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# Analysis of Derivative Free Rational Scheme

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**Abstract** In this research paper we present analysis of the derivative free rational one step scheme for solving initial value problems (IVPs) of first order Ordinary differential Equations (ODEs). The scheme is consistent and stability property resembles of the trapezoidal method, which is A-stable. This method has been applied to problems with singularities and the one which are considered to be stiff. Numerical results show that the scheme is suitable for solving both stiff problems and the one whose solutions possess singularities.

**Keywords** Rational Methods;Ordinary Differential Equations;step size;Initial Value Problems;stiffness;Singularities

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## 1 Introduction

In this research work, we shall consider numerical methods for solving initial value problems (IVPs) for ordinary differential equations (ODEs). These are usually written in the form

$$\frac{dy}{dx} = f(x,y), \quad a \leq x \leq b, \quad y(0) = y_0, \quad f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \tag{1}$$

In the literature, classical methods for solving first order ordinary differential equations are based on polynomial interpolation in $h$. Some of these methods are the well documented Runge Kutta methods introduced by Runge, improved further by Kutta [1] and the commonly known Linear Multistep methods (LMM) such as backward differentiation formulas (BDFs). According to Ikhile [2], Van Niekerk [3] and Teh *et al.* [4], [5] these methods are said to perform poorly when the solution of the initial value Problem possess singularities. These authors presented different alternative methods based on rational interpolation which overstep the singular point smoothly. These rational methods mostly depend on calculating higher derivatives of the state function. A derivative free rational scheme was proposed in [6] which was a modification to Van Niekerk' one step order two scheme presented in [5]. The main aim of this paper is to analyse the theoretical properties of the scheme proposed in [6] for issues of consistency, stability and suitability for solving stiff IVPs.

## 2 Derivative Free Rational Scheme

The derivative free rational scheme that was proposed in [6] is given by;

$$y_{n+1} = y_n + \frac{2h(f(x_n,y_n))^2}{3f(x_n,y_n) - f(x_n+h, y_n + hf(x_n,y_n))} \tag{2}$$

which is a modification of Van Niekerk's method presented as

$$y_{n+1} = y_n + \frac{2h(y'_n)^2}{2y'_n - hy''_n} \tag{3}$$

where the denominator in (3), has been approximated as

$$\begin{aligned} 2y'_n - hy''_n &= 3f(x_n, y_n) - [f(x_n, y_n) + hf'(x_n, y_n)] \\ &\approx 3f(x_n, y_n) - f(x_n + h, y_n + hf(x_n, y_n)). \end{aligned}$$

## 3 Local Truncation error

In this section, the local truncation error of the proposed scheme is presented.

**Definition 1** The local truncation error at $x_{n+1}$ of the general explicit one step method is defined to be $T_{n+1}$, where;

$$T_{n+1} = y(x_{n+1}) - y(x_n) - h\phi(x_n, y(x_n), h) \tag{4}$$

and $y(x_n)$ is the theoretical solution of the initial value problem.

If we make the localizing assumption that no previous errors have been made, then $y_n = y(x_n)$. It follows that local truncation error of one step method is given by

$$T_{n+1} = y(x_{n+1}) - y_{n+1}. \tag{5}$$

**Definition 2** The method (1) is said to be of order $p$ if $p$ is the largest integer for which $T_{n+1} = O(h^{p+1})$ for every $n$ and $p \geq 1$.

If $y(x)$ is assumed to be sufficiently differentiable, the local truncation error for non linear one step method is given by

$$T_{n+1} = \psi(x_n, y(x_n))h^{p+1} + O(h^{p+2}) \tag{6}$$

where $\psi(x, y)$ is the principal error function and $\psi(x_n, y(x_n))h^{p+1}$ is the principal local error, Lambert [7].

