MATEMATIKA, 2016, Volume 32, Number 1, 13–29

Global existence of solutions for a class of singular fractional differential equations with impulse effects

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Abstract We establish sufficient conditions for the global existence of solutions for a class of impulsive singular fractional differential equations by constructing a weighted Banach space and a completely continuous operator and using the fixed point theorem in the Banach space. An example is given to illustrate the efficiency of the main theorems.

Keywords Global existence; singular fractional differential equation; impulsive effect; fixed point theorem.

2010 Mathematics Subject Classification 92D25, 34A08, 26A33, 39B99, 45G10, 34B37, 34B15, 34B16

1 Introduction

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments, we refer the reader to [1]. It is well known that

$$\begin{cases} x'(t) = f(t, x(t)),\ t \geq 0,\ \ t \neq t_k, k = 1, 2, \cdots, \\ \Delta x(t_k) = \lim\limits_{t \to t_k^+} x(t) - \lim\limits_{t \to t_k^-} x(t) = I_k(t_k, x(t_k)),\ k = 1, 2, \cdots, \end{cases} \tag{1.1}$$

models many kinds of ecological systems [1], where f and $I_k(k = 1, 2 \cdots)$ are continuous functions. The existence of solutions of this system and the global attractivity of solutions were studied by many authors, see [2] and the references therein. In mathematics, a functional equation is any equation that specifies a function in implicit form [3]. Studying the existence of solutions of the following functional equation with continuous variable and impulse effects

$$\begin{cases} x'(t) = f(t, x(t)),\ t \geq 0,\ \ t \neq t_k, k = 1, 2, \cdots, \\ \Delta x(t_k) = \lim\limits_{t \to t_k^+} x(t) - \lim\limits_{t \to t_k^-} x(t) = I_k(t_k, x(t_k)),\ k = 1, 2, \cdots, \end{cases} \tag{1.2}$$

is interesting.

The topic of fractional calculus (derivatives and integrals of arbitrary orders) is enjoying growing interest not only among mathematicians, but also among physicists and engineers [4]. Differential equations of fractional order have proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering [5–9]. Consequently, considerable attention has been given to the solvability of fractional differential equations of physical interest.

The differential equations in (1.1) and (1.2) can be unified by the following one with fractional differential derivative

$$D_{0^+}^{\alpha}x(t) = f(t, x(t)),\ t \geq 0 \tag{1.3}$$

where $0 \leq \alpha \leq 1$ and D^{α} is a fractional derivative of order α. The existence and the uniqueness of solutions and numerical solutions of model (1.3) has been studied by many authors see [4, 10, 11]. The solvability of initial value problems of fractional differential equations have been studied in [4, 12–15]. Recently, the authors in papers [10, 16–22] studied the existence of solutions of the different initial value problems for the impulsive fractional differential equations on a finite interval and in mentioned papers the nonlinearities in the fractional differential equations and the impulse functions are continuous and bounded ones. To our knowledge, the existence of solutions of initial value problems of impulsive fractional differential equations on infinite interval with singular nonlinearities has not been studied.

In this paper, we discuss the following initial value problem for nonlinear singular fractional differential equation with impulse effects

$$\begin{cases} D_{0^+}^{\alpha}u(t) = m(t)f(t, u(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0, \\ \lim\limits_{t \to 0^+} t^{1-\alpha}u(t) = \int_0^{+\infty} \phi(s)F(s, u(s))ds, \\ \lim\limits_{t \to t_i^+} (t - t_i)^{1-\alpha}u(t) = I(t_i, u(t_i)), i \in \mathbb{N}, \end{cases} \tag{1.4}$$

where

(a) $0 < \alpha < 1$, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α, $0 = t_0 < t_1 < t_2 < \cdots$ with $\lim\limits_{i \to +\infty} t_i = +\infty$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $\mathbb{N} = \{1, 2, \cdots\}$,

(b) $m : (0, +\infty) \to \mathbb{R}$ satisfy $m|_{(t_k, t_{k+1}]} \in C^0(t_k, t_{k+1}](k = 0, 1, 2, \cdots)$, m may be singular at $t = 0$, m satisfies that there exist constants $L > 0$, $k > -\alpha$, such that $|m(t)| \leq Lt^k$ for all $t \in (0, +\infty)$,

(c) $\phi : (0, +\infty) \to \mathbb{R}$ satisfy $\phi \in L^1(0, +\infty)$ and,

(d) $f, F : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, $I : \{t_i : i \in \mathbb{N}\} \times \mathbb{R} \to \mathbb{R}$ is a discrete Carathéodory function.

A function x with $x : (0, +\infty) \to \mathbb{R}$ is said to be a solution of (1.4), if $x|_{(t_i, t_{i+1}]} \in C^0(t_i, t_{i+1}](i \in \mathbb{N}_0)$ and x satisfies all equations in (1.4).

We obtain the results on the existence of solutions of (1.4). An example is given to illustrate the efficiency of the main theorems.

The remainder of this paper is as follows: in Section 2, we present preliminary results. In Section 3, the main theorems and their proof are given. In Section 4, two examples are given to illustrate the main results.

2 Preliminary results

For the convenience of the readers, we present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the monograph [8].

Definition 2.1[8] Let $c \in \mathbb{R}$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (c, \infty) \to \mathbb{R}$ is given by $I_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} g(s)ds$, provided that the right-hand side exists.

Definition 2.2[8] Let $c \in \mathbb{R}$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (c, \infty) \to \mathbb{R}$ is given by $D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_c^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds$, where $n-1 \le \alpha < n$, provided that the right-hand side exists.

Remark 2.1 One sees that $I_{c^+}^{\alpha} D_{c^+}^{\alpha} x(t) = x(t) + \sum\limits_{i=1}^{n} c_i (t-c)^{\alpha-i}$ for some $c_i \in \mathbb{R}$ and $D_{c^+}^{\alpha} I_{c^+}^{\alpha} x(t) = x(t)$.

Definition 2.3 Let k be defined in (c), $\sigma > k+1$. We call $K : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function if it satisfies the followings:

(i) $t \to K\left(t, \frac{(t-t_i)^{\alpha-1}}{1+t^{\sigma}} x\right)$ is measurable on $(t_i, t_{i+1}] (i \in \mathbb{N}_0)$,

(ii) $x \to K\left(t, \frac{(t-t_i)^{\alpha-1}}{1+t^{\sigma}} x\right)$ is continuous on R for almost all $t \in (t_i, t_{i+1}], i \in \mathbb{N}_0$,

(iii) for each $r > 0$ there exists a constant $A_r > 0$ such that $\left|K\left(t, \frac{(t-t_i)^{\alpha-1}}{1+t^{\sigma}} x\right)\right| \le A_r,\ t \in (t_i, t_{i+1}], i \in \mathbb{N}_0, |x| \le r$.

