Direct and inverse theorems of approximation theory in $L^2(\mathbb{R}^d, w_l(x)dx)$

Radouan Daher, Salah El Ouadih and Mohamed El Hamma

Department of Mathematics, Faculty of Sciences Ain Chock, University Hassan II, Casablanca, Morocco

e-mail: 1 rjdaher024@gmail.com, 2 salahwadih@gmail.com, 3 m-elhamma@yahoo.fr

Abstract In this paper, we prove analogues of direct and some inverse theorems for the Dunkl harmonic analysis, using the function with bounded spectrum and generalized spherical mean operator.

Keywords Generalized continuity modulus; Bernstein theorem; Jackson theorem; best approximation.

AMS Mathematics Subject Classification 42B37, 42B10

1 Introduction

Yet by the year 1912, S. Bernstein obtained the estimate inverse to Jakson’s inequality in the space of continuous functions for some special cases [1], later Stechkin [2], Timan [3], proved such inverse estimates, including the case of the space $L_p$, $1 < p < \infty$.

Approximation problems for functions in the space $L^2(\mathbb{R}^d, w_l(x)dx)$, where $w_l$ is a weight function invariant under the action of an associated reflection groups, using the function with bounded spectrum, are studied in this paper. Applying the Dunkl transform, Dunkl Laplacian operator and generalized spherical mean operator, we obtain analogs of the Bernstein inequality for function with bounded spectrum, direct and inverse theorem of Jackson type [4],[2],[5], where the modulus of smoothness is constructed on the basis of generalized spherical mean operator.

2 The Dunkl transform and its basic properties

Dunkl [3] defined a family of first-order differential-difference operators related to some reflection groups. These operators generalize in a certain manner the usual differentiation and have gained considerable interest in various fields of mathematics and also in physical applications. The theory of Dunkl operators provides generalizations of various multivariable analytic structures. Among others, we cite the exponential function, the Fourier transform and the translation operator. For more details about these operators see [6], [7], [1] and [8].

Let $R$ be a root system in $\mathbb{R}^d$, $W$ the corresponding reflection group, $R_+$ a positive subsystem of $R$ and $l$ a non-negative and $W$-invariant function defined on $R$. The Dunkl operator is defined for $f \in C^1(\mathbb{R}^d)$ by

$$D_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} l(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, x \in \mathbb{R}^d.$$ 

Here $\langle , \rangle$ is the usual Euclidean scalar product on $\mathbb{R}^d$ with the associated norm $|.|$ and $\sigma_\alpha$ the reflection with respect to the hyperplane $H_\alpha$ orthogonal to $\alpha$. We consider the weight
function
\[ w_l(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2l(\alpha)}, \]
where \( w_l \) is \( W \)-invariant and homogeneous of degree \( 2\gamma \) where
\[ \gamma = \sum_{\alpha \in R_+} l(\alpha). \]

We let \( \eta \) be the normalized surface measure on the unit sphere \( S^{d-1} \) in \( \mathbb{R}^d \) and set
\[ d\eta_l(y) = w_l(y) d\eta(y). \]

Then \( \eta_l \) is a \( W \)-invariant measure on \( S^{d-1} \), and we let \( d\lambda = \eta_l(S^{d-1}) \).

The Dunkl kernel \( E_l \) on \( \mathbb{R}^d \times \mathbb{R}^d \) has been introduced by Dunkl in [9]. For \( y \in \mathbb{R}^d \) the function \( x \mapsto E(x, y) \) can be viewed as the solution on \( \mathbb{R}^d \) of the following initial problem:
\[
\begin{cases}
D_j u(x, y) = y_j u(x, y), & \text{if } 1 \leq j \leq d, \\
u(0, y) = 0, & \text{for all } y \in \mathbb{R}^d.
\end{cases}
\]

This kernel has a unique holomorphic extension to \( \mathbb{C}^d \times \mathbb{C}^d \).

