

## Radio Number for Special Family of Graphs with Diameter 2, 3 and 4

**Murugan Muthali**

School of Science, Tamil Nadu Open University  
577, Anna Salai, Chennai – 600 015, India  
e-mail: muruganganesan@yahoo.in

**Abstract** A radio labeling of a graph  $G$  is an injective function  $f: V(G) \rightarrow N \cup \{0\}$  such that for every  $u, v \in V(G)$ ,  $|f(u) - f(v)| \geq \text{diam}(G) - d(u, v) + 1$ . The span of  $f$  is the difference of the largest and the smallest channels used, that is,

$$\max_{u, v \in V(G)} \{f(u) - f(v)\}.$$

The radio number of  $G$  is defined as the minimum span of a radio labeling of  $G$  and denoted as  $rn(G)$ . In this paper, we present algorithms to get the radio labeling of special family of graphs like double cones, books and  $nC_4$  with a common vertex whose diameters are 2, 3 and 4 respectively.

**Keywords** Radio labeling, radio number; channel assignment; distance two labeling.

**2010 Mathematics Subject Classification** 05C78, 05C12, 05C15.

### 1 Introduction

The channel assignment problem is a telecommunication problem in which our aim is to assign a channel (non-negative integer) to each TV or Radio station so that we do not have any interference in the communication. The level of interference between the TV or Radio stations correlates with the geographic locations of these stations. Earlier designers of TV networks considered close locations and very close locations so that the transmitters at the close locations receive different channels and the transmitters at very close locations are at least two apart for clear communication.

This can be modeled by graph models in which vertices represent the stations and two vertices are joined by an edge if they are very close and they are joined by a path of length 2 if they are close.

The mathematical abstraction of the above concept is distance two labeling or  $L(2,1)$ -labeling. A  $L(2,1)$ -labeling of a graph  $G$  is an assignment  $f$  from the vertex set  $V(G)$  to the set of non-negative integers such that  $|f(x) - f(y)| \geq 2$  if  $x$  and  $y$  are adjacent and  $|f(x) - f(y)| \geq 1$  if  $x$  and  $y$  are at distance 2, for all  $x$  and  $y$  in  $V(G)$ .

Practically speaking the interference among channels may go beyond two levels. So we have to extend the interference level from two to the largest possible - the diameter of the corresponding graph. So the concept of  $L(2,1)$  labeling was generalized to radio labeling.

Radio labeling was originally introduced by Chartrand *et al.* [1] in 2001. A radio labeling of a graph  $G$  is an injective function  $f: V(G) \rightarrow N \cup \{0\}$  such that for every  $u, v \in V(G)$ ,  $|f(u) - f(v)| \geq \text{diam}(G) - d(u, v) + 1$ .

The span of  $f$  is the difference of the largest and the smallest channels used, that is,

$$\max_{u,v \in V(G)} \{f(u) - f(v)\}.$$

The radio number of  $G$  is defined as the minimum span of a radio labeling of  $G$  and denoted as  $rn(G)$ .

## 2 Some Existing Results

Now, we present some known results on radio labeling.

- Liu and Zhu [2] have discussed the radio number for paths and cycles.
- Liu [3] has obtained lower bounds for the radio number of trees and characterized trees achieving this bound.
- Mustapha *et al.* [4] have discussed radio  $k$ -labeling for cartesian products of graphs.
- The radio labeling of cube and fourth power of cycles have been discussed by Sooryanarayana and Raghunath [5,6].
- The radio labeling of  $k^{\text{th}}$  power of a path is discussed by Rao *et al.* [7].
- The radio number for the split graph and the middle graph of cycle  $C_n$  is discussed by Vaidya and Vihol [8].
- In the survey of literature available on radio labeling, we found that only two types of problems are considered in this area till this date.
  - (i) To investigate bounds for the radio number of a graph.
  - (ii) To completely determine the radio number of a graph.

## 3 Main Results

In this section, we develop and justify algorithms to get Radio labeling of the family of graphs - double cones whose diameter is 2, books whose diameter is 3 and  $nC_4$  with a common vertex whose diameter is 4.

We note that the concept of radio labeling and  $L(2,1)$  labeling coincides when the diameter of the graph is 2. In general, to prove a labeling is a radio labeling, we have to prove that every  $u, v \in V(G)$ ,  $|f(u) - f(v)| \geq \text{diam}(G) - d(u, v) + 1$ .