Definition 2.4 $G : \{t_i : i \in \mathbb{N}\} \times R\}$ is called a discrete Carathéodory function if G satisfies the following items:

(i) $x \to G\left(t_i, \frac{(t_i-t_{i-1})^{\alpha-1}}{1+t_i^{\sigma}} x\right)$ is continuous on R for most all $i \in \mathbb{N}_0$,

(ii) for each $r > 0$ there exists $A_{r,i} \ge 0$ such that $\left|G\left(t_i, \frac{(t_i-t_{i-1})^{\alpha-1}}{1+t_i^{\sigma}} x\right)\right| \le A_{r,i},\ i \in \mathbb{N}, |x| \le r$ and $\sum\limits_{i=1}^{+\infty} A_{r,i} < +\infty$.

To obtain the main results, we need the abstract existence theorem.

Lemma 2.1[15]Leray-Schauder Nonlinear Alternative

Let X be a Banach space and $T : X \to X$ be a completely continuous operator. Suppose Ω is a nonempty open subset of X centered at zero. Then either there exists $x \in \partial\Omega$ and $\lambda \in (0,1)$ such that $x = \lambda Tx$ or there exists $x \in \overline{\Omega}$ such that $x = Tx$.

For $\phi \in L^1(0, \infty)$, denote $||\phi||_1 = \int_0^{\infty} |\phi(s)|ds$. Let the Gamma and Beta functions $\Gamma(\alpha)$ and $\mathbf{B}(p,q)$ be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx,\ \ \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx.$$

Let k be defined in (b) and $\sigma > \max\{1-\alpha, k+1\}$, $\delta(t) = (t-t_i)^{1-\alpha}$ for $t \in (t_i, t_{i+1}](i \in N_0)$. Choose

$$X = \left\{ x : \begin{array}{c} x|_{(t_i,t_{i+1}]} \in C^0(t_i, t_{i+1}](i \in \mathbb{N}_0), \\ \lim_{t \to t_i^+}(t-t_i)^{1-\alpha}x(t)(i \in \mathbb{N}_0), \ \lim\limits_{t\to+\infty} \frac{\delta(t)}{1+t^\sigma}x(t) \text{ are finite} \end{array} \right\}.$$

For $x \in X$, define the norm by $||x|| = ||x||_X = \sup\limits_{t\in(0,+\infty)} \frac{\delta(t)}{1+t^\sigma}|x(t)|$. It is easy to show that X is a real Banach space.

Lemma 2.2 *Suppose that (a)-(d) hold, $x \in X$. Then $u \in X$ is a solution of*

$$\begin{cases} D_{0^+}^\alpha u(t) = m(t)f(t, x(t)), \quad t \in (t_i, t_{i+1}], i \in \mathbb{N}_0, \cdots, \\ \lim\limits_{t\to0^+} t^{1-\alpha}u(t) = \int_0^1 \phi(s)F(s, x(s))ds, \\ \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}u(t) = I(t_i, x(t_i)), i \in \mathbb{N} \end{cases} \tag{2.1}$$

if and only if $u \in X$ satisfies the integral equation

$$\begin{aligned} x(t) = t^{\alpha-1} \int_0^{+\infty} \phi(s)F(s, x(s))ds + \sum_{j=1}^{i}(t-t_j)^{\alpha-1}I(t_j, x(t_j)) \\ + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(u)f(u, x(u))du, t \in (t_i, t_{i+1}], i \in \mathbb{N}_0. \end{aligned} \tag{2.2}$$

Proof If $x \in X$, then there exists $r > 0$ such that $||x|| = \sup\limits_{t\in(0,\infty)} \frac{\delta(t)}{1+t^\sigma}|x(t)| \le r$. Since f is a Carathéodory function, there exists $A_r \ge 0$ such that

$$|f(t, x(t))| = \left| f\left(t, (t-t_i)^{\alpha-1}(1+t^\sigma)\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}x(t)\right)\right| \le A_r, t \in (t_i, t_{i+1}], i \in \mathbb{N}_0.$$

Similarly from F, I are Carathéodory function and discrete Carathéodory function, we get that there exist constants $A_r', A_{r,i} \ge 0 (i \in \mathbb{N})$ such that

$$|F(t, x(t))| \le A_r', \ |I(t_i, x(t_i))| \le A_{r,i}, \ t \in (t_i, t_{i+1}], i \in \mathbb{N}, \ \sum_{i=1}^{+\infty} A_{r,i} < +\infty.$$

Use $D_{0^+}^\alpha u(t) = m(t)f(t, x(t)), \quad t \in (t_i, t_{i+1}], i \in \mathbb{N}_0$, we will prove by mathematical induction method that there exist constants $c_\sigma \in \mathbb{R}(\sigma \in \mathbb{N}_0)$ such that

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)f(s, x(s))ds + \sum_{\sigma=0}^{i} c_\sigma(t-t_\sigma)^{\alpha-1}, t \in (t_i, t_{i+1}], i \in \mathbb{N}_0. \tag{2.3}$$

It is easy to know for $t \in (t_0, t_1]$ that there exists a constant $c_0 \in \mathbb{R}$ such that

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)f(s, x(s))ds + c_0t^{\alpha-1}. \tag{2.4}$$

So (2.3) holds for $i=0$. Now suppose that (2.3) holds for $i=0,1,\cdots,j$. We prove that (2.3) holds for $i=j+1$. In fact, assume

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}m(s)f(s,x(s))ds+\sum_{\sigma=0}^{j}c_\sigma(t-t_\sigma)^{\alpha-1}+\Phi(t),t\in(t_{j+1},t_{j+2}]. \quad (2.5)$$

