sqrt{6}}{945},\\
&a_{25}=\frac{25\left(360-\sqrt{5\left(30769-448\sqrt{30}\right)}\right)}{84672}, a_{26}=\tfrac{2592-1019\sqrt{6}}{35280}, a_{27}=\tfrac{-293+96\sqrt{6}}{60480},\\
&a_{31}=\tfrac{2425-61\sqrt{5}}{105000}, a_{32}=\frac{6\left(375+\sqrt{30\left(6149+140\sqrt{30}\right)}\right)}{30625}, a_{33}=\tfrac{625-\sqrt{5}}{5880}, a_{34}=\tfrac{4}{35}-\tfrac{264}{875\sqrt{5}},\\
&a_{35}=\tfrac{625-253\sqrt{5}}{5880}, a_{36}=\frac{6\left(-375+\sqrt{30\left(6149-140\sqrt{30}\right)}\right)}{30625}, a_{37}=\tfrac{325-61\sqrt{5}}{105000}, a_{41}=\tfrac{193}{6720},\\
&a_{42}=\tfrac{3\left(96+35\sqrt{6}\right)}{3920}, a_{43}=\tfrac{25\left(40+21\sqrt{5}\right)}{9408}, a_{44}=\tfrac{4}{35}, a_{45}=\tfrac{25\left(40-21\sqrt{5}\right)}{9408},\\
&a_{46}=\tfrac{3\left(96-35\sqrt{6}\right)}{3920}, a_{47}=-\tfrac{17}{6720}, a_{51}=\tfrac{2425+61\sqrt{5}}{105000}, a_{52}=\frac{6\left(375+\sqrt{30\left(6149-140\sqrt{30}\right)}\right)}{30625},\\
&a_{53}=\tfrac{625+253\sqrt{5}}{5800}, a_{54}=\tfrac{4\left(125+66\sqrt{5}\right)}{4375}, a_{55}=\tfrac{625+\sqrt{5}}{5880},\\
&a_{56}=\frac{6\left(375-\sqrt{30\left(6149+140\sqrt{30}\right)}\right)}{30625}, a_{57}=\tfrac{325+61\sqrt{5}}{105000}, a_{61}=\tfrac{1877-96\sqrt{6}}{60480},\\
&a_{62}=\tfrac{2592+1019\sqrt{6}}{35280}, a_{63}=\frac{25\left(360+\sqrt{5\left(30769-448\sqrt{30}\right)}\right)}{84672}, a_{64}=\tfrac{108+41\sqrt{6}}{945},\\
&a_{65}=\frac{25\left(360+\sqrt{5\left(30769+448\sqrt{30}\right)}\right)}{84672}, a_{66}=\tfrac{2592+109\sqrt{6}}{35280}, a_{67}=\tfrac{-293-96\sqrt{6}}{60480}, a_{71}=\tfrac{11}{420},\\
&a_{72}=\tfrac{36}{245}, a_{73}=\tfrac{125}{588}, a_{74}=\tfrac{8}{35}, a_{75}=\tfrac{125}{588}, a_{76}=\tfrac{36}{245}, a_{77}=\tfrac{11}{420}\Big\}.
\end{aligned}
\tag{31}
$$

On substituting the values in (19) and (31) with $s = 7$ into (2) and (3), we obtained the 7-stage tenth order Gauss-Kronrod-Lobatto IIIA method, or in brief as GKLM(7,10)-IIIA. GKLM(7,10)-IIIA has proved to possess tenth order of accuracy because the values in (19) satisfy all the order conditions in $B(10)$. In addition, the values in (19) and (31) also satisfy the order conditions in

$$D(3): \quad \sum_{i=1}^{7} b_i c_i^{k-1} a_{ij} = \frac{b_j}{k}\left(1 - c_j^k\right), \ j = 1, \ldots, 7, \ k = 1, \ldots, 3.$$

Since GKLM(7,10)-IIIA satisfies $C(7)$, then we can claim that GKLM(7,10)-IIIA has stage order 7.