Rösler has proved in [8] the following integral representation for the Dunkl kernel,
\[ E_l(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\mu_x(y), \quad x \in \mathbb{R}^d, \quad z \in \mathbb{C}^d, \]
where \( \mu_x \) is a probability measure on \( \mathbb{R}^d \) with support in the closed ball \( B(0, |x|) \) of center 0 and radius \( |x| \).

**Proposition 1** [6] Let \( z, w \in \mathbb{C}^d \) and \( \lambda \in \mathbb{C} \)
(i) \( E_l(z, 0) = 1, E_l(z, w) = E_l(w, z), E_l(\lambda z, w) = E_l(z, \lambda w) \).
(ii) For all \( \nu = (\nu_1, ..., \nu_d) \in \mathbb{N}, x \in \mathbb{R}^d, z \in \mathbb{C}^d \), we have
\[ |D^\nu z E_k(x; z)| \leq |x|^{|\nu|} \exp(|x||Rez|), \]
where
\[ D^\nu z = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} ... \partial x_d^{\nu_d}} |\nu| = \nu_1 + ... + \nu_d. \]

In particular \( |D^\nu z E_l(ix; z)| \leq |x|^{|\nu|} \) for all \( x, z \in \mathbb{R}^d \).

We denote by \( L^2_l(\mathbb{R}^d) = L^2(\mathbb{R}^d, w_l(x) dx) \) the space of measurable functions on \( \mathbb{R}^d \) such that
\[ \|f\|_{2,l} = \left( \int_{\mathbb{R}^d} |f(x)|^2 w_l(x) dx \right)^{1/2}, \]
and \( D_l \) the Dunkl Laplacian defined by
\[ D_l = \sum_{i=1}^d D^2_j. \]
The scalar product in the Hilbert space $L_2^2(\mathbb{R}^d)$ obeys the formula

$$\langle f, g \rangle := \int_{\mathbb{R}^d} \overline{f(x)} g(x) w_1(x) dx, \quad f, g \in L_2^2(\mathbb{R}^d).$$

By the partial integration one can verify the correlation

$$\langle D_l f, g \rangle = \langle g, D_l f \rangle,$$

for any functions $f, g \in \mathcal{D}$ ($\mathcal{D}$ denotes the set of infinitely differentiable functions with a compact support).

As usual, we endow the space $\mathcal{D}$ with a topology; this turns it into a topological vector space [10]. Let $\mathcal{D}'$ stand for the set of generalized functions, i.e., linear continuous functionals on the space $\mathcal{D}$. We denote the value of a functional $f \in \mathcal{D}'$ on a function $\varphi \in \mathcal{D}$ by $\langle f, \varphi \rangle$. The space $L_2^2(\mathbb{R}^d)$ is embedded into $\mathcal{D}'$, provided that for $f \in L_2^2(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}$ we put

$$\langle f, \varphi \rangle := \int_{\mathbb{R}^d} f(x) \varphi(x) w_1(x) dx.$$

One can extend (in a natural way) the action of the Dunkl Laplacian operator $D_l$ onto the space of generalized functions $\mathcal{D}'$, putting

$$\langle D_l f, \varphi \rangle := \langle f, D_l \varphi \rangle, \quad f \in \mathcal{D}', \varphi \in \mathcal{D}.$$ 

In particular, the action of the operator $D_l f$ is defined for any function $f \in L_2^2(\mathbb{R}^d)$ but, generally speaking, $D_l f$ is a generalized function.

