Weak-FPI-rings

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Abstract We define a particular case of a FPI-ring called a weak FPI-ring. We investigate the transfer of the weak FPI-ring to trivial ring extensions, pullbacks, subring retracts, amalgamated duplication of a ring along an ideal, and direct product of rings.

Keywords FPI-rings; weak FPI-rings; coherent rings; trivial ring extension; pullbacks; amalgamated duplication of a ring along an ideal; direct product of rings.

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1 Introduction

We define a particular case of an FPI-ring called a weak FPI-ring. A ring $R$ is called weak FPI-rings if, for every two ideals $I \subseteq J$ of $R$ such that $I$ is finitely generated flat, $J$ is projective proper ideal, then $J$ is projective (Definition 1).

In Proposition 1 the existence of a relationship between FPI-ring and weak FPI-ring is demonstrated. Meanwhile, in Proposition 2 we prove that every coherent ring is a weak FPI-ring. Naturally, every FPI-ring is a weak FPI-ring. In Theorem 1(i), the existence of a converse relationship FPI-ring and weak FPI-ring is supported by a sufficient condition, and in Theorem 1(ii), we show that if $R$ is a local, then $R$ is weak FPI-rings, and in Theorem 2 we study and validate the transfer of the weak FPI-ring to trivial ring extension.

In addition, a condition that let the descent of the weak FPI-ring holds in the extension of the ring in Proposition 3. Namely, if $R \subseteq T$ with $T$ is a faithfully flat $R$-module, and $T$ is a weak FPI-ring then that $R$ is a weak FPI-ring. In Theorem 3, we study the notion of weak FPI-rings in direct products of rings. In Theorem 4 we study the transfer of the weak FPI-ring to pullbacks.

2 Main Results

Recall that a ring $R$ is called a FPI-ring if every finitely generated flat ideal is projective. This paper investigates a generalization of FPI-ring as follows:

Definition 1 A ring $R$ is called weak FPI-rings if, for every two ideals $I \subseteq J$ of $R$ such that $I$ is finitely generated flat, $J$ projective proper ideal implies that $I$ is projective.
The following proposition shows that the relationship between weak FPI-ring and weak-hereditary.

**Proposition 1** Let $R$ be a weak FPI-ring of $\text{wdim} \leq 1$. Then $R$ is weak-hereditary.

**Proof** Let $R$ be a weak FPI-ring with $\text{wdim}(R) \leq 1$ and let $I \subseteq J$ be two ideals of $R$ such that $I$ is finitely generated, $J$ projective proper ideal. Then $I$ is flat since $\text{wdim}(R) \leq 1$. So $I$ is finitely projective since $R$ is a weak FPI-ring. Therefore, $R$ is weak weak-hereditary.

The following proposition shows that the every coherent ring is weak FPI-ring.

**Proposition 2** Any coherent ring is a weak FPI-ring.

**Proof** Assume that $R$ is a coherent ring. We must show that it is a weak FPI-ring. Let $I \subseteq J$ be two ideals of $R$ such that $I$ is finitely generated flat, $J$ projective proper ideal. Then $I$ is a finitely presented since $R$ is coherent. Hence, $I$ is projective.

**Theorem 1** Let $R$ be a ring. Then:

(i) If $R$ contains a regular element, then $R$ is a weak FPI-ring if and only if $R$ is a FPI-ring.

(ii) If $R$ is a local, then $R$ is a weak FPI-ring.

**Proof**

(i) If $R$ is a FPI-ring, then $R$ weak FPI-ring. Conversely, suppose that $J$ is finitely generated flat proper ideal of $R$. Let $x \in R$ regular element of $R$, so $xJ \subseteq xR$. On the other hand, $xR$ proper ideal and $xR \cong R$, then $xR$ is free implies projective. So $xJ$ is projective ideal, since $R$ is a weak FPI-ring. But $xJ \cong J$, then $J$ is projective.

(ii) Let $R$ be a local. We claim that $R$ is a weak FPI-ring. Assume that $J_1 \subseteq J_2 \subseteq M$, where $M$ is a maximal ideal of $R$, $J_2$ is a proper projective ideal and $J_1$ is finitely generated flat ideal of $R$. Then $J_1$ is free (since $R$ is local), hence $J_2$ is a proper projective ideal of $R$. Then $R$ is a weak FPI-ring.

The following example shows that the weak FPI-ring is not necessarily in general a coherent ring.

**Example 1** Let $T$ be a field and $A := T \times T^\infty$ be the trivial ring extension of $T$ by $T^\infty$. Then:

(i) By Theorem 1(ii), $A$ is a weak FPI-ring (since $A$ is local).

(ii) By [3, Theorem 2.1], $A$ is not a coherent ring.

**Example 2** Let $(A, M)$ be any local ring with $M^2 = 0$. Since $A = Q(A)$ is local, then by Theorem 1(ii), $A$ is weak FPI-ring.