On substituting the values in (19) and (31) into (25), the stability function for GKLM(7,10)-IIIA is given by

$$R(z)_{\mathrm{GKLM(7,10)-IIIA}} = \frac{604800 + 302400z + 68880z^2 + 9240z^3 + 780z^4 + 40z^5 + z^6}{604800 - 302400z + 68880z^2 - 9240z^3 + 780z^4 - 40z^5 + z^6}. \tag{32}$$

Figure 3 is the plot of the stability function (32). The shaded region in Figure 3 is the region of absolute stability of GKLM(7,10)-IIIA. We have observed that the region of absolute stability of GKLM(7,10)-IIIA contains the whole left-half complex plane, which suggest that GKLM(7,10)-IIIA is A-stable. However, it is not L-stable since $\left|R(z)_{\mathrm{GKLM(7,10)-IIIA}}\right| \to 1$ as $\mathrm{Re}(z) \to -\infty$.

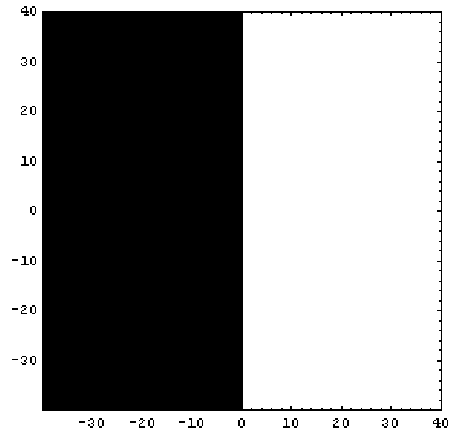

Figure 3: Stability Region of GKLM(7,10)-IIIA

3.4 7-stage Tenth Order Gauss-Kronrod-Lobatto IIIB Method

In order to complete the development of the last implicit Runge-Kutta method, we choose the a_{ij}, $i, j = 1(1)7$ to satisfy all the 49 order conditions of

$$D(7): \quad \sum_{i=1}^{7} b_i c_i^{k-1} a_{ij} = \frac{b_j}{k}\left(1 - c_j^k\right), \ j = 1, \ldots, 7, \ k = 1, \ldots, 7. \tag{33}$$

On substituting the weights and abscissas in (19) into (33) and solve these 49 equations simultaneously using *MATHEMATICA 5.0*, yield the solution of the parameters a_{ij}, $i, j = 1(1)7$ as shown below:

$$\begin{aligned}
&\Big\{a_{11} = \tfrac{11}{420}, a_{12} = \tfrac{-293-96\sqrt{6}}{10780}, a_{13} = \tfrac{325+61\sqrt{5}}{12936}, a_{14} = -\tfrac{17}{770}, a_{15} = \tfrac{325-61\sqrt{5}}{12936},\\
&a_{16} = \tfrac{-293+96\sqrt{6}}{10780}, a_{17} = 0, a_{21} = \tfrac{11}{420}, a_{22} = \tfrac{2592+109\sqrt{6}}{35280},\\
&a_{23} = \frac{375-\sqrt{30\left(6149+140\sqrt{30}\right)}}{3528}, a_{24} = \tfrac{4}{35} - \tfrac{1}{4\sqrt{6}}, a_{25} = \frac{375-\sqrt{30\left(6149-140\sqrt{30}\right)}}{3528},\\
&a_{26} = \tfrac{2592-1019\sqrt{6}}{35280}, a_{27} = 0, a_{31} = \tfrac{11}{420}, a_{32} = \frac{360+\sqrt{5\left(30769+448\sqrt{30}\right)}}{4900}, a_{33} = \tfrac{625+\sqrt{5}}{5880},\\
&a_{34} = \tfrac{4}{35} - \tfrac{3}{10\sqrt{5}}, a_{35} = \tfrac{625-253\sqrt{5}}{5880}, a_{36} = \frac{360-\sqrt{5\left(30769-448\sqrt{30}\right)}}{4900}, a_{37} = 0, a_{41} = \tfrac{11}{420},\\
&a_{42} = \tfrac{108+41\sqrt{6}}{1470}, a_{43} = \tfrac{125+66\sqrt{5}}{1176}, a_{44} = \tfrac{4}{35}, a_{45} = \tfrac{125-66\sqrt{5}}{1176}, a_{46} = \tfrac{108-41\sqrt{6}}{1470},\\
&a_{47} = 0, a_{51} = \tfrac{11}{420}, a_{52} = \frac{360+\sqrt{5\left(30769-448\sqrt{30}\right)}}{4900}, a_{53} = \tfrac{625+253\sqrt{5}}{5800}, a_{54} = \tfrac{4}{35} + \tfrac{3}{10\sqrt{5}},\\
&a_{55} = \tfrac{625-\sqrt{5}}{5880}, a_{56} = \frac{360-\sqrt{5\left(30769+448\sqrt{30}\right)}}{4900}, a_{57} = 0, a_{61} = \tfrac{11}{420}, a_{62} = \tfrac{2592+1019\sqrt{6}}{35280},\\
&a_{63} = \frac{375+\sqrt{30\left(6149-140\sqrt{30}\right)}}{3528}, a_{64} = \tfrac{4}{35} + \tfrac{1}{4\sqrt{6}}, a_{65} = \frac{375+\sqrt{30\left(6149+140\sqrt{30}\right)}}{3528},\\
&a_{66} = \tfrac{2592-109\sqrt{6}}{35280}, a_{67} = 0, a_{71} = \tfrac{11}{420}, a_{72} = \tfrac{1877-96\sqrt{6}}{10780}, a_{73} = \tfrac{2425+61\sqrt{5}}{12936}, a_{74} = \tfrac{193}{770},\\
&a_{75} = \tfrac{2425-61\sqrt{5}}{12936}, a_{76} = \tfrac{1877+96\sqrt{6}}{10780}, a_{77} = 0\Big\}.
\end{aligned} \tag{34}$$