Double Cones are graphs obtained by joining two isolated vertices to every vertex of  $C_n$ . A Double Cone on  $n + 2$  vertices is denoted by  $CO_{n+2} = C_n + 2K_1$ . We note that  $CO_{n+2}$  has two vertices with maximum degree  $n$ .

### Algorithm 1

Input : A Double Cone  $CO_{n+2} = C_n + 2K_1$ ,  $n \geq 5$ .

Output : Radio labeling of  $CO_{n+2}$ .

Step 1: Arrange the vertices of  $C_n$  as  $v_1, v_2, \dots, v_n$  and let  $u_1, u_2$  denotes  $2K_1$ .

Step 2: First label  $v_i$ s of odd  $i$  and then label  $v_i$ s of even  $i$  starting from the label 3 and ending with  $n + 2$  consecutively.

Step 3: Label  $u_1$  with 0 and  $u_2$  with 1.

The justification of the desired output is proved in Theorem 1.

**Theorem 1** For the double cone  $CO_{n+2}$ ,  $n \geq 5$ ,  $rn(CO_{n+2}) = n + 2$ .

**Proof:** Consider the double cone  $CO_{n+2}$ ,  $n \geq 5$ . Let  $V(C_n) = \{v_1, v_2, \dots, v_n\}$  and  $u_1, u_2$  denotes  $2K_1$ . Define  $f: V(CO_{n+2}) \rightarrow N \cup \{0\}$  such that

$$f(u_1) = 0$$

$$f(u_2) = 1$$

$$f(v_1) = 3$$

$$f(v_i) = f(v_1) + \frac{i-1}{2}, \text{ if } i \text{ is odd and } i > 1.$$

$$f(v_2) = f(v_1) + \left\lceil \frac{n}{2} \right\rceil$$

$$f(v_i) = f(v_1) + \left\lceil \frac{n}{2} \right\rceil + \frac{i-2}{2}, \text{ if } i \text{ is even and } i > 2.$$

When  $d(v_i, v_j) = 1$ , we have  $|f(v_i) - f(v_j)| = \left\lceil \frac{n}{2} \right\rceil$  or  $\left\lceil \frac{n}{2} \right\rceil - 1, n - 1$ .

That is, when  $d(v_i, v_j) = 1$ , we have  $|f(v_i) - f(v_j)| \geq 2$ .

When  $d(v_i, v_j) = 2$ , we have  $|f(v_i) - f(v_j)| \geq 1$ , since  $f(v_i)$ s are distinct.

Since  $f(v_i) \geq 3$  and  $f(u_1) = 0$ , we have  $|f(v_i) - f(u_1)| \geq 3$ .

Since  $f(v_i) \geq 3$  and  $f(u_2) = 1$ , we have  $|f(v_i) - f(u_2)| \geq 2$ .

Hence  $f$  is a radio labeling and  $rn(CO_{n+2}) \leq n + 2$ .

Since  $CO_{n+2}$  has two vertices  $u_1$  and  $u_2$  with maximum degree  $n$  and that too, they are adjacent with the same vertices  $v_1, v_2, \dots, v_n$ , the consecutive  $n + 2$  labels  $0, 1, 2, \dots, n + 1$  are not sufficient to produce a radio-labeling.

Therefore,  $rn(CO_{n+2}) \geq n + 2$ .

Hence  $rn(CO_{n+2}) = n + 2$ .

Book  $B_n$  is the product of the star  $K_{1,n}$  with  $K_2$ . A book  $B_n$  has  $p = 2n + 2$  vertices,  $q = 3n + 1$  edges,  $n$  pages, maximum degree  $n + 1$  and diameter 3. Next we present an algorithm to get a radio labeling of a book  $B_n$ .

### Algorithm 2

Input : Book  $B_n$ ,  $n \geq 4$ , with  $p$  vertices.

Output : Radio labeling of  $B_n$ .

Step 1: Let the vertices of the  $n^{\text{th}}$  page of  $B_n$  be  $v, v_n, w_n$  and  $w$  where  $v$  and  $w$  lie on the common edge  $vw$ .

Step 2: Label  $v$  with the label  $p + 1$  and  $w$  with the label 0.

Step 3: Label  $v_i$  with  $2i$ ,  $i = 1, 2, \dots, n$ .

Step 4: Label  $w_i$  with  $f(v_i) + 5$ ,  $i = 1, 2, \dots, n - 2$  and  $w_{n-1}$  with the label 3 and  $w_n$  with the label 5, where  $f(v_i)$  is the label of  $v_i$ .

The justification of the desired output is proved in Theorem 2.