Following definition given above, we define the local truncation error of (2) at $x_{n+1}$ to be the residual when $y_{n+1}$ is replaced by $y(x_{n+1})$; that is

$$T_{n+1} = [y(x_{n+1}) - y(x_n)](aw_1 + bw_2) - 2h(w_1)^2 \tag{7}$$

where

$$\begin{aligned} w_1 &= f(x_n, y(x_n)) \\ w_2 &= f(x_n + \mu h, y(x_n) + \mu hf(x_n, y(x_n))). \end{aligned}$$

Expanding $y(x_{n+1})$ using Taylor series and substituting into equation (7), we get

$$\begin{aligned} T_{n+1} &= [hy'(x_n) + \frac{1}{2}h^2y''(x_n) + \frac{1}{6}h^3y'''(x_n) + O(h^4)][ay'(x_n) \\ &\quad +bf(x_n + \mu h, y(x_n) + \mu hf(x_n, y(x_n)))] - 2h(y'(x_n))^2. \end{aligned} \quad (8)$$

When we expand the term

$$f(x_n + \mu h, y(x_n) + \mu hf(x_n, y(x_n)))$$

using bivariate Taylor's series and using the relations

$$\begin{aligned} y' &= f(x, y) \\ y'' &= f_x(x, y) + f_y(x, y)y', \end{aligned}$$

we get

$$\begin{aligned} T_{n+1} &= \left(hy'(x_n) + \frac{1}{2}h^2y''(x_n) + \frac{1}{6}h^3y'''(x_n) + O(h^4)\right)(ay' + b[f + \mu hf_x + \mu hy'f_y]) \\ &\quad - 2h(y'(x_n))^2 \\ &= (hy'(x_n) + \frac{1}{2}h^2y''(x_n)\frac{1}{6}h^3y'''(x_n) + O(h^4))(ay'(x_n) + b(y'(x_n) + \mu hy''(x_n)) \\ &\quad - 2h(y'(x_n))^2 \\ &= (a + b - 2)h(y')^2 + \left(\frac{a+b}{2} + \mu b\right)y'(x_n)y''(x_n)h^2 \\ &\quad + \left(\frac{1}{2}b\mu(y''(x_n))^2 + \frac{(a+b)}{6}y'(x_n)y'''(x_n)\right)h^3 + O(h^4). \end{aligned} \quad (9)$$

Where $y$ and its derivatives are evaluated at $x_n$ and $f$ and its derivatives are evaluated at $(x_n, y(x_n))$.
We observe that the local truncation error in (9) implies that the general one step scheme is of order two, where the principal local error is given by;

$$\left(\frac{1}{2}b\mu(y''(x_n))^2 + \frac{(a+b)}{6}y'(x_n)y'''(x_n)\right).$$

For the general function $f$, constants $a$, $b$ and $\mu$ cannot be chosen such that the $O(h^3)$ term disappear. Hence we must choose these parameters so that the terms $O(h)$ and $O(h^2)$ are zero. The proposed derivative free scheme would be satisfied when

$$a + b = 2$$

and

$$a + b + 2\mu b = 0$$

which implies that $\mu b = -1$, where $\mu \in [0, 1]$.
In case where $\mu = 0$ we realize that the scheme (2) collapse to the Euler' one step method .

To illustrate this proposal, consider the case where $a = 102$, $b = -100$ and $\mu = 0.01$. In this case the proposed one step scheme will be;

$$y_{n+1} = y_n + \frac{2h(f(x_n, y_n))^2}{102f(x_n, y_n) - 100f(x_n + \mu h, y_n + \mu hf(x_n, y_n))} \tag{10}$$

The general derivative free rational one step scheme is given by;

$$y_{n+1} = y_n + 2h\frac{(f(x_n, y_n))^2}{af(x_n, y_n) + bf(x_n + \mu h, y_n + \mu hf(x_n, y_n))} \tag{11}$$

where $\mu = \frac{-1}{b}$. In case of equation (2), the local truncation error would be given by

$$T_{n+1} = \left(-\frac{1}{2}(y'')^2 + \frac{1}{3}y'y'''\right)h^3 + O(h^4) \tag{12}$$

where $a = 3, \quad b = -1$ and $\mu = 1$.

## 4 Properties of the scheme

In this section we analyse the consistency and stability properties of the method proposed.

### 4.1 Consistency Property

A scheme is said to be consistent if the difference equation of the computation formula exactly approximate the differential equation it intends to solve as the step size tends to zero. To prove consistency property of our proposed scheme (2), we subtract $y_n$ both sides and divide by $h$ to get;

$$\frac{y_{n+1} - y_n}{h} = \frac{2(f(x, y))^2}{3f(x, y) - f(x + h, y + hf(x, y))}. \tag{13}$$

Taking the limit as $h \to 0$ on both sides of (13) we have

$$\lim_{h\to 0} \frac{y_{n+1} - y_n}{h} = \lim_{h\to 0} \frac{2(f(x, y))^2}{3f(x, y) - f(x + h, y + hf(x, y))} \longrightarrow f(x, y) = y'(x, y). \tag{14}$$

which indicates that the scheme satisfy the consistency property, hence it implies that it converges.