On substituting the values in (19) and (34) with $s = 7$ into (2) and (3), we obtained the 7-stage tenth order Gauss-Kronrod-Lobatto IIIB method, or in brief as GKLM(7,10)-IIIB. GKLM(7,10)-IIIB has proved to possess tenth order of accuracy because the values in (19) satisfy all the order conditions in $B(10)$. In addition, the values in (19) and (34) also satisfy the order conditions in

$$C(3): \quad \sum_{j=1}^{7} a_{ij} c_j^{k-1} = \frac{c_i^k}{k},\ i = 1, \ldots, 7,\ k = 1, \ldots, 3.$$

Since GKLM(7,10)-IIIB satisfies $C(3)$, then we can claim that GKLM(7,10)-IIIB has stage order 3.

On substituting the values in (19) and (34) into (25), the stability function for GKLM(7,10)-IIIB is given by

$$R(z)_{\text{GKLM}(7,10)-\text{IIIB}} = \frac{604800 + 302400z + 68880z^2 + 9240z^3 + 780z^4 + 40z^5 + z^6}{604800 - 302400z + 68880z^2 - 9240z^3 + 780z^4 - 40z^5 + z^6}. \quad (35)$$

We note that both GKLM(7,10)-IIIA and GKLM(7,10)-IIIB possess the same stability function (as in (32) and (35)). Therefore, Figure 3 also represents the plot of stability function (35). It follows that the shaded region in Figure 3 is the region of absolute stability of GKLM(7,10)-IIIB and the method is found to be A-stable but not L-stable.

4 Numerical Experiments and Comparisons

In the first half of this section, some test problems are used to check the performance of GKLM(7,10)-III, GKLM(7,10)-IIIA, GKLM(7,10)-IIIB and GKLM(7,10)-IIIC using different numbers of integration steps. We presented the maximum absolute errors over the integration interval given by $\max_{0 \le n \le N} \{|y(x_n) - y_n|\}$ where N is the number of integration steps. We note that $y(x_n)$ and y_n represent the exact solution and numerical solution of a test problem at point x_n, respectively. The numerical results obtained from these Kronrod-Lobatto methods are compared with the numerical results obtained from the classical 5-stage tenth order Gauss-Legendre method.

Problem 1 [17]

$$y'(x) = -100y(x) + 99e^{2x}, y(0) = 0, x \in [0, 10].$$

The exact solution is given by $y(x) = \frac{33}{34}\left(e^{2x} - e^{-100x}\right)$.

Problem 2 [18]

$$y''(x) + 101y'(x) + 100y(x) = 0, y(0) = 1.01, y'(0) = -2, x \in [0, 10].$$

The exact solution is given by $y(x) = 0.01e^{-100x} + e^{-x}$. *Problem 2* can also be written as a system, i.e.

$$y_1'(x) = y_2(x), y_1(0) = 1.01, x \in [0, 10];$$

$$y_2'(x) = -100y_1(x) - 101y_2(x), y_2(0) = -2, x \in [0, 10].$$

The exact solutions of this system are given by $y_1(x) = y(x) = 0.01e^{-100x} + e^{-x}$ and $y_2(x) = y'(x) = -e^{-100x} - e^{-x}$.

