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Weak-FPI-rings

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Abstract We define a particular case of a FPI-ring called a weak FPI-ring. We investigate the transfer of the weak FPI-ring to trivial ring extensions, pullbacks, subring retracts, amalgamated duplication of a ring along an ideal, and direct product of rings.

Keywords FPI-rings; weak FPI-rings; coherent rings; trivial ring extension; pullbacks; amalgamated duplication of a ring along an ideal; direct product of rings.

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1 Introduction

We define a particular case of an FPI-ring called a weak FPI-ring. A ring R is called weak FPI-rings if, for every two ideals $J_1 \subseteq J_2$ of R such that J_1 is finitely generated flat, J_2 is projective proper ideal, then J_1 is projective (Definition 1).

In Proposition 1 the existence of a relationship between FPI-ring and weak FPI-ring is demonstrated. Meanwhile, in Proposition 2 we prove that every coherent ring is a weak FPI-ring. Naturally, every FPI-ring is a weak FPI-ring. In Theorem 1(i), the existence of a converse relationship FPI-ring and weak FPI-ring is supported by a sufficient condition, and in Theorem 1(ii), we show that if R is a local, then R is weak FPI-rings, and in Theorem 2 we study and validate the transfer of the weak FPI-ring to trivial ring extension. Remember that for a ring R_1 and an R_1-module M, a ring $R := R_1 \propto M$ of pairs (x_1, m_1) whose underlying group is $R_1 \times M$ with pairwise addition and multiplication given by $(x_1, m_1)(x_2, m_2) = (x_1x_2, x_1m_2 + x_2m_1)$ is said to be trivial ring extension of R_1 by M. See for instance [1, 2].

In addition, a condition that let the descent of the weak FPI-ring holds in the extension of the ring in Proposition 3. Namely, if $R \subseteq T$ with T is a faithfully flat R-module, and T is a weak FPI-ring then that R is a weak FPI-ring. In Theorem 3, we study the notion of weak FPI-rings in direct products of rings. In Theorem 4 we study the transfer of the weak FPI-ring to pullbacks.

2 Main Results

Recall that a ring R is called a FPI-ring if every finitely generated flat ideal is projective. This paper investigates a generalization of FPI-ring as follows:

Definition 1 A ring R is called weak FPI-rings if, for every two ideals $I \subseteq J$ of R such that I is finitely generated flat, J projective proper ideal implies that I is projective.

The following proposition shows that the relationship between weak FPI-ring and weak-hereditary.

Proposition 1 *Let R be a weak FPI-ring of $wdim \leq 1$. Then R is weak-hereditary.*

Proof Let R be a weak FPI-ring with $wdim(R) \leq 1$ and let $I \subseteq J$ be two ideals of R such that I is finitely generated, J projective proper ideal. Then I is flat since $wdim(R) \leq 1$. So I is finitely projective since R is a weak FPI-ring. Therefore, R is weak weak-hereditary. □

The following proposition shows that the every coherent ring is weak FPI-ring.

Proposition 2 *Any coherent ring is a weak FPI-ring.*

Proof Assume that R is a coherent ring. We must show that it is a weak FPI-ring. Let $I \subseteq J$ be two ideals of R such that I is finitely generated flat, J projective proper ideal. Then I is a finitely presented since R is coherent. Hence, I is projective. □

Theorem 1 *Let R be a ring. Then:*

(i) *If R contains a regular element, then R is a weak FPI-ring if and only if R is a FPI-ring.*

(ii) *If R is a local, then R is a weak FPI-ring.*

Proof

(i) If R is a FPI-ring, then R weak FPI-ring. Conversely, suppose that J is finitely generated flat proper ideal of R. Let $x \in R$ regular element of R, so $xJ \subseteq xR$. On the other hand, xR proper ideal and $xR \cong R$, then xR is free implies projective. So xJ is projective ideal, since R is a weak FPI-ring. But $xJ \cong J$, then J is projective.

(ii) Let R be a local. We claim that R is a weak FPI-ring. Assume that $J_1 \subseteq J_2 \subseteq M$, where M is a maximal ideal of R, J_2 is a proper projective ideal and J_1 is a finitely generated flat ideal of R. Then J_1 is free (since R is local), hence J_2 is a proper projective ideal of R. Then R is a weak FPI-ring. □

The following example shows that the weak FPI-ring is not necessarily in general a coherent ring.

Example 1 Let T be a field and $A := T \propto T^{\infty}$ be the trivial ring extension of T by T^{∞}. Then:

(i) By Theorem 1(ii), A is a weak FPI-ring (since A is local).