**Theorem 2** For book  $B_n$ ,  $n \geq 4$ ,  $rn(B_n) = p + 1$ .

**Proof:** Consider a book  $B_n$ . A book  $B_n$  has  $p = 2n + 2$  vertices  $q = 3n + 1$  edges,  $n$  pages, maximum degree  $n + 1$  and diameter 3. Let the vertices of the  $n^{\text{th}}$  page of  $B_n$  be  $v, v_n, w_n$  and  $w$  where  $v$  and  $w$  lie on the common edge  $vw$ . Define  $f: V(B_n) \rightarrow N \cup \{0\}$  such that

$$f(v) = p + 1$$

$$f(w) = 0$$

$$f(v_i) = 2i, i = 1, 2, \dots, n$$

$$f(w_i) = f(v_i) + 5, i = 1, 2, \dots, n - 2$$

$$f(w_{n-1}) = 3$$

$$f(w_n) = 5$$

We note that  $d(v, v_i) = 1, d(w, w_i) = 1, d(v_i, w_i) = 1, i = 1, 2, \dots, n$  and  $d(v, w) = 1$ .

Now,  $|f(v) - f(v_i)| = p + 1 - 2i = 2n + 2 + 1 - 2i = 2n + 3 - 2i \geq 3, i = 1, 2, \dots, n$ .

$$|f(w) - f(w_i)| = f(v_i) + 5 = 2i + 5 \geq 3, i = 1, 2, \dots, n - 2.$$

$$|f(w) - f(w_{n-1})| = 3.$$

$$|f(w) - f(w_n)| = 5.$$

Consider  $|f(v_i) - f(w_i)| = 2i + 5 - 2i = 5, i = 1, 2, \dots, n - 2$ .

$$|f(v_{n-1}) - f(w_{n-1})| = 2n - 2 - 3 = 2n - 5 \geq 3.$$

$$|f(v_n) - f(w_n)| = n - 5 \geq 3.$$

Also  $|f(v) - f(w)| = p + 1 = 2n + 3$ .

Thus,  $|f(s_i) - f(t_i)| \geq 3$  when  $d(s_i, t_i) = 1, s_i, t_i \in V(B_n)$ .

We note that  $d(v, w_i) = 2, d(w, v_i) = 2, i = 1, 2, \dots, n, d(v_i, v_j) = 2, i \neq j, i, j = 1, 2, \dots, n$  and  $d(w_i, w_j) = 2, i \neq j, i, j = 1, 2, \dots, n$ .

When  $d(v, w_i) = 2, i = 1, 2, \dots, n$ ,

$$|f(v) - f(w_i)| = p + 1 - i - 5 = 2n + 2 + 1 - 2i - 5 = 2n - 2 - 2i \geq 2, i = 1, 2, \dots, n - 2.$$

$$|f(v) - f(w_{n-1})| = p + 1 - 3 = p - 2 \geq 2.$$

$$|f(v) - f(w_n)| = p + 1 - 5 = p - 4 \geq 2.$$

When  $d(w, v_i) = 2, i = 1, 2, \dots, n$ ,

$$|f(w) - f(v_i)| = 2i \geq 2, i = 1, 2, \dots, n.$$

When  $d(v_i, v_j) = 2, i \neq j, i, j = 1, 2, \dots, n$ ,

$$\text{consider for } i > j, |f(v_i) - f(v_j)| = 2i - 2j \geq 2.$$

When  $d(w_i, w_j) = 2, i \neq j, i, j = 1, 2, \dots, n - 2$ ,

$$\text{consider for } i > j, |f(w_i) - f(w_j)| = |f(v_i) - f(v_j)| = 2i - 2j \geq 2.$$

Since  $f(w_{n-1}) = 3$  and  $f(w_n) = 5, |f(w_i) - f(w_j)| \geq 2$ , for all  $i, j = 1, 2, \dots, n$ .

Thus,  $|f(s_i) - f(t_i)| \geq 2$  when  $d(s_i, t_i) = 2, s_i, t_i \in V(B_n)$ .

Since all the vertex labels are distinct, we have,

$$|f(s_i) - f(t_i)| \geq 1 \text{ when } d(s_i, t_i) = 3, s_i, t_i \in V(B_n).$$

Hence  $f$  is a radio labeling and  $rn(B_n) \leq p + 1$ .

Since there are  $p$  vertices, there should be  $p$  distinct labels. The  $p$  labels  $0, 1, 2, \dots, p - 1$  are not sufficient to produce a radio labeling of the Book  $B_n$ .