From Table 1, we could see that GKLM(7,10)-IIIA with stage order 7 generated the smallest absolute errors for $N = 160$ when solving *Problem 1*. For $N = 320$, GKLM(7,10)-III and GKLM(7,10)-IIIA are found to have comparable accuracy, and more accurate than GKLM(7,10)-IIIB, GKLM(7,10)-IIIC and the classical 5-stage tenth order Gauss-Legendre method. For $N = 640$, all methods in comparison are found to have comparable accuracy. For $N = 160$ and $N = 320$, GKLM(7,10)-IIIB was the least accurate method because its stage order is the lowest (i.e. 3) among the methods in comparison. From here, we have observed that, if the stage order was significantly lower than the order of the Runge-Kutta method, then the values Y_i from (3) were much less accurate due to lower stage order, and affecting the accuracy of the final results computed via formula (2).

Table 1: Maximum Absolute Errors for Various Tenth Order Methods with Respect to Number of Steps (*Problem 1*)

N	5-stage tenth order Gauss-Legendre method	GKLM(7,10)-III	GKLM(7,10)-IIIA	GKLM(7,10)-IIIB	GKLM(7,10)-IIIC
160	2.54095(-04)	1.69611(-04)	3.97925(-05)	7.55789(-02)	1.27208(-03)
320	1.47579(-06)	4.92589(-07)	1.78814(-07)	1.38760(-04)	3.03984(-06)
640	1.19209(-07)	2.38419(-07)	2.38419(-07)	3.57628(-07)	2.38419(-07)

From Table 2, the effects of stage order were not apparent, but all four Kronrod-Lobatto methods were more accurate than the classical 5-stage tenth order Gauss-Legendre method. *Problem 2* could be expressed in the form of $y' = \lambda y$, $\mathrm{Re}(\lambda) < 0$, which is exactly the Dahlquist's test equation. All stability functions for Runge-Kutta methods could be derived from the application of the Dahlquist's test equation to the Runge-Kutta methods. Since the stability functions for GKLM(7,10)-IIIA and GKLM(7,10)-IIIB were identical (as in (32) and (35)), therefore the numerical results generated by these two methods were found to be identical.

Table 2: Maximum absolute errors for various tenth order methods with respect to number of steps (*Problem 2*)

N	5-stage tenth order Gauss-Legendre method	GKLM(7,10)-III	GKLM(7,10)-IIIA	GKLM(7,10)-IIIB	GKLM(7,10)-IIIC
160	2.61795(-06)	1.74751(-06)	4.09984(-07)	4.09984(-07)	2.14734(-07)
320	1.52051(-08)	5.07516(-09)	1.75659(-09)	1.75659(-09)	1.66448(-09)
640	2.99030(-11)	7.11864(-12)	3.10929(-12)	3.10929(-12)	4.03089(-12)

For the second half of this section, we have considered the numerical solutions of two real-world problems i.e. the Van der Pol oscillator and the Brusselator.

Problem 3 [9]
The Van der Pol oscillator is used to model a nonlinear diode oscillator in electrical circuit. The oscillator is given by

$$y_1'(x) = y_2(x), y_1(0) = 2, x \in [0, 2.5];$$

$$y_2'(x) = \frac{\left(1 - y_1(x)^2\right) y_2(x) - y_1(x)}{\varepsilon}, \varepsilon > 0, y_2(0) = 0, x \in [0, 2.5].$$

In this study, we have chosen $\varepsilon = 0.003$.

Problem 4 [10]
The Brusselator is used to describe the laws of chemical kinetics for certain types of multi-molecular reactions. The Brusselator considered in this study is given by

$$y_1'(x) = 1 + y_1(x)^2 y_2(x) - 4y_1(x), y_1(0) = 1.5, x \in [0, 20];$$

$$y_2'(x) = 3y_1(x) - y_1(x)^2 y_2(x), y_2(0) = 3, x \in [0, 20].$$

We note that both *Problem 3* and *Problem 4* possess no analytical solutions, and hence only approximate solutions can be obtained. Figure 4 showed the numerical solution of *Problem 3* generated by GKLM(7,10)-IIIA using 1000 fixed steps over the interval $0 \le x \le 2.5$. The other three Kronrod-Lobatto methods also generated exactly the same result as depicted in Figure 4.

