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A Note On Some Diophantine Equations

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Abstract In this study, we solve the equations $(x+y+1)^2 = 5xy$, $(x+y-1)^2 = 5xy$, and $(x-y+1)^2 = 5xy$. We find all positive integer solutions of these equations in terms of Fibonacci and Lucas sequences. By using the solutions of these equations we give all positive integer solutions of the equations $x^2+y^2-3xy+x = 0$, $x^2+y^2-3xy-x = 0$, and $x^2+y^2-7xy-x = 0$. Moreover, it is shown that the equation $x^2+y^2-7xy+x = 0$ has no positive integer solutions.

Keywords Fibonacci numbers; Lucas numbers; Diophantine equation.

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1 Introduction

The Fibonacci sequence (F_n) is defined by $F_0 = 0$, $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. F_n is called the nth Fibonacci number. Fibonacci numbers for negative subscripts are defined as $F_{-n} = (-1)^{n+1}F_n$ for $n \geq 1$. The Lucas sequence (L_n) is defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. L_n is called the nth Lucas number. Lucas numbers for negative subscripts are defined as $L_{-n} = (-1)^n L_n$. For more information about Fibonacci and Lucas sequences one can consult [1].

2 Preliminaries

It is well known that $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$ for every $n \in \mathbb{Z}$ where $\alpha = \left(1+\sqrt{5}\right)/2$ and $\beta = \left(1-\sqrt{5}\right)/2$ are the roots of the polynomial $x^2 - x - 1$. These are known as Binet's Formula. By using Binet's Formula, we can prove the following identities.

$$L_n^2 - 5F_n^2 = 4(-1)^n \tag{1}$$

$$L_{n-1} + L_{n+1} = 5F_n,$$

$$(L_{2n}^2 + 5F_{2n+1}^2 + 1)^2 = 25L_{2n}^2 F_{2n+1}^2, \tag{2}$$

$$(L_{2n+2}^2 + 5F_{2n+1}^2 + 1)^2 = 25L_{2n+2}^2 F_{2n+1}^2, \tag{3}$$

$$(L_{2n+1}^2 + 5F_{2n} - 1)^2 = 25L_{2n+1}^2 F_{2n}^2, \tag{4}$$

$$(L_{2n-1}^2 + 5F_{2n} - 1)^2 = 5L_{2n-1}^2 F_{2n}^2, \tag{5}$$

$$(F_{2n+1}^2 + F_{2n+2}^2 + 1)^2 = F_{2n+1}^2 (5F_{2n+2}^2 + 4), \tag{6}$$

$$(F_{2n+1}^2 + F_{2n}^2 + 1)^2 = F_{2n+1}^2 (5F_{2n}^2 + 4), \tag{7}$$

$$(F_{2n}^2 + F_{2n+1}^2 - 1)^2 = F_{2n}^2 (5F_{2n+1}^2 - 4), \tag{8}$$

$$(F_{2n}^2 + F_{2n-1}^2 - 1)^2 = F_{2n}^2(5F_{2n-1}^2 - 4), \tag{9}$$

and

$$3 \mid F_n \Leftrightarrow 4 \mid n. \tag{10}$$

Now we give the following theorems from [2].

Theorem 1 *All positive integer solutions of the equation* $x^2 - 5y^2 = 4$ *are given by* $(x, y) = (L_{2n}, F_{2n})$ *with* $n \geq 1$.

Theorem 2 *All positive integer solutions of the equation* $x^2 - 5y^2 = -4$ *are given by* $(x, y) = (L_{2n+1}, F_{2n+1})$ *with* $n \geq 1$.

3 Main Theorems

Theorem 3 *All positive integer solutions of the equation* $(x + y + 1)^2 = 5xy$ *are given by* $(x, y) = (L_{2n}^2, 5F_{2n+1}^2)$ *or* $(L_{2n+2}^2, 5F_{2n+1}^2)$ *with* $n \geq 1$.

Proof Let $(x+y+1)^2 = 5xy$ for some positive integers x and y. It can be seen that x and y are relatively prime integers. The equation $(x + y + 1)^2 = 5xy$ is symmetric with respect to x and y. So, if (x, y) is a solution of the equation, then (y, x) is also a solution of the equation. Therefore, without loss of generality we may suppose that $x = a^2$ and $y = 5b^2$ for some positive integers a and b. This implies that $(a^2 + 5b^2 + 1)^2 = 25a^2b^2$. Thus we get $a^2 + 5b^2 - 5ab = -1$. Multiplying both side of the equation by 4 and completing the square, we get $(2a - 5b)^2 - 5b^2 = -4$. Then by Theorem 2, we obtain $|2a - 5b| = L_{2n+1}$ and $b = F_{2n+1}$ for some $n \geq 1$. If $2a - 5b = L_{2n+1}$, then we have

$$a = \frac{(L_{2n+1} + 5b)}{2} = \frac{(L_{2n+1} + 5F_{2n+1})}{2} = \frac{(L_{2n} + L_{2n+2} + L_{2n+1})}{2} = L_{2n+2}.$$

Similarly, if $2a-5b = -L_{2n+1}$, then we get $a = L_{2n}$. Thus it follows that $(x, y) = (a^2, 5b^2) = (L_{2n}^2, 5F_{2n+1}^2)$ or $(L_{2n+2}^2, 5F_{2n+1}^2)$.

Conversely, if $(x, y) = (L_{2n}^2, 5F_{2n+1}^2)$ or $(L_{2n+2}^2, 5F_{2n+1}^2)$, then we obtain $(x + y + 1)^2 = 5xy$ by (2) and (3). □

Corollary 1 *All positive integer solutions of the equations* $x^2 + y^2 - 3xy + x = 0$ *are given by* $(x, y) = (F_{2n+1}^2, F_{2n+2}^2 + 1)$ *or* $(F_{2n+1}^2, F_{2n}^2 + 1)$ *with* $n \geq 1$.