For $t\in(t_{j+1},t_{j+2}]$, we have by Definition 2.2

$$\begin{aligned}
&m(t)f(t,x(t))=D_{0^+}^\alpha u(t)=\tfrac{1}{\Gamma(1-\alpha)}\left[\int_0^t(t-s)^{-\alpha}u(s)ds\right]'\\
&=\tfrac{1}{\Gamma(1-\alpha)}\left[\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}(t-s)^{-\alpha}\left(\tfrac{1}{\Gamma(\alpha)}\int_0^s(s-u)^{\alpha-1}m(u)f(u,x(u))du+\sum_{\sigma=0}^{j}c_\sigma(s-t_\sigma)^{\alpha-1}\right)ds\right]'\\
&+\tfrac{1}{\Gamma(1-\alpha)}\left[\int_{t_j}^t(t-s)^{-\alpha}\left(\tfrac{1}{\Gamma(\alpha)}\int_0^s(s-u)^{\alpha-1}m(u)f(u,x(u))du+\sum_{\sigma=0}^{j}c_\sigma(s-t_\sigma)^{\alpha-1}+\Phi(s)\right)ds\right]'\\
&=D_{t_{j+1}^+}^\alpha\Phi(t)+\tfrac{1}{\Gamma(1-\alpha)}\left[\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}(t-s)^{-\alpha}\sum_{\sigma=0}^{j}c_\sigma(s-t_\sigma)^{\alpha-1}ds\right]'\\
&+\tfrac{1}{\Gamma(1-\alpha)}\left[\int_0^t(t-s)^{-\alpha}\tfrac{1}{\Gamma(\alpha)}\int_0^s(s-u)^{\alpha-1}m(u)f(u,x(u))duds\right]'\\
&+\tfrac{1}{\Gamma(1-\alpha)}\left[\int_{t_j}^t(t-s)^{-\alpha}\sum_{\sigma=0}^{j}c_\sigma(s-t_\sigma)^{\alpha-1}ds\right]'\\
&=D_{t_{j+1}^+}^\alpha\Phi(t)+\tfrac{1}{\Gamma(1-\alpha)}\left[\sum_{i=0}^{j-1}\sum_{\sigma=0}^{j}c_\sigma\int_{t_i}^{t_{i+1}}(t-s)^{-\alpha}(s-t_\sigma)^{\alpha-1}ds\right]'\\
&+\tfrac{1}{\Gamma(1-\alpha)}\tfrac{1}{\Gamma(\alpha)}\left[\int_0^t\int_u^t(t-s)^{-\alpha}(s-u)^{\alpha-1}dsm(u)f(u,x(u))du\right]'\\
&+\tfrac{1}{\Gamma(1-\alpha)}\left[\sum_{\sigma=0}^{j}c_\sigma\int_{t_j}^t(t-s)^{-\alpha}(s-t_\sigma)^{\alpha-1}ds\right]'\\
&=D_{t_{j+1}^+}^\alpha\Phi(t)+\tfrac{1}{\Gamma(1-\alpha)}\left[\sum_{i=0}^{j-1}\sum_{\sigma=0}^{j}c_\sigma\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{\frac{t_{i+1}-t_\sigma}{t-t_\sigma}}(1-w)^{-\alpha}w^{\alpha-1}dw\right]'\\
&+\tfrac{1}{\Gamma(1-\alpha)}\tfrac{1}{\Gamma(\alpha)}\left[\int_0^t\int_0^1(1-w)^{-\alpha}w^{\alpha-1}dwm(u)f(u,x(u))du\right]'\\
&+\tfrac{1}{\Gamma(1-\alpha)}\left[\sum_{\sigma=0}^{j}c_\sigma\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^1(1-w)^{-\alpha}w^{\alpha-1}dw\right]'\\
&=D_{t_{j+1}^+}^\alpha\Phi(t)+\tfrac{1}{\Gamma(1-\alpha)}\left[\sum_{\sigma=0}^{j-1}\sum_{i=\sigma}^{j-1}c_\sigma\int_{\frac{t_i-t_\sigma}{t-t_\sigma}}^{\frac{t_{i+1}-t_\sigma}{t-t_\sigma}}(1-w)^{-\alpha}w^{\alpha-1}dw\right]'\\
&+\tfrac{1}{\Gamma(1-\alpha)}\tfrac{1}{\Gamma(\alpha)}\left[\int_0^t\int_0^1(1-w)^{-\alpha}w^{\alpha-1}dwm(u)f(u,x(u))du\right]'\\
&+\tfrac{1}{\Gamma(1-\alpha)}\left[\sum_{\sigma=0}^{j}c_\sigma\int_{\frac{t_j-t_\sigma}{t-t_\sigma}}^1(1-w)^{-\alpha}w^{\alpha-1}dw\right]'\\
&=D_{t_{j+1}^+}^\alpha\Phi(t)+\tfrac{1}{\Gamma(1-\alpha)}\left[\sum_{\sigma=0}^{j}c_\sigma\int_0^1(1-w)^{-\alpha}w^{\alpha-1}dw\right]'+\left[\int_0^t m(u)f(u,x(u))du\right]'\\
&=D_{t_{j+1}^+}^\alpha\Phi(t)+m(t)f(t,x(t)).
\end{aligned}$$

It follows that $D_{t_{j+1}^+}^\alpha\Phi(t)=0$ for $t\in(t_{j+1},t_{j+2}]$. So there exists a constant $c_{j+1}\in\mathbb{R}$ such that $\Phi(t)=c_{j+1}(t-t_{j+1})^{\alpha-1}$. Thus

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)f(s,x(s))ds+\sum_{\sigma=0}^{j+1}c_\sigma(t-t_\sigma)^{\alpha-1}, t\in(t_{j+1},t_{j+2}].$$

Hence (2.3) holds for $i=j+1$. By mathematical induction method, we know that (2.3) is valid.

From $\lim\limits_{t\to 0^+} t^{1-\alpha}x(t)=\int_0^{+\infty}\phi(s)F(s,x(s))ds$, we get $c_0=\int_0^{+\infty}\phi(s)F(s,x(s))ds$.

From $\lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}u(t)=I(t_i,x(t_i)), i\in\mathbb{N}$, we get that $c_i=I(t_i,x(t_i)), i\in\mathbb{N}$.

Hence, we have

$$u(t)=\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds$$
$$+t^{\alpha-1}\int_0^\infty \phi(s)F(s,x(s))ds+\textstyle\sum_{s=1}^i(t-t_s)^{\alpha-1}I(t_s,x(t_s)), t\in(t_i,t_{i+1}], i\in\mathbb{N}_0.$$

Hence u satisfies (2.2). Furthermore

$$\begin{aligned}&t^{1-\alpha}\left|\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds\right|\\&\le A_rt^{1-\alpha}L\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^kds\\&=A_rt^{1+k}L\int_0^1\tfrac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^kdw\to 0 \text{ as } t\to 0^+.\end{aligned}$$

Then $t\to\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds$ is convergent and continuous on $(0,+\infty)$. Furthermore, the limits $\lim\limits_{t\to 0^+}t^{1-\alpha}u(t)=0$ and $\lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}u(t)=I(t_i,x(t_i)), i\in\mathbb{N}$ and for $t\in(t_i,t_{i+1}]$ we have

$$\begin{aligned}&\tfrac{\delta(t)}{1+t^\sigma}|u(t)|\le\tfrac{(t-t_i)^{1-\alpha}}{1+t^\sigma}t^{\alpha-1}||\phi||_1A'_r+\sum_{j=1}^i\tfrac{(t-t_i)^{1-\alpha}}{1+t^\sigma}(t-t_j)^{\alpha-1}A_{r,j}\\&+LA_r\tfrac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\int_0^t\tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u^kdu\\&\le\tfrac{1}{1+t^\sigma}||\phi||_1A'_r+\sum_{j=1}^i\tfrac{1}{1+t^\sigma}A_{r,j}+LA_r\tfrac{(t-t_i)^{1-\alpha}}{1+t^\sigma}t^{\alpha+k}\int_0^1\tfrac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^kdw\\&\le\tfrac{1}{1+t^\sigma}||\phi||_1A'_r+\tfrac{1}{1+t^\sigma}\sum_{j=1}^{+\infty}A_{r,j}+LA_r\tfrac{t^{k+1}}{1+t^\sigma}\int_0^1\tfrac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^kdw\to 0 \text{ as } t\to+\infty.\end{aligned}$$

So $u\in X$.