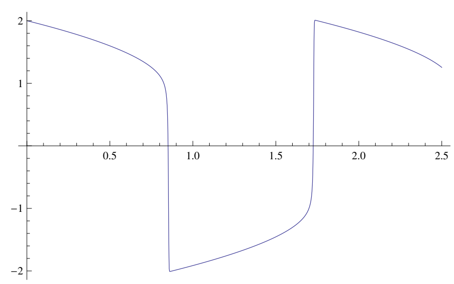

Figure 4: Numerical Solution of *Problem 3* using GKLM(7,10)-IIIA

Most importantly, Figure 4 is found to be comparable to the numerical solution graphed on page 25 of Hairer *et al.* [9]. On the other hand, Figure 5 and Figure 6 showed the numerical solutions of *Problem 4* generated by GKLM(7,10)-IIIC using 1000 fixed steps over the interval $0 \le x \le 20$.

The other three Kronrod-Lobatto methods also generated exactly the same result as depicted in Figure 5 and Figure 6. Figure 5 and Figure 6 are valid because they are found to be comparable to the numerical solutions graphed on page 170 of Hairer *et al.* [10].

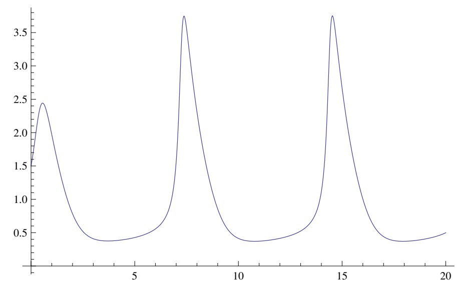

Figure 5: Numerical Solution of $y_1(x)$ of *Problem 4* using GKLM(7,10)-IIIC

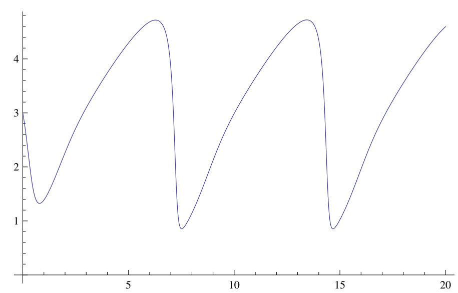

Figure 6: Numerical Solution of $y_2(x)$ of *Problem 4* using GKLM(7,10)-IIIC

5 Conclusions

In this paper, we have constructed four 7-stage tenth order implicit Runge-Kutta methods that are based on 7-point Gauss-Kronrod-Lobatto quadrature formula. The resulting implicit methods are 7-stage tenth order Gauss-Kronrod-Lobatto III method (GKLM(7,10)-III), 7-stage tenth order Gauss-Kronrod-Lobatto IIIA method (GKLM(7,10)-IIIA), 7-stage tenth order Gauss-Kronrod-Lobatto IIIB method (GKLM(7,10)-IIIB) and 7-stage tenth order Gauss-Kronrod-Lobatto IIIC method (GKLM(7,10)-IIIC).

Theoretical analyses showed that the stage order for GKLM(7,10)-III, GKLM(7,10)-IIIA, GKLM(7,10)-IIIB and GKLM(7,10)-IIIC are 6, 7, 3 and 4, respectively. In terms of absolute stability analyses, GKLM(7,10)-III is not an A-stable method and GKLM(7,10)-

IIIC is a L-stable method. On the other hand, GKLM(7,10)-IIIA and GKLM(7,10)-IIIB shared the same stability function and they are found to be A-stable only.

Numerical experiments and comparisons in Section 4 showed that implicit Runge-Kutta methods based on Gauss-Kronrod-Lobatto quadrature formula worked well for the numerical solution of first order initial value problem (1). We noticed that Kronrod-Lobatto type implicit Runge-Kutta methods with higher stage order give more accurate numerical solutions. In addition, all the proposed methods are promising in solving two examples of real-world problems i.e. the Van der Pol oscillator and the Brusselator. Future study will start to investigate the non-linear stability properties for the implicit Runge-Kutta methods proposed in this paper and those reported in Teh and Yaacob [12, 13].

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