**Theorem 2** Let $A$ be a domain which is not a field, $K = qf(A)$, $E$ be a $K$-vector space and $R := A \times E$. Then, $R$ is a weak FPI-ring.
Lemma 1 Let $T := K \alpha E$ be the trivial ring extension of a field $K$ by a $K$-vector space $E$. Then there exists no proper flat ideal of $T$.

Proof Let $J := 0 \alpha \hat{E}$ be a proper ideal of $T$ where $\hat{E}(\subseteq E)$ is a $K$-vector space. We claim that $J$ is not flat. Deny. Let $\{f_i\}_{i \in I}$ be a basis of the $K$-vector space $\hat{E}$ and consider the $T$-map $T^{(l)} \rightarrow J$ defined by $u((a_i, e_i)_{i \in I}) = (0, \sum_{i \in I} a_i f_i)$. Clearly, $\text{Ker}(u) = 0 \alpha E^{(l)} = (0 \alpha E)^{(l)}$. Hence, by [4, Lemma 2.5], we obtain $(0 \alpha E)^{(l)} = (0 \alpha E^{(l)}) \cap (0 \alpha E)T^{(l)} = (0 \alpha E)^{(l)}(0 \alpha E) = 0$, a contradiction. Hence, $J$ is not flat. \hfill $\Box$

Proof of Theorem 2.

Let $J_1 \subseteq J_2$ be two ideals of $R$ with $J_2$ is proper projective and $J_1$ is a finitely generated flat ideal. Set $T := K \alpha E$ which is a flat $R$-module since $T = S^{-1}R$, where $S = A - \{0\}$. Thus, $J_2 \otimes_R T = J_2 T$ is proper projective, and $J_1 \otimes R T = J_1 T$ is a finitely generated flat ideal since $T$ is a flat $R$-module. Hence $J_1 T = K \alpha E$ by Lemma 1. On the other hand, we have $J_1 T \subseteq J_2 T$ since $J_1 \subseteq J_2$, then $J_1 T$ is projective since $T$ is weak FPI-rings. Therefore, there exists $(a, e) \in J$ such that $a \neq 0$ which implies that $J_1 = I_1 \alpha E = I_1 \otimes_A R$ for some nonzero ideal $I_1$ of $A$. We claim that $I_1$ is a projective ideal of $A$. For any $A$-module $N$, we have by [5, p.118]

$$\text{Ext}_A(I_1, N \otimes A R) \cong \text{Ext}_R(I_1 \otimes A R, N \otimes A R) = 0.$$ 

On the other hand, $N$ is a direct summand of $N \otimes_A R$ since $A$ is a direct summand of $R$. Therefore, $\text{Ext}_D(I_1, N) = 0$ for all $A$-module $N$. This means that $I_1$ is a projective ideal of $A$. \hfill $\Box$

Corollary 1 Let $A$ be a domain which is not a field, $K = qf(A)$, and $R := A \alpha K$ be the trivial ring extension of $A$ by $K$. Then, $R$ is a FPI-ring.

Proof By Theorem 2 $R$ is a weak FPI-ring. Let $0 \neq a \in A$, then $(a, e)$ is a regular element of $R$. Thus by Theorem 1, $R$ is a FPI-ring. \hfill $\Box$

For two rings $A_1 \subseteq A_2$, we say that $A_1$ is a module retract of $A_2$ if there exists an $A_1$-module homomorphism $\phi : A_2 \rightarrow A_1$ such that $\phi|_{A_1} = id|_{A_1}$; $\phi$ is called a module retraction map. If such a map $\phi$ exists, $A_2$ contains $A_1$ as an $A_1$-module direct summand. See for instance ([2,6,7]).

Proposition 3 Let $A \rightarrow S$ be a faithfully flat ring homomorphism, such that each ideal $I$ of $A$, $IS \neq S$. If $S$ is a weak FPI-ring, then $A$ is a weak FPI-ring.

Proof Assume that $S$ is a weak FPI-ring. Let $I_1 \subseteq I_2$ be a two ideals of $A$ with $I_2$ is proper projective, and $I_1$ is a finitely generated flat ideal. Since $S$ is faithfully flat over $A$, $I_2 \otimes_A S = I_2 S$ is a proper projective ideal of $S$ and $I_1 \otimes_A S = I_1 S$ is a finitely generated flat ideal of $S$. On the other hand, we have $I_1 S \subseteq I_2 S$ and $S$ is a weak FPI-ring then, $I_1 S$
is projective. We claim that \( I_1 \) is a projective ideal of \( A \). Indeed, for any \( A \)-module \( L \), we have by [5, p.118],

\[
\text{Ext}_A(I_1, L \otimes_A S) \cong \text{Ext}_S(I_1 \otimes_A S, L \otimes_A S) = 0.
\]

On the other hand, \( L \) is a direct summand of \( L \otimes_A S \) since \( A \) is a direct summand of \( S \). Therefore, \( \text{Ext}_A(I_1, L) = 0 \) for every \( A \)-module \( L \). This means that \( I_1 \) is a projective ideal of \( A \), as desired.