On the other hand, if $x\in X$ and u is a solution of (2.2), then we can show $u\in X$ and $\lim\limits_{t\to 0^+}t^{1-\alpha}x(t)=\int_0^{+\infty}\phi(s)F(s,x(s))ds=:c_0$ and $\lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}u(t)=I(t_i,x(t_i))=c_i, i\in\mathbb{N}$. We rewrite (2.2) by

$$x(t)=\sum_{j=0}^i(t-t_j)^{\alpha-1}c_j+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0.$$

Furthermore, we have

$$
\begin{aligned}
&D_{0^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\left[\int_0^t(t-s)^{-\alpha}x(s)\right]'\\
&=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}}(t-s)^{-\alpha}\left(\sum_{\sigma=0}^{i}(s-t_\sigma)^{\alpha-1}c_\sigma+\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}m(u)f(u,x(u))du\right)\right]'\\
&+\frac{1}{\Gamma(1-\alpha)}\left[\int_{t_j}^t(t-s)^{-\alpha}\left(\sum_{\sigma=0}^{j}(s-t_\sigma)^{\alpha-1}c_\sigma+\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}m(u)f(u,x(u))du\right)\right]'.
\end{aligned}
$$

Use above method, we have $D_{0^+}^{\alpha}x(t)=m(t)f(t,x(t))$. Then $u\in X$ is a solution of (2.1). The proof is completed. □

Now, we define the operator T on X by

$$
\begin{aligned}
(Tx)(t)&=t^{\alpha-1}\int_0^{+\infty}\phi(s)F(s,x(s))ds+\sum_{j=1}^{i}(t-t_j)^{\alpha-1}I(t_j,x(t_j))\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds,\ t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0.
\end{aligned}
\tag{2.6}
$$

Lemma 2.3 *By Lemma 2.2, $x\in X$ is a solution of IVP(1.10) if and only if $x\in X$ is a fixed point of the operator T.*

Lemma 2.4 *Suppose that (a)-(d) hold. Then $T:X\to X$ is well defined and is completely continuous.*

Proof The proof is very long, so we list the steps. Firstly, we prove that $T:X\to X$ is well defined; secondly, we prove that T is continuous and finally, we prove that T is compact. So T is completely continuous. Thus the proof is divided into three steps.

Step(i) Prove that $T:X\to X$ is well defined.

For $x\in X$, we have $||x||=r>0$. then $||x||=\sup_{t\in(0,+\infty)}\frac{\delta(t)}{1+t^\sigma}|x(t)|\le r$. Since f is a Carathéodory function, there exists $A_r\ge 0$ such that

$$
|f(t,x(t))|=\left|f\left(t,(t-t_i)^{\alpha-1}(1+t^\sigma)\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}x(t)\right)\right|\le A_r,t\in(t_i,t_{i+1}],i\in\mathbb{N}_0.
\tag{2.7}
$$

Similarly from F,I are Carathéodory function and discrete Carathéodory function, we get that there exist constants $A_r',A_{r,i}\ge 0(i\in\mathbb{N})$ such that

$$
|F(t,x(t))|\le A_r',\ |I(t_i,x(t_i))|\le A_{r,i},\ t\in(t_i,t_{i+1}],i\in\mathbb{N},\ \sum_{i=1}^{+\infty}A_{r,i}<+\infty.
\tag{2.8}
$$

Similarly to the proof of Lemma 2.2, we have $Tx\in X$. Then $T:X\to X$ is well defined.

Step(ii) We prove that T is continuous. Let $x_n \in X$ with $x_n \to x_0$ as $n \to \infty$. We will show that $Tx_n \to Tx_0$ as $n \to \infty$.

In fact, we have $r > 0$ such that $||x_n|| \le r > 0(n = 0, 1, 2, \cdots)$. Then

$$||x_n|| = \sup_{t \in (0,\infty)} \tfrac{\delta(t)}{1+t^\sigma}|x_n(t)| \le r, \text{ and } \sup_{t \in (0,+\infty)} \tfrac{\delta(t)}{1+t^\sigma}|x_n(t) - x_0(t)| \to 0 \tag{2.9}$$

as $n \to +\infty$. Since f, F are Carathéodory functions, I is a discrete Carathéodory function, we get that there exist constants $A'_r, A_{r,i} \ge 0(i \in \mathbb{N})$ such that (2.7) and (2.8) hold.

By

$$\begin{aligned}(Tx_n)(t) = \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)f(s, x_n(s))ds + t^{\alpha-1}\int_0^{+\infty} \phi(s)F(s, x_n(s))ds \\ +\sum_{j=1}^i (t-t_j)^{\alpha-1} I(t_j, x_n(t_j)), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0.\end{aligned} \tag{2.10}$$

From the Lebesgue dominated convergence theorem, we get

$$\sup_{t \in (0,+\infty)} \tfrac{\delta(t)}{1+t^\sigma}|(Tx_n)(t) - (Tx_0)(t)| \to 0 \tag{2.11}$$

as $n \to +\infty$. It follows that T is continuous.

Step(iii) We prove that T is compact, i.e., for each nonempty open bounded subset Ω of X, prove that $T(\overline{\Omega})$ is relatively compact.

Let Ω be a bounded open subset of X. We have $r > 0$ such that then $||x|| = \sup_{t \in (0,+\infty)} \frac{\delta(t)}{1+t^\sigma}|x(t)| \le r$ for all $x \in \overline{\Omega}$. Since f, F are Carathéodory functions, I is a discrete Carathéodory function, we get that there exist constants $A'_r, A_{r,i} \ge 0(i \in N)$ such that (2.7) and (2.8) hold.