Proof Assume that $x^2 + y^2 - 3xy + x = 0$ for some positive integers x and y. Then we have $(x + y)^2 = x(5y - 1)$, which implies that $(5x + 5y - 1 + 1)^2 = 5(5x(5y - 1))$. Since $5 \nmid L_n$ by (1), it follows that $(5y - 1, 5x) = (L_{2n}^2, 5F_{2n+1}^2)$ or $(L_{2n+2}^2, 5F_{2n+1}^2)$ for some $n \geq 1$ by Theorem 3. Let $(5y - 1, 5x) = (L_{2n}^2, 5F_{2n+1}^2)$. Then we get $x = F_{2n+1}^2$ and $y = (\ L_{2n}^2 + 1)/5 = (5F_{2n}^2 + 4 + 1) = F_{2n}^2 + 1$ by (1). Thus $(x, y) = (F_{2n+1}^2, F_{2n}^2 + 1)$. If $(5y-1, 5x) = (L_{2n+2}^2, 5F_{2n+1}^2)$, then in a similar way, we obtain $(x, y) = (F_{2n+1}^2, F_{2n+2}^2 + 1)$.

Conversely, if $(x, y) = (F_{2n+1}^2, F_{2n+2}^2 + 1)$ or $(F_{2n+1}^2, F_{2n}^2 + 1)$, then we have $(x + y)^2 = x(5y - 1)$ by (6) and (7). □

Theorem 4 *All positive integers solutions of the equation* $(x + y - 1)^2 = 5xy$ *are given by* $(x, y) = (L_{2n+1}^2, 5F_{2n}^2)$ *or* $(L_{2n-1}^2, 5F_{2n}^2)$ *with* $n \geq 1$.

Proof Let $(x+y-1)^2 = 5xy$ for some positive integers x and y. Similarly, it can be seen that x and y are relatively prime integers . The equation $(x+y-1)^2 = 5xy$ is symmetric with respect to x and y. So, if (x, y) is a solution of the equation, then (y, x) is also a solution of the equation. Therefore, without loss of generality we may suppose that $x = a^2$ and $y = 5b^2$ for some positive integers a and b. This implies that $(a^2+5b^2-1)^2 = 25a^2b^2$. Thus we get $a^2+5b^2-5ab = 1$. Multiplying both side of the equation by 4 and completing the square, we get $(2a-5b)^2-5b^2 = 4$. Then by Theorem 1, we obtain $|2a-5b| = L_{2n}$ and $b = F_{2n}$ for some $n \geq 1$. If $2a-5b = L_{2n}$, then we have $a = (L_{2n}+5b)/2 = (L_{2n}+5F_{2n})/2 = (L_{2n}+L_{2n-1}+L_{2n+1})/2 = L_{2n+1}$. Similarly, if $2a-5b = -L_{2n}$, then we get $a = L_{2n-1}$. Thus it follows that $(x, y) = (a^2, 5b^2) = (L^2_{2n-1}, 5F^2_{2n})$ or $(L^2_{2n+1}, 5F^2_{2n})$. Conversely if $(x, y) = (L^2_{2n-1}, 5F^2_{2n})$ or $(L^2_{2n+1}, 5F^2_{2n})$, then we obtain $(x+y-1)^2 = 5xy$ by (4) and (5). □

By using the above theorem and the identities (8) and (9), we can give the following corollary easily.

Corollary 2 *All positive integer solutions of the equation* $x^2+y^2-3xy-x = 0$ *are given by* $(x, y) = (F^2_{2n}, F^2_{2n+1}-1)$ *or* $(F^2_{2n}, F^2_{2n-1}-1)$ *with* $n \geq 1$.

Since the proofs of the following theorem and corollary can be done similarly, we omit their proofs. The identity (10) will be used in the proof of the theorem.

Theorem 5 *All positive integer solutions of the equation* $(x-y+1)^2 = 5xy$ *are given by* $(x, y) = (5F^2_{4n}/9, L^2_{4n+2}/9)$ *or* $(5F^2_{4n}/9, L^2_{4n-2}/9)$ *with* $n \geq 1$.

Corollary 3 *All positive integer solutions of the equation* $x^2+y^2-7xy-x = 0$ *are given by* $(x, y) = (F^2_{4n}/9, (F^2_{4n+2}-1)/9)$ *or* $(F^2_{4n}/9, (F^2_{4n-2}-1)/9)$ *with* $n \geq 1$

Corollary 4 *The equation* $x^2+y^2-7xy+x = 0$ *has no positive integer solutions.*

Proof Assume that $x^2+y^2-7xy+x = 0$ for some positive integer x and y. Then $(y-x)^2 = x(5y-1)$, which implies that $(5y-1-5x+1)^2 = 5(5x(5y-1))$. Then by Theorem 5, we get $5y-1 = 5F^2_{4n}/9$ for some $n \geq 1$, which is impossible. □

4 Conclusion

In this study, we give all positive integer solutions of the equations $(x+y+1)^2 = 5xy$, $(x+y-1)^2 = 5xy$, and $(x-y+1)^2 = 5xy$ in terms of the Fibonacci and Lucas sequences. Moreover, we solve some other diophantine equations. We think that the Diophantine equations $(x+y+1)^2 = 10xy$, $(x+y-1)^2 = 10xy$, and $(x-y+1)^2 = 10xy$ can be solved in a similar argument.

References

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