Now we study the notion of weak FPI-rings in direct products of rings.

**Theorem 3** Let \((R_i)_{i=1,2,...,n}\) be a family of rings and let \( R := \prod_{i=1}^n R_i \). If \( R \) is a weak FPI-ring, then so is \( R_i \) for each \( i = 1, \ldots, n \).

We need the following lemma before proving Theorem 3.

**Lemma 2** [8, Lemma 2.5] Let \((R_i)_{i=1,2}\) be a family of rings and let \( E_i \) be an \( R_i \)-module for \( i = 1, 2 \). Then:

(i) \( N_1 \prod N_2 \) is a finitely generated \( A_1 \prod A_2 \)-module if and only if \( N_i \) is a finitely generated \( A_i \)-module for \( i = 1, 2 \).

(ii) \( N_1 \prod N_2 \) is a flat \( A_1 \prod A_2 \)-module if and only if \( N_i \) is a flat \( A_i \)-module for \( i = 1, 2 \).

(iii) \( N_1 \prod N_2 \) is a projective \( A_1 \prod A_2 \)-module if and only if \( N_i \) is a projective \( A_i \)-module for \( i = 1, 2 \).

**Proof of Theorem 3** We prove the result for \( i = 1, 2 \), and the Theorem will be established by induction on \( n \).

Assume that \((A_1 \times A_2)\) is a weak FPI-ring. We wish to show that \( A_1 \) and \( A_2 \) are weak FPI-rings. Let \( I_1 \) and \( J_1 \) be two ideals of \( A_1 \) and let \( I_1 \subseteq J_1 \) with \( J_1 \) is a projective proper ideal and \( I_1 \) is a finitely generated flat ideal. Then \( I_1 \times A_2 \) is a finitely generated flat ideal of \((A_1 \times A_2)\) and \( J_1 \times A_2 \) is a projective proper ideal of \((A_1 \times R_2)\). Since \((A_1 \times A_2)\) is a weak FPI-ring and \( I_1 \times A_2 \subseteq J_1 \times A_2 \), then \( I_1 \times A_2 \) is a projective ideal. Then \( I_1 \) is a projective ideal.

**Theorem 4** Let \( A \subseteq B(=S^{-1}A) \) be an extension of rings, where \( S \) is a multiplicative subset of \( A \), and \( Q \) is an ideal of both \( A \) and \( B \). Assume that \( B \) is a local weak FPI-ring. Then \( A \) is a weak FPI-ring provided \( A/Q \) is a weak FPI-ring.

We need the following lemma before proving Theorem 4.

**Lemma 3** [9, Lemma 2.7] Let \( A, B, S \) and \( Q \) be as in Theorem 4. Assume that \( B \) is a local ring and let \( I \) be any finitely generated flat ideal of \( A \). Then there exists \( 0 \neq x \in B \) and an ideal \( I' \supseteq Q \) of \( A \) such that \( I \otimes A/Q \cong I'/Q \) as \( A/Q \)-modules and \( I = xI' \cong I' \) as \( A \)-modules.
Proof of Theorem 4 Let $A \subseteq B := S^{-1}A$ be an extension of rings, where $S$ is a multiplicative subset of $A$, let also $Q$ is an ideal of both $A$ and $B$ and $B$ is a local weak FPI-ring. Assume that $A/Q$ is an FPI-ring and let $I \subseteq J$ two ideals of $A$ such that $I$ is finitely generated flat and $J$ is a projective proper. Then $I \otimes_A B := IB$ is a finitely generated flat and $I \otimes_A (A/Q) \cong I'/Q$ is a finitely generated flat ideal of $A/Q$. On the other hand, $J \otimes_A B := JB$ is a projective proper and $J \otimes_A (A/Q) \cong J'/Q$ is a projective proper ideal of $A/Q$. Now we have $I \otimes_A B := IB \subseteq J \otimes_A B := JB$. Then $I \otimes_A B := IB$ is projective since $B$ is a weak FPI-ring. Since $A/Q$ is a weak FPI-ring and $I \otimes_A (A/Q) \cong I'/Q \subseteq J \otimes_A (A/Q) \cong J'/Q$, then $I \otimes_A (A/Q) \cong I'/Q$ is projective. Therefore, $I$ is a projective ideal.

Theorem 4 enriches the literature with new examples of weak FPI-rings.

Example 3 Let $D$ be a non-local integral domain, $K := qf(D)$, $T := K[X]/(X^n) = K + M$, where $X$ is an indeterminate over $K$, $n$ is a positive integer, $M = XT$ is a maximal ideal of a local ring $T$ and $R = D + M$. Then:

(i) $R$ is a weak FPI-ring.
(ii) $R$ is not local since $D$ is not local.
(iii) $R$ is not Noetherian since $D$ is not Noetherian and $R$ is a faithfully flat $D$-module.

References