Sub-step (iii1) Prove that $\left\{t \to \frac{\delta(t)}{1+t^\sigma}(Tx)(t)) : x \in \overline{\Omega}\right\}$ is uniformly bounded.

In fact, for $t \in (t_i, t_{i+1}](i \in \mathbb{N}_0)$, we have

$$\begin{aligned}&\tfrac{\delta(t)}{1+t^\sigma}|(Tx))(t)| \le \tfrac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s, x(s))|ds \\ &+\tfrac{(t-t_i)^{1-\alpha}t^{1-\alpha}}{1+t^\sigma}\int_0^\infty |\phi(s)F(s, x(s))|ds + \tfrac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\sum_{j=1}^i (t-t_j)^{\alpha-1}|I(t_j, x(t_j))| \\ &\le A_r L \tfrac{t^{1+k}}{1+t^\sigma}\int_0^1 \tfrac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^k dw + ||\phi||_1 A_r + \sum_{s=1}^{+\infty} A_{r,s} \\ &\le A_r L \tfrac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)} + ||\phi||_1 A_r + \sum_{j=1}^{+\infty} A_{r,j} < \infty.\end{aligned}$$

It follows that

$$\sup_{t \in (0,+\infty)} \tfrac{\delta(t)}{1+t^\sigma}|(Tx))(t)| < A_r L \tfrac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)} + ||\phi||_1 A_r + \sum_{j=1}^{+\infty} A_{r,j}$$

for all $x \in \overline{\Omega}$. It is easy to see that $T(\overline{\Omega})$ is uniformly bounded in X.

Sub-step (iii2) Prove that $\left\{t \to \frac{\delta(t)}{1+t^\sigma}(Tx)(t)) : x \in \overline{\Omega}\right\}$ is equi-continuous on each interval $(t_i, t_{i+1}], i \in \mathbb{N}_0$.

For each $s_1, s_2 \in (t_i, t_{i+1}]$ $s_2 > s_1$, we have

$$\begin{aligned}
&\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}(Tx)(s_1) - \frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}(Tx)(s_2)\right|\\
&\leq \left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\int_0^{s_1}\frac{(s_1-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds - \frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\int_0^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds\right|\\
&+\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma} - \frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\right|\int_0^{+\infty}|\phi(s)F(s,x(s))|ds\\
&+\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\sum_{j=1}^{i}(s_1-t_j)^{\alpha-1}I(t_j,x(t_j)) - \frac{(s_2-t_i)^{1-\alpha}}{1+s_1^\sigma}\sum_{j=1}^{i}(s_2-t_j)^{\alpha-1}I(t_j,x(t_j))\right|\\
&\leq \left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\int_0^{s_1}\frac{(s_1-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds - \frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\int_0^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds\right|\\
&+A_r'||\phi||_1\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma} - \frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\right|\\
&+\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\sum_{j=1}^{i}(s_1-t_j)^{\alpha-1}I(t_j,x(t_j)) - \frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\sum_{j=1}^{i}(s_2-t_j)^{\alpha-1}I(t_j,x(t_j))\right|.
\end{aligned}$$

We have

$$\begin{aligned}
&\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\int_0^{s_1}\frac{(s_1-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds - \frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\int_0^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds\right|\\
&\leq \left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma} - \frac{s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\right|\int_0^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,x(s))|ds\\
&+\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\int_{s_1}^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,x(s))|ds\\
&+\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\int_0^{s_1}\frac{|(s_1-s)^{\alpha-1}-(s_2-s)^{\alpha-1}|}{\Gamma(\alpha)}|m(s)f(s,x(s))|ds\\
&\leq LA_r\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma} - \frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\right|\int_0^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^k ds\\
&+LA_r\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\int_{s_1}^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^k ds + LA_r\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\int_0^{s_1}\frac{(s_1-s)^{\alpha-1}-(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^k ds\\
&\leq LA_r\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma} - \frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\right|s_2^{\alpha+k}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\\
&+LA_r\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}s_2^{\alpha+k}\int_{\frac{s_1}{s_2}}^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\\
&+LA_r\left[s_1^{\alpha+k}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw - s_2^{\alpha+k}\int_0^{\frac{s_1}{s_2}}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\right]\\
&\leq LA_r\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma} - \frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\right|t_{i+1}^{\alpha+k}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\\
&+LA_r(t_{i+1}-t_i)^{1-\alpha}t_{i+1}^{\alpha+k}\int_{\frac{s_1}{s_2}}^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\\
&+LA_r\left[s_1^{\alpha+k}\int_{\frac{s_1}{s_2}}^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw - [s_2^{\alpha+k}-s_1^\alpha]\int_0^{\frac{s_1}{s_2}}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\right]\\
&\leq LA_r\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma} - \frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\right|t_{i+1}^{\alpha+k}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\\
&+LA_r(t_{i+1}-t_i)^{1-\alpha}t_{i+1}^{\alpha+k}\int_{\frac{s_1}{s_2}}^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\\
&+LA_r\left[t_{i+1}^{\alpha+k}\int_{\frac{s_1}{s_2}}^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw - [s_2^{\alpha+k}-s_1^\alpha]\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\right]\\
&\to 0 \text{ uniformly as } s_1 \to s_2 \text{ by } l > -\alpha, s_1, s_2 \in (t_i, t_{i+1}]
\end{aligned}$$

and

$$\begin{array}{l}\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\sum_{j=1}^{i}(s_1-t_j)^{\alpha-1}I(t_j,x(t_j))-\frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\sum_{j=1}^{i}(s_2-t_j)^{\alpha-1}I(t_j,x(t_j))\right|\\ \le\left|\frac{1}{1+s_1^\sigma}-\frac{1}{1+s_2^\sigma}\right|A_{r,i}+\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}-\frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\right|\sum_{j=1}^{i-1}(s_2-t_j)^{\alpha-1}A_{r,j}\\ +\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\sum_{j=1}^{i-1}\left[\frac{1}{(s_1-t_j)^{1-\alpha}}-\frac{1}{(s_2-t_j)^{1-\alpha}}\right]A_{r,j}\\ \le\left|\frac{1}{1+s_1^\sigma}-\frac{1}{1+s_2^\sigma}\right|A_{r,i}+\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}-\frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\right|(t_i-t_{i-1})^{\alpha-1}\sum_{j=1}^{i-1}A_{r,j}\\ +\frac{(t_{i+1}-t_i)^{1-\alpha}}{(t_i-t_{i-1})^{2(1-\alpha)}}\sum_{j=1}^{i-1}\left[(s_2-t_j)^{1-\alpha}-(s_1-t_j)^{1-\alpha}\right]A_{r,j}.\end{array}$$