Suppose they produce a radio labeling. We note that  $B_n$  has 2 adjacent vertices  $v$  and  $w$  such that each is adjacent with a different set of  $\frac{p-2}{2}$  vertices, say,  $A$  and  $B$  respectively such that (i) each vertex in  $A$  should have label difference at least 3 with  $v$  and among themselves they should have label difference at least 2, (ii) each vertex in  $B$  should have label difference at least 3 with  $w$  and among themselves they should have label difference at least 2 and (iii) the label difference of  $v$  and  $w$  should be 3. This is not possible with  $p$  consecutive non-negative integers  $0, 1, 2, \dots, p - 1$ . Therefore  $B_n$  should have more than

these  $p$  labels  $0, 1, 2, \dots, p-1$ . Since we have 2 partitions of the vertex set of  $B_n$ , say,  $A \cup \{v\}$  and  $B \cup \{w\}$  with the above three properties,  $B_n$  should have at least 2 more labels to have a radio labeling. Therefore,  $rn(B_n) \geq p+1$ . Hence  $rn(B_n) = p+1$ .

Now we consider the graph, the collection of  $n$  copies of  $C_4$ s, all of which have a common vertex, that is,  $nC_4$  with a common vertex.

### Algorithm 3

Input :  $nC_4$  with a common vertex,  $n \geq 3$ .

Output : Radio labeling of the graph,  $nC_4$  with a common vertex.

Step 1: Let the common vertex be  $u$  and the vertices of the  $i^{\text{th}}$  cycle be  $u, v_{i1}, v_{i2}$  and  $v_{i3}$ ,  $i = 1, 2, \dots, n$ .

Step 2: Label the common vertex  $u$  with  $7n+2$ .

Step 3: Label  $v_{i2}$  with label  $i-1$ ,  $i = 1, 2, \dots, n$ .

Step 4: Arrange the remaining vertices as  $v_{11}, v_{13}, v_{21}, v_{23}, \dots, v_{n1}, v_{n3}$ .  
Label these vertices starting from  $n+1, n+4, n+7, \dots, 7n-2$ .

The justification of the desired output is proved in Theorem 3.

**Theorem 3** Let  $G$  denotes  $nC_4$  with a common vertex. Then  $rn(G) = 7n+2, n \geq 3$ .

**Proof:** Consider the graph,  $nC_4$  with a common vertex. Let the common vertex be  $u$  and the vertices of the  $i^{\text{th}}$  cycle be  $u, v_{i1}, v_{i2}$  and  $v_{i3}$ ,  $i = 1, 2, \dots, n$ . Define  $f: V(G) \rightarrow N \cup \{0\}$  such that

$$f(u) = 7n+2$$

$$f(v_{i2}) = i-1, i = 1, 2, \dots, n$$

$$f(v_{i1}) = n+1 + (i-1)6, i = 1, 2, \dots, n$$

$$f(v_{i3}) = n+4 + (i-1)6, i = 1, 2, \dots, n$$

We note that  $d(u, v_{ij}) = 1, i = 1, 2, \dots, n$  and  $j = 1, 3$ . Also  $d(v_{i1}, v_{i2}) = 1$  and  $d(v_{i3}, v_{i2}) = 1, i = 1, 2, \dots, n$ .

Now,  $|f(u) - f(v_{i1})| = 7n+2 - n - 1 - (i-1)6 = 6n+1 - (i-1)6 = 6n+7-6i \geq 4, i = 1, 2, \dots, n$ .

Now,  $|f(u) - f(v_{i3})| = 7n+2 - n - 4 - (i-1)6 = 6n-2 - (i-1)6 = 6n+4-6i \geq 4, i = 1, 2, \dots, n$ .

Consider  $|f(v_{i1}) - f(v_{i2})| = n+1 + (i-1)6 - (i-1) = n+1 + (i-1)5 \geq 4, i = 1, 2, \dots, n$ .

Consider  $|f(v_{i3}) - f(v_{i2})| = n+4 + (i-1)6 - (i-1) = n+4 + (i-1)5 \geq 4, i = 1, 2, \dots, n$ .

We note that, for  $i = 1, 2, \dots, n$  and  $j = 2, d(u, v_{ij}) = 2$  and for  $i, \alpha = 1, 2, \dots, n$  and  $j, k = 1, 3, d(v_{ij}, v_{\alpha k}) = 2(i, \alpha, j, k$  all are not equal).