(ii) By [3, Theorem 2.1], A is not a coherent ring.

Example 2 Let (A, M) be any local ring with $M^2 = 0$. Since $A = Q(A)$ is local, then by Theorem 1(ii), A is weak FPI-ring.

Theorem 2 *Let A be a domain which is not a field, $K = qf(A)$, E be a K-vector space and $R := A \propto E$. Then, R is a weak FPI-ring.*

We need the following lemma before proving Theorem 2.

Lemma 1 *Let $T := K \propto E$ be the trivial ring extension of a field K by a K-vector space E. Then there exists no proper flat ideal of T.*

Proof Let $J := 0 \propto \acute{E}$ be a proper ideal of T where $\acute{E}(\subseteq E)$ is a K-vector space. We claim that J is not flat. Deny. Let $\{fi\}_{i\in I}$ be a basis of the K-vector space $\acute{E}$ and consider the T-map $T^{(I)} \longrightarrow J$ defined by $u((a_i, e_i)_{i\in I}) = (0, \sum_{i\in I} a_i f_i)$. Clearly, $Ker(u) = 0 \propto E^{(I)} = (0 \propto E)^{(I)}$. Hence, by [4, Lemma 2.5], we obtain $(0 \propto E)^{(I)} = (0 \propto E^{(I)}) \bigcap (0 \propto E)T^{(I)} = (0 \propto E)^{(I)})(0 \propto E) = 0$, a contradiction. Hence, J is not flat. □

Proof of Theorem 2.

Let $J_1 \subseteq J_2$ be two ideals of R with J_2 is proper projective and J_1 is a finitely generated flat ideal. Set $T := K \propto E$ which is a flat R-module since $T = S^{-1}R$, where $S = A - \{0\}$. Thus, $J_2 \bigotimes_R T = J_2T$ is proper projective, and $J_1 \bigotimes_R T = J_1T$ is a finitely generated flat ideal since T is a flat R-module. Hence $J_1T = K \propto E$ by Lemma 1. On the other hand, we have $J_1T \subseteq J_2T$ since $J_1 \subseteq J_2$, then J_1T is projective since T is weak FPI-rings. Therefore, there exists $(a, e) \in J$ such that $a \neq 0$ which implies that $J_1 = I_1 \propto E = I_1 \bigotimes_A R$ for some nonzero ideal I_1 of A. We claim that I_1 is a projective ideal of A. For any A-module N, we have by [5, p.118]

$$Ext_A(I_1, N \bigotimes_A R) \cong Ext_R(I_1 \bigotimes_A R, N \bigotimes_A R) = 0.$$

On the other hand, N is a direct summand of $N \bigotimes_A R$ since A is a direct summand of R. Therefore, $ExtD(I_1, N) = 0$ for all A-module N. This means that I_1 is a projective ideal of A. □

Corollary 1 *Let A be a domain which is not a field, $K = qf(A)$, and $R := A \propto K$ be the trivial ring extension of A by K. Then, R is a FPI-ring.*

Proof By Theorem 2 R is a weak FPI-ring. Let $0 \neq a \in A$, then (a, e) is a regular element of R. Thus by Theorem 1, R is a FPI-ring. □

For two rings $A_1 \subseteq A_2$, we say that A_1 is a module retract of A_2 if there exists an A_1-module homomorphism $\phi : A_2 \longrightarrow A_1$ such that $\phi|_{A_1} = id|_{A_1}$; ϕ is called a module retraction map. If such a map ϕ exists, A_2 contains A_1 as an A_1-module direct summand. See for instance ([2, 6, 7]).

Proposition 3 *Let $A \longrightarrow S$ be a faithfully flat ring homomorphism, such that each ideal I of A, $IS \neq S$. If S is a weak FPI-ring, then A is a weak FPI-ring.*

Proof Assume that S is a weak FPI-ring. Let $I_1 \subseteq I_2$ be a two ideals of A with I_2 is proper projective, and I_1 is a finitely generated flat ideal. Since S is faithfully flat over A, $I_2 \bigotimes_A S = I_2S$ is a proper projective ideal of S and $I_1 \bigotimes_A S = I_1S$ is a finitely generated flat ideal of S. On the other hand, we have $I_1S \subseteq I_2S$ and S is a weak FPI-ring then, I_1S

is projective. We claim that I_1 is a projective ideal of A. Indeed, for any A-module L, we have by [5, p.118],

$$Ext_A(I_1, L \bigotimes_A S) \cong Ext_S(I_1 \bigotimes_A S, L \bigotimes_A S) = 0.$$

On the other hand, L is a direct summand of $L \bigotimes_A S$ since A is a direct summand of S. Therefore, $Ext_A(I_1, L) = 0$ for every A-module L. This means that I_1 is a projective ideal of A, as desired. □

Now we study the notion of weak FPI-rings in direct products of rings.