Since there exists $\delta>0$ such that

$$\begin{array}{l}|(s_2-t_j)^{1-\alpha}-(s_1-t_j)^{1-\alpha}|<\epsilon \text{ for every } j\in N[1,i-1],|s_1-s_2|\le\delta,\\ \left|\frac{1}{1+s_1^\sigma}-\frac{1}{1+s_2^\sigma}\right|<\epsilon,|s_1-s_2|<\delta,\\ \left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}-\frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\right|<\epsilon,|s_1-s_2|<\delta,\end{array}$$

then

$$\begin{array}{l}\left|\frac{(s_1-t_i)^{1-\alpha}}{1+s_1^\sigma}\sum_{j=1}^{i}(s_1-t_j)^{\alpha-1}I(t_j,x(t_j))-\frac{(s_2-t_i)^{1-\alpha}}{1+s_2^\sigma}\sum_{j=1}^{i}(s_2-t_j)^{\alpha-1}I(t_j,x(t_j))\right|\\ \le\epsilon A_{r,i}+\epsilon(t_i-t_{i-1})^{\alpha-1}\sum_{j=1}^{+\infty}A_{r,j}+(t_{i+1}-t_i)^{1-\alpha}\epsilon\sum_{j=1}^{+\infty}A_{r,j}.\end{array}$$

It follows that $\left\{t\to\frac{\delta(t)}{1+t^\sigma}(Tx)(t)):x\in\overline{\Omega}\right\}$ is equi-continuous on each interval $(t_i,t_{i+1}],i\in\mathbb{N}_0$.

Sub-step (iii3) Prove that $\left\{t\to\frac{\delta(t)}{1+t^\sigma}(Tx)(t)):x\in\overline{\Omega}\right\}$ is equi-convergent as $t\to+\infty$.

We have for $t\in(t_i,t_{i+1}]$

$$\begin{array}{l}\left|\frac{\delta(t)}{1+t^\sigma}(Tx)(t)\right|\le\frac{(t-t_i)^{1-\alpha}t^{\alpha-1}}{1+t^\sigma}||\phi||_1A_r'+\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\sum_{j=1}^{i}(t-t_j)^{\alpha-1}A_{r,j}\\ +\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}LA_r\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^kds\\ \le\frac{1}{1+t^\sigma}||\phi||_1A_r'+\frac{1}{1+t^\sigma}\sum_{j=1}^{i}A_{r,j}+LA_r\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}t^{\alpha+k}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^kdw\\ \le\frac{1}{1+t^\sigma}||\phi||_1A_r'+\frac{1}{1+t^\sigma}\sum_{j=1}^{i}A_{r,j}+LA_r\frac{t^{1+k}}{1+t^\sigma}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^kdw\to 0 \text{ uniformly as } t\to+\infty.\end{array}$$

Hence we get that $\left\{t\to\frac{\delta(t)}{1+t^\sigma}(Tx)(t)):x\in\overline{\Omega}\right\}$ is equi-convergent as $t\to+\infty$.

So $T(\overline{\Omega})$ is relatively compact in X. Then T is completely continuous. The proofs are completed. □

3 Main results

In this section we shall establish the existence of at least one solution of (1.10).

Theorem 3.1 *Suppose that (a)-(d) hold. Furthermore, suppose that there exist nonnegative numbers $c_f, b_f, C_F, B_F, C_{I,k}, B_{I,k}$ such that*

$$\begin{array}{l}\left|f\left(t,\frac{1+t^\sigma}{(t-t_i)^{1-\alpha}}U\right)\right|\le c_f+b_f|U|, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0, U\in\mathbb{R},\\ \left|F\left(t,\frac{1+t^\sigma}{(t-t_i)^{1-\alpha}}U\right))\right|\le C_F+B_F|U|, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0, U\in\mathbb{R},\\ \left|I\left(t_i,\frac{1+t_i^\sigma}{(t_i-t_{i-1})^{1-\alpha}}U\right)\right|\le C_{I,i}+B_{I,i}|U|, i\in\mathbb{N}, U\in\mathbb{R},\\ \sum\limits_{j=1}^{+\infty}C_{I,j},\ \sum\limits_{j=1}^{+\infty}B_{I,j}\ \text{are convergent.}\end{array}\tag{3.1}$$

Then (1.4) has at least one solution if

$$L\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}b_f+||\phi||_1B_F+\sum_{j=1}^{+\infty}B_{I,j}<1.\tag{3.2}$$

Proof To apply Lemma 2.1, we should define an open bounded subset Ω of X centered at zero such that assumptions in Lemma 2.1 hold.

Let $\Omega_1=\{x\in X:\ x=\lambda Tx \text{ for some } \lambda\in(0,1)\}$. We prove that Ω_1 is bounded. For $x\in\Omega_1$, we get $x=\lambda T(x)$. So

$$\begin{aligned}x(t)&=\lambda(Tx)(t)\\ &=\lambda t^{\alpha-1}\int_0^{+\infty}\phi(s)F(s,x(s))ds+\lambda\sum_{j=1}^{i}(t-t_j)^{\alpha-1}I(t_j,x(t_j))\\ &\quad+\lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s))ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0.\end{aligned}\tag{3.3}$$

It is easy to see from (i) that

$$\begin{aligned}|f(t,x(t))|&=\left|f\left(t,\frac{1+t^\sigma}{(t-t_i)^{1-\alpha}}\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}x(t)\right)\right|\\ &\le c_f+b_f\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}|x(t)|\\ &\le c_f+b_f||x||, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0.\end{aligned}$$

Similarly we get

$$|F(t,x(t))|\le C_F+B_F||x||, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0,$$

$$|I(t_i,x(t_i))|\le C_{I,i}+B_{I,i}||x||, i\in\mathbb{N}.$$

From (3.3), note $\sigma>\max\{1-\alpha,k+1\}$, for $t\in(t_i,t_{i+1}]$, we have

$$\frac{\delta(t)}{1+t^\sigma}|x(t)|\leq\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}|(Tx)(t)|\leq\frac{t^{1-\alpha}}{1+t^\sigma}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,x(s))|du$$

$$+\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\int_0^\infty|\phi(s)F(s,x(s))|ds+\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\sum_{s=1}^{i}(t-t_s)^{\alpha-1}|I_s(t_s,x(t_s))|$$

$$\leq\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}Ls^k\left[c_f+b_f||x||\right]ds+\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\int_0^{+\infty}|\phi(s)|\left[C_F+B_F||x||\right]ds$$

$$+\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\sum_{s=1}^{i}(t-t_s)^{\alpha-1}\left[C_{I,s}+B_{I,s}||x||\right]$$