Since  $f(u) = 7n+2$  and for  $i = 1, 2, \dots, n$  and  $j = 2, f(v_{ij}) = i-1$ , we have,

$$|f(u) - f(v_{ij})| = 7n+2 - i + 1 = 7n+3 - i \geq 3, i = 1, 2, \dots, n \text{ and } j = 2.$$

Now, since, for  $i = 1, 2, \dots, n, f(v_{i1}) = n+1 + (i-1)6$  and  $f(v_{i3}) = n+4 + (i-1)6$ , we have,

$$\text{for } i, \alpha = 1, 2, \dots, n \text{ and } j, k = 1, 3, (i, \alpha, j, k \text{ all are not equal}), |f(v_{ij}) - f(v_{\alpha k})| \geq 3.$$

Next, we note that, for  $i = 1, 2, \dots, n, d(v_{i2}, v_{jk}) = 3$  where  $j = 1, 2, \dots, n$  &  $j \neq i$  and  $k = 1, 3$ .

$$\text{In this case, } |f(v_{j1}) - f(v_{i2})| = n+1 + (j-1)6 - (i-1) = n+6j - i - 1 \geq 2.$$

$$\text{Also, } |f(v_{j3}) - f(v_{i2})| = n+4 + (j-1)6 - (i-1) = n+6j - i - 1 \geq 2.$$

Finally, for  $i \neq j$ ,  $d(v_{i2}, v_{j2}) = 4$ .

Since all the labels of  $v_{i2}$ ,  $i = 1, 2, \dots, n$  are distinct,  $|f(v_{i2}) - f(v_{j2})| \geq 1$ .

Hence  $f$  is a radio labeling and  $rn(G) \leq 7n+2$ .

Now we find the optimal value of  $rn(G)$ . The minimum possible label of  $u$  is 0. Since the distance between  $u$  and  $v_{11}$  is 1, the minimum label we can use at  $v_{11}$  is at least 4. Since the vertices  $v_{13}, v_{21}, v_{23}, v_{31}, v_{33}, \dots, v_{n1}, v_{n3}$  are at a distance 2 from  $v_{11}$ , each of this should have a label difference 3 with  $v_{11}$  and among themselves. And so the minimum range of the labels for these vertices is  $\{7, 10, \dots, 6n+1\}$ . So, without loss of generality, we assume that the minimum label  $v_{n3}$  should receive is  $6n+1$ . There are  $n$  more vertices, namely,  $v_{12}, v_{22}, \dots, v_{n2}$ . We cannot use the label  $6n+2$  to any of these vertices because these vertices are at a distance 3 from  $v_{n3}$  and hence the label difference at these vertices should be at least 2. So the minimum label we can use to any one of these vertices is  $6n+3$  and there are  $n$  such vertices. Therefore the minimum label we required to label all these vertices is  $6n+1 + (n+1) = 7n+2$ .

Therefore, the minimum label which can give a radio labeling for the graph  $G$  is  $7n+2$ . That is,  $rn(G) \geq 7n+2$ . Hence,  $rn(G) = 7n+2$ .

## 4 Conclusion

In this paper, we have found Radio number for the families like double cones, books and  $nC_4$  with a common vertex whose diameters are 2,3 and 4 respectively.

## References

- [1] Chartrand, G., Erwin, D., Harary, F. and Zhang, P. Radio labelings of graphs. *Bulletin of Inst. Combin. Appl.* 2001. 33: 77-85.
- [2] Liu, D. D. and Zhu, X. Multilevel distance labelings for paths cycles. *SIAM J. Discrete Math.* 2005. 19: 610-621.
- [3] Liu, D. D. Radio number for trees. *Discrete Mathematics.* 2007. 308(7): 1153-1164.
- [4] Mustapha K., Khennoufs, K. and Togni, O. Radio  $k$ -Labelings for Cartesian Products of Graphs. *Discussiones Mathematicae Graph Theory.* 2008. 28(1): 165-178.
- [5] Sooryanarayana, B. and Raghunath, P. On the radio labeling of cube of a cycle. *Far East J. Appl. Math.* 2007. 29(1): 113-147.
- [6] Sooryanarayana, B. and Raghunath, P. On the radio labeling of fourth power of a cycle. *Journal of Applied Mathematical Analysis and Applications.* 2007. 3(2): 195-228.
- [7] Devadas Rao, P. Sooryanarayana, B. and Chandru Hegde. On the radio labeling of  $k^{\text{th}}$  power of a path. *Journal of graphs and Combinatorics (Springer)*. submitted.
- [8] Vaidya, S. K. and Vihol, P. L. Radio labeling for some cycle related graphs. *International Journal of Mathematics and Soft computing.* 2012. 2(2): 11-24.