### 4.2 Stability

In order to examine the stability for the proposed scheme, let us consider the differential equation;

$$y' = \lambda y, \quad Re(\lambda) < 0.$$

From this, equation (2) can be expressed as

$$y_{n+1} = y_n + \frac{2h(\lambda y_n)^2}{3\lambda y_n - \lambda(1 + \lambda h)y_n} = \left(\frac{2 + \lambda h}{2 - \lambda h}\right)y_n. \tag{15}$$

Setting $z = \lambda h$ in the above equation, the amplification factor is therefore;

$$R(z) = \frac{2+z}{2-z}. \tag{16}$$

Plotting the stability function (16), we have the region as described by Figure 1. This stability region has the modulus less than one on the left-half complex plane and thus the method (2) is A-stable.

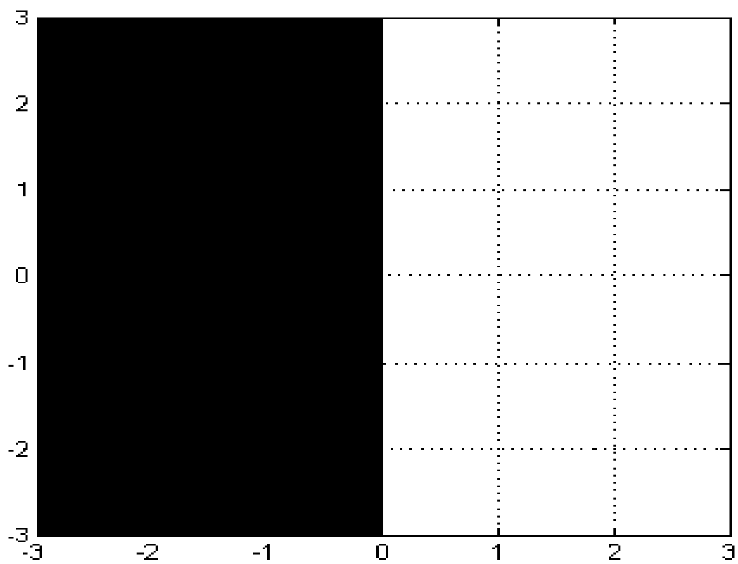


Figure 1: Stability Region for the Proposed Rational Scheme (2)

## 5 Numerical Results

In our numerical results were apply the proposed scheme to a problem with singularities and the one which is stiff, where comparison is made with Van Niekerk method (3).

### 5.1 Solution near singularity

The nonlinear initial value problem in consideration is given by,

$$y' = 1 + y^2, y(0) = 1$$

where the theoretical solution is

$$y(x) = \tan(x + \pi/4).$$

In Table 1, we compare the methods before the point of singularity and the behaviour after a point of singularity. In this case, the methods were applied with constant step size of $h = 0.01$ and the errors are absolute errors.

Table 1: Absolute Errors for Solution with Singularity ,$y' = 1 + y^2, y(0) = 1, h = 0.01$

| x | y(x) | Van Niekerk(3) | Derivative Free(11) $\mu = 0.1$ | Derivative Free(11) $\mu = 0.001$ |
|---|---|---|---|---|
| 0.1 | 1.223048880449865 | 2.07655E-04 | 1.41959E-04 | 2.07589E-04 |
| 0.2 | 1.508497647121400 | 5.44973E-04 | 3.49859E-04 | 5.44778E-04 |
| 0.3 | 1.895765122854009 | 1.14622E-03 | 6.68959E-04 | 1.14574E-03 |
| 0.4 | 2.464962756722603 | 2.35322E-03 | 1.16834E-03 | 2.35203E-03 |
| 0.5 | 3.408223442335828 | 5.24136E-03 | 1.84997E-03 | 5.23797E-03 |
| 0.6 | 5.331855223458727 | 1.46533E-02 | 1.07251E-03 | 1.46398E-02 |
| 0.7 | 11.681373800310254 | 7.95207E-02 | 5.65539E-02 | 7.93873E-02 |
| 0.75 | 28.238252850141599 | 4.89624E-01 | 1.03306E+00 | 4.88214E-01 |
| 0.80 | -68.479668345576044 | 3.27142E+00 | 2.26089E+01 | 3.23726E+00 |