$$\leq L\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}t^{\alpha+k}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^kdw\left[c_f+b_f||x||\right]+\left(\frac{t-t_i}{t}\right)^{1-\alpha}\frac{t^{1-\alpha}}{1+t^\sigma}||\phi||_1\left[C_F+B_F||x||\right]$$

$$+\sum_{s=1}^{i}\frac{(t-t_i)^{1-\alpha}}{(t-t_s)^{1-\alpha}}\left[C_{I,s}+B_{I,s}||x||\right]$$

$$\leq L\frac{(t-t_i)^{1-\alpha}}{t^{1-\alpha}}\frac{t^{1+k}}{1+t^\sigma}\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}\left[c_f+b_f||x||\right]+||\phi||_1\left[C_F+B_F||x||\right]+\sum_{s=1}^{i}\left[C_{I,s}+B_{I,s}||x||\right]$$

$$\leq L\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}c_f+||\phi||_1C_F+\sum_{s=1}^{+\infty}C_{I,s}+\left[L\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}b_f+||\phi||_1B_F+\sum_{s=1}^{+\infty}B_{I,s}\right]||x||.$$

It follows from (3.2) that

$$||x||\leq\frac{L\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}c_f+||\phi||_1C_F+\sum_{s=1}^{+\infty}C_{I,s}}{1-\left(L\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}b_f+||\phi||_1B_F+\sum_{s=1}^{+\infty}B_{I,s}\right)}.\tag{3.4}$$

It follows that Ω_1 is bounded.

To apply Lemma 2.1, let

$$\Omega=\left\{x\in X:||x||<\frac{L\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}c_f+||\phi||_1C_F+\sum_{s=1}^\infty t_s^{1-\alpha}C_{I,s}}{1-\left(L\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}b_f+||\phi||_1B_F+\sum_{s=1}^\infty t_s^{1-\alpha}B_{I,s}\right)}+1\right\}.$$

Then Ω is a non-empty open bounded subset of X and $\Omega\supset\overline{\Omega_1}$ centered at zero.

It is easy to see from Lemma 2.3 that T is a completely continuous operator. One can see from (3.4) that

$$x\neq\lambda Tx\text{ for all }x\in\partial\Omega\text{ and }\lambda\in(0,1).$$

Thus, from Lemma 2.1, $x=Tx$ has at least one solution $x\in\overline{\Omega}$. So x is a solution of (1.4). The proof of Theorem 3.1 is complete. $\square$

For easy referencing, we list the conditions needed as follows:

(C)$_\mu$ there exist positive number $\mu > 0$ and positive functions ψ_i, Ψ_i $(i = 1, 2), K_s$ $(s = 1, 2, \cdots)$ and real numbers $L_i \geq 0 (i = 1, 2), J_s (s = 1, 2, \cdots)$ such that

$$\left| f\left(t, \tfrac{1+t^\sigma}{(t-t_i)^{1-\alpha}} U\right) - \psi_1(t) \right| \leq L_1 |U|^\mu, \quad t \in (t_i, t_{i+1}], \ i \in \mathbb{N}_0, \ U \in \mathbb{R},$$

$$\left| F\left(t, \tfrac{1+t^\sigma}{(t-t_i)^{1-\alpha}} U\right) - \Psi_1(t) \right| \leq L_2 |U|^\mu, \quad t \in (t_i, t_{i+1}], \ i \in \mathbb{N}_0, \ U \in \mathbb{R},$$

$$\left| I\left(t_i, \tfrac{1+t_i^\sigma}{(t_i-t_{i-1})^{1-\alpha}} U\right) - K_i \right| \leq J_i |U|^\mu, \quad i \in \mathbb{N}, \ U \in \mathbb{R}, \ \sum_{i=1}^{+\infty} J_i < \infty.$$

Theorem 3.2 *Suppose that (a)-(e), (A)-(B) hold. Then, (1.4) has at least one solution if (C)$_\mu$ holds for*

(i) $\mu > 1$ *and*

$$\frac{\|\psi_0\|^{1-\mu}(\mu-1)^{\mu-1}}{\mu^\mu} \geq \mathbf{B}(\alpha, k+1) L L_1 + ||\phi||_1 L_2 + \sum_{s=1}^{+\infty} J_s$$

where

$$\psi_0(t) = \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)\psi_1(s) ds + t^{\alpha-1} \int_0^\infty \phi(s)\Psi_1(s) ds$$
$$+ \sum_{s=1}^{i} K_s (t-t_s)^{\alpha-1}, t \in (t_i, t_{i+1}], i \in N_0. \qquad (3.5)$$

(ii) $\sigma \in (0, 1)$, *or*

(iii) $\sigma = 1$ *and* $\mathbf{B}(\alpha, k+1) L L_1 + ||\phi||_1 L_2 + \sum_{s=1}^{+\infty} J_s < 1.$

Proof Let the Banach space X and its norm be defined as in Section 2. Define the nonlinear operator T by (2.6).

We have from Lemma 2.3 that $T : X \to X$ is well defined and is completely continuous.

Let ψ_0 be defined by (3.5). It is easy to show that $\psi_0 \in X$. Let $r > 0$ and define $M_r = \{x \in X : \|x - \psi_0\| \leq r\}$.

For $x \in M_r$, we find for $t \in (t_i, t_{i+1}]$

$$\tfrac{(t-t_i)^{1-\alpha}}{1+t^\sigma} |(Tx)(t) - \psi_0(t)| = \tfrac{(t-t_i)^{1-\alpha}}{1+t^\sigma} \left| \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)[f(s, x(s)) - \psi_1(s)] ds \right.$$

$$\left. + t^{\alpha-1} \int_0^\infty \phi(s)[F(s, x(s)) - \Psi_1(s)] ds + \sum_{s=1}^{i} [I_s(t_s, x(t_s)) - J_s(t_s)](t-t_s)^{\alpha-1} \right|$$

$$\leq \sup_{t \in (0,\infty)} \tfrac{t^{1-\alpha}}{1+t^\sigma} \left[\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |m(s)||f(s, x(s)) - \psi_1(s)| ds \right.$$

$$\left. + t^{\alpha-1} \left(\int_0^\infty |\phi(s)||F(s, x(s)) - \Psi_1(s)| ds + \sum_{s=1}^k t_s^{1-\alpha} |I_s(t_s, x(t_s)) - J_s(t_s)| \right) \right]$$

$$\leq \frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}LL_1 s^k||x||^\mu ds$$

$$+\frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}t^{\alpha-1}||\phi||_1L_2||x||^\mu + \frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\sum_{s=1}^{i}(t-t_s)^{\alpha-1}J_s||x||^m u$$

$$\leq \left(\frac{\mathbf{B}(\alpha,k+1)LL_1}{\Gamma(\alpha)} + ||\phi||_1L_2 + \sum_{s=1}^{+\infty}J_s\right)||x||^\mu.$$