Theorem 3 *Let $(R_i)_{i=1,2,\ldots,n}$ be a family of rings and let $R := \prod_{i=1}^n R_i$. If R is a weak FPI-ring, then so is R_i for each $i = 1, \ldots, n$.*

We need the following lemma before proving Theorem 3.

Lemma 2 [8, Lemma 2.5] *Let $(R_i)_{i=1,2}$ be a family of rings and let E_i an R_i-module for $i = 1, 2$. Then:*

(i) *$N_1 \prod N_2$ is a finitely generated $A_1 \prod A_2-$module if and only if N_i is a finitely generated A_i-module for i = 1, 2.*

(ii) *$N_1 \prod N_2$ is a flat $A_1 \prod A_2-$module if and only if N_i is a flat A_i-module for i = 1, 2.*

(iii) *$N_1 \prod N_2$ is a projective $A_1 \prod A_2-$module if and only if N_i is a projective A_i-module for i = 1, 2.*

Proof of Theorem 3 We prove the result for $i = 1, 2$, and the Theorem will be established by induction on n.
Assume that $(A_1 \times A_2)$ is a weak FPI-rings. We wish to show that A_1 and A_2 are weak FPI-rings. Let I_1 and J_1 be two ideals of A_1 and let $I_1 \subseteq J_1$ with J_1 is a projective proper ideal and I_1 is a finitely generated flat ideal. Then $I_1 \times A_2$ is a finitely generated flat ideal of $(A_1 \times A_2)$ and $J_1 \times A_2$ is a projective proper ideal of $(A_1 \times R_2)$. Since $(A_1 \times A_2)$ is a weak FPI-ring and $I_1 \times A_2 \subseteq J_1 \times A_2$, then $I_1 \times A_2$ is a projective ideal. Then I_1 is a projective ideal. □

Theorem 4 *Let $A \subseteq B(:= S^{-1}A)$ be an extension of rings, where S is a multiplicative subset of A, and Q is an ideal of both A and B. Assume that B is a local weak FPI-ring. Then A is a weak FPI-ring provided A/Q is a weak FPI-ring.*

We need the following lemma before proving Theorem 4.

Lemma 3 [9, Lemma 2.7] *Let A,B, S and Q be as in Theorem 4. Assume that B is a local ring and let I be any finitely generated flat ideal of A. Then there exists $0 \neq x \in B$ and an ideal $I^{'} \supseteq Q$ of A such that $I \bigotimes A/Q \cong I^{'}/Q$ as A/Q-modules and $I = xI^{'} \cong I^{'}$ as A-modules.*

Proof of Theorem 4 Let $A \subseteq B(:= S^{-1}A)$ be an extension of rings, where S is a multiplicative subset of A, let also Q is an ideal of both A and B and B is a local weak FPI-ring. Assume that A/Q is an FPI-ring and let $I \subseteq J$ two ideals of A such that I is finitely generated flat and J is a projective proper. Then $I \bigotimes_A B := IB$ is a finitely generated flat and $I \bigotimes_A (A/Q) \cong I'/Q$ is a finitely generated flat ideal of A/Q. On the other hand, $J \bigotimes_A B := JB$ is a projective proper and $J \bigotimes_A (A/Q) \cong J'/Q$ is a projective proper ideal of A/Q. Now we have $I \bigotimes_A B := IB \subseteq J \bigotimes_A B := JB$. Then $I \bigotimes_A B := IB$ is projective since B is a weak FPI-ring. Since A/Q is a weak FPI-ring and $I \bigotimes_A (A/Q) \cong I'/Q \subseteq J \bigotimes_A (A/Q) \cong J'/Q$, then $I \bigotimes_A (A/Q) \cong I'/Q$ is projective . Therefore, I is a projective ideal. □

Theorem 4 enriches the literature with new examples of weak FPI-rings.

Example 3 Let D be a non-local integral domain, $K := qf(D)$, $T := K[X]/(X^n) = K + M$, where X is an indeterminate over K, n is a positive integer, $M = XT$ is a maximal ideal of a local ring T and $R = D + M$. Then:

(i) R is a weak FPI-ring.

(ii) R is not local since D is not local.

(iii) R is not Noetherian since D is not Noetherian and R is a faithfully flat D-module.

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