### 5.2 A Stiff Equation

The test problem considered is the one given by Frank and Ueberhuber [3] which is described as,

$$y' = \lambda(y - g(x)) + g'(x)$$

with $x \in [0, 1], g(0) = 3, g(x) = \sin(0.1x) + 2$, where $\lambda$ is the stiffness ratio and the analytic solution is given by;

$$y(x) = g(x) + (y(0) - g(0))e^{\lambda x}.$$

Table 2 considers a mildly stiff case, where $\lambda = -10$ and $h = 0.01$ while Table 3 is a case where the problem is very stiff, in which $\lambda = -10^3$ and $h = 0.001$.

## 6 Conclusion

In this study, we have presented analysis of derivative free rational one step scheme. The theoretical analysis shows that the method is consistent and also A-stable. The numerical results in Table 1, indicates that our proposed scheme produces comparable results to those of Van Niekerk for solving problems whose solutions possess singularities as it overstep singular point at $x = \frac{\pi}{4}$. Similarly, the scheme indicates to be capable of solving stiff differential equations as presented in Table 2 and 3. The proposed scheme produces solutions of better accuracy for $\mu \ll 1$. The novelty of the proposed method is that one does not need to calculate the first derivative of the function when solving first order ordinary differential equations. For further test of the scheme one would need to analyse the performance of the proposed approach when solving stiff systems of differential equations.

Table 2: Absolute Errors for Mildly Stiff Problem, $\lambda = -10, h = 0.01$

| x | y(x) | Van Niekerk(3) | Derivative Free (11) $\mu = 0.1$ | Derivative Free (11) $\mu = 0.001$ |
|---|---|---|---|---|
| 0.1 | 2.377879274505609 | 3.222E-04 | 3.22212E-04 | 3.22212E-04 |
| 0.2 | 2.155333949929946 | 2.471E-04 | 2.47113E-04 | 2.47113E-04 |
| 0.3 | 2.079782568570360 | 1.496E-04 | 1.49576E-04 | 1.49576E-04 |
| 0.4 | 2.058304973075368 | 9.345E-05 | 9.34461E-05 | 9.34452E-05 |
| 0.5 | 2.056717116269764 | 5.755E-05 | 5.75564E-05 | 5.75549E-05 |
| 0.6 | 2.062442758656111 | 1.013E-05 | 1.01357E-05 | 1.01340E-05 |
| 0.7 | 2.070854729303087 | 1.588E-06 | 1.58957E-06 | 1.58767E-06 |
| 0.8 | 2.080250156597075 | 7.819E-08 | 7.60553E-08 | 7.81668E-08 |
| 0.9 | 2.090001959002098 | 2.591E-07 | 2.56736E-07 | 2.59084E-07 |
| 1.0 | 2.099878816576590 | 1.791E-07 | 1.76437E-07 | 1.79033E-07 |

Table 3: Absolute Errors for Highly Stiff Problem $\lambda = -10^3, h = 0.001$

| x | y(x) | Van Niekerk(3) | Derivative Free(11) $\mu = 1$ | Derivative Free (11) $\mu = 0.001$ |
|---|---|---|---|---|
| 0.01 | 2.001045399763096 | 3.236E-05 | 3.236E-05 | 3.236E-05 |
| 0.02 | 2.002000000727820 | 1.281E-08 | 1.280E-08 | 1.281E-08 |
| 0.03 | 2.002999995500096 | 1.476E-11 | 9.015E-14 | 1.495E-11 |
| 0.04 | 2.003999989333342 | 3.402E-13 | 1.932E-11 | 5.995E-14 |
| 0.1 | 2.009999833334167 | 3.402E-13 | 4.933E-11 | 1.701E-13 |
| 0.3 | 2.029995500202496 | 3.300E-13 | 1.493E-10 | 1.701E-13 |
| 0.5 | 2.049979169270678 | 3.402E-13 | 2.492E-10 | 1.701E-13 |
| 0.7 | 2.069942847337533 | 3.300E-13 | 3.490E-10 | 1.701E-13 |
| 0.9 | 2.089878549198011 | 3.300E-13 | 4.487E-10 | 1.696E-13 |
| 1.0 | 2.099833416646828 | 3.402E-13 | 4.985E-10 | 1.599E-13 |

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