Case (i) $\mu > 1$. Let $r = r_0 = \frac{\|\psi_0\|}{\mu-1}$. By assumption,

$$\frac{r_0}{(r_0+\|\psi_0\|)^\mu} = \frac{\|\psi_0\|^{1-\mu}(\mu-1)^{\mu-1}}{\mu^\mu} \geq \mathbf{B}(\alpha,k+1)LL_1 + ||\phi||_1L_2 + \sum_{s=1}^{\infty}t_s^{1-\alpha}J_s.$$

Then, for $x \in M_{r_0}$ we have

$$\|Tx-\psi_0\| \leq \left(\mathbf{B}(\alpha,k+1)LL_1 + ||\phi||_1L_2 + \sum_{s=1}^{\infty}t_s^{1-\alpha}J_s\right)(r_0+\|\psi_0\|)^\mu \leq r_0.$$

Hence, we have a bounded subset $M_{r_0} \subseteq X$ such that $T(M_{r_0}) \subseteq M_{r_0}$. Then, Schauder fixed point theorem implies that T has a fixed point $x \in M_{r_0}$. Hence, x is a bounded solution of (1.4).

Case (ii) $\mu \in (0,1)$. Choose $r > 0$ sufficiently large such that

$$\left(\mathbf{B}(\alpha,k+1)LL_1 + ||\phi||_1L_2 + \sum_{s=1}^{\infty}t_s^{1-\alpha}J_s\right)(r+\|\psi_0\|)^\sigma \leq r.$$

Then, for $x \in M_r$ we have

$$\|Tx-\psi_0\| \leq \left(\mathbf{B}(\alpha,k+1)LL_1 + ||\phi||_1L_2 + \sum_{s=1}^{\infty}t_s^{1-\alpha}J_s\right)(r+\|\psi_0\|)^\sigma \leq r.$$

So $T(M_r) \subseteq M_r$ and Schauder fixed point theorem implies that T has a fixed point $x \in M_r$. This x is a bounded solution of (1.4).

Case (iii) $\mu = 1$. We choose

$$r \geq \frac{\mathbf{B}(\alpha,k+1)LL_1 + ||\phi||_1L_2 + \sum_{s=1}^{\infty}t_s^{1-\alpha}J_s}{1-\left(\mathbf{B}(\alpha,k+1)LL_1 + ||\phi||_1L_2 + \sum_{s=1}^{\infty}t_s^{1-\alpha}J_s\right)}.$$

Then, for $x \in M_r$ we have

$$\|Tx-\psi_0\| \leq \left(\mathbf{B}(\alpha,k+1)LL_1 + ||\phi||_1L_2 + \sum_{s=1}^{\infty}t_s^{1-\alpha}J_s\right)(r+\|\psi_0\|) \leq r.$$

Hence, as in earlier cases we conclude that T has a fixed point $x \in M_r$, which is a bounded solution of (1.4).

From above discussion, the proof is complete. $\square$

4 An example

To illustrate the usefulness of our main results, we present an example that Theorem 3.1 can be readily applied.

Example 4.1 Consider the following impulsive problem

$$\begin{cases} D_{0^+}^{\frac{2}{5}} u(t) = t^{-\frac{1}{2}} \left[c_0 + b_0 \frac{(t-t_i)^{3/5}}{1+t^{2/3}} u(t) \right], t \in (t_i, t_{i+1}], i \in \mathbb{N}_0, \\ \lim\limits_{t \to 0} t^{\frac{1}{3}} u(t) = B_0 \int_0^\infty e^{-s} \frac{s^{3/5}}{1+s^{2/3}} u(s) ds, \\ \Delta u(t_i) = \lim\limits_{t \to t_i^+} u(t) - \lim_{t \to t_i^-} u(t) = 2^{-i}, i \in \mathbb{N}, \end{cases} \tag{4.1}$$

where c_0, b_0, B_0 are constants and $t_i = i (i \in \mathbb{N})$.

Corresponding to (1.4), we have
(a) $\alpha = \frac{2}{5}$,
(b) $0 = t_0 < t_1 = 1 < \cdots < t_k = k < \cdots$ with $\lim_{k \to \infty} t_k = \infty$,
(c) $m(t) = t^{-\frac{1}{2}}$ satisfies that $|m(t)| \le Lt^k$ with $L = 1$, $k = -\frac{1}{2}$,
(d) $\phi(t) = e^{-t}$ satisfies $\phi \in L^1(0, \infty)$ and,
(e) f, F, I are defined by

$$f(t, x) = c_0 + b_0 \frac{t^{3/5}}{1+t^{2/3}} x,$$

$$F(t, x) = B_0 \frac{s^{3/5}}{1+s^{2/3}} x,$$

$$I(t_i, x) = 2^{-i} (i = 1, 2, \cdots).$$

Choose $\sigma = \frac{2}{3}$. Then $\sigma > \max\{k+1, 1-\alpha\}$. It is easy to show that
(A). f, F are Carathéodory functions.

(B). I is a discrete Carathéodory function.

It is easy to see that the inequalities

$$\left| f\left(t, \frac{1+t^\sigma}{(t-t_i)^{1-\alpha}} U\right) \right| \le c_f + b_f |U|,$$

$$\left| F\left(t, \frac{1+t^\sigma}{(t-t_i)^{1-\alpha}} U\right) \right| \le C_F + B_F |U|,$$

$$\left| I\left(t_i, \frac{1+t_i^\sigma}{(t_i-t_{i-1})^{1-\alpha}} U\right) \right| \le C_{I,i} + B_{I,i} |U|,$$

$$\sum_{s=1}^{+\infty} C_{I,s}, \quad \sum_{s=1}^{+\infty} B_{I,s} \text{ are convergent.}$$

hold for all $U \in R$, $t \in (0, \infty)$ with $c_f = |c_0|, b_f = |b_0|$, $C_F = 0, B_F = |B_0|$, $C_{I,s} = 2^{-s}, B_{I,s} = 0 (s = 1, 2, \cdots)$.

Then Theorem 3.1 implies that (4.1) has at least one solution if

$$\frac{\mathbf{B}(2/5, 1/2)}{\Gamma(2/5)}|b_0| + |B_0| < 1. \tag{4.2}$$

Remark 4.1 It is easy to see that (4.1) has at least one solution for sufficiently small $|B_0|$ and $|b_0|$.

Acknowledgement

The author was supported by the Natural Science Foundation of Guangdong province (No:S2011010001900) and the Foundation for High-level talents in Guangdong Higher Education Project.

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