The Dunkl transform is defined for $f \in L_1^1(\mathbb{R}^d) = L^1(\mathbb{R}^d, w_1(x) dx)$

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = c_l^{-1} \int_{\mathbb{R}^d} f(x) E_i(-i\xi, x) w_1(x) dx,$$

where the constant $c_l$ is given by

$$c_l = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2}} w_1(x) dx.$$

The inverse Dunkl transform is defined by the formula

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) E_i(ix, \xi) w_1(\xi) d\xi, \quad x \in \mathbb{R}^d.$$ 

From [11], we have that if $f \in L_2^2(\mathbb{R}^d)$

$$D_l f(\xi) = -|\xi|^2 \widehat{f}(\xi).$$

The Dunkl transform shares several properties with its counterpart in the classical case. We mention here, in particular that Parseval theorem holds in $L_2^2(\mathbb{R}^d)$. As in the classical case, a generalized translation operator is defined in the Dunkl [5, 12]. Namely, for $f \in L_2^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we define $\tau_x f$ to be the unique function in $L_2^2(\mathbb{R}^d)$ satisfying

$$\widehat{\tau_x f}(y) = E_i(ix, y) \widehat{f}(y) \quad \text{a.e.} \quad y \in \mathbb{R}^d.$$
Form to Parseval theorem and proposition 1.1, we see that
\[ \| \tau_x f \|_{2, l} \leq \| f \|_{2, l} \text{ for all } x \in \mathbb{R}^d. \]
For \( \alpha > -\frac{1}{2} \), let \( j_\alpha(x) \) be a normalized Bessel function of the first kind, i.e.,
\[ j_\alpha(x) = \frac{2^{\alpha} \Gamma(\alpha + 1) J_\alpha(x)}{x^\alpha}, \]
where \( J_\alpha(x) \) is a Bessel function of the first kind [13].

The function \( J_\alpha(x) \) is infinitely differentiable, \( J_\alpha(0) = 1 \).

**Proposition 2** ([14] or [15]) For \( x \in \mathbb{R} \) the following inequalities are fulfilled
(i) \( | j_\alpha(x) | \leq 1 \).
(ii) \( | 1 - j_\alpha(x) | \geq c_\alpha \) with \( |x| \geq 1 \), where \( c_\alpha > 0 \) is a certain constant which depend only on \( \alpha \).
(iii) \( | 1 - j_\alpha(x) | \leq c_1 x^2 \), where \( c_1 \) is a constant.

The generalized spherical mean value of \( f \in L^2_1(\mathbb{R}^d) \) is defined by
\[ M_h f(x) = \frac{1}{d_l} \int_{S^{d-1}} \tau_x f(hy) d\eta(y), \quad x \in \mathbb{R}^d, \quad h > 0. \]
We have
\[ \|M_h f\|_{2, l} \leq \|f\|_{2, l}. \quad (2) \]

**Proposition 3** ([16]) Let \( f \in L^2_1(\mathbb{R}^d) \) and fix \( h > 0 \). Then \( M_h f \in L^2_1(\mathbb{R}^d) \) and
\[ \tilde{M}_h f(\xi) = j_{\alpha + \frac{d}{2} - 1}(h|\xi|) \hat{f}(\xi), \quad \xi \in \mathbb{R}^d. \quad (3) \]
Let the function \( f \in L^2_1(\mathbb{R}^d) \). We define differences of the order \( k \) (\( k = 1, 2, ... \)) with a step \( h > 0 \).
\[ \Delta^k_h f(x) = (I - M_h)^k f(x), \]
where \( I \) is the unit operator.
For any positive integer \( k \), we define the generalized module of smoothness of the \( k \)th order by the formula
\[ \omega_k(f, \delta)_{2, l} = \sup_{0 < h \leq \delta} \| \Delta^k_h f \|_{2, l}, \delta > 0. \]
Let \( W^k_{2, l} \) be the Sobolev space constructed by the operator \( D_l \), i.e.,
\[ W^k_{2, l} = \{ f \in L^2_1(\mathbb{R}^d) : D_l^j f \in L^2_1(\mathbb{R}^d); j = 1, 2, ..., k \}, \]
where \( D_l^0 f = f, D_l^j f = D_l(D_l^{j-1} f). \)
For any \( f \in L^2_1(\mathbb{R}^d) \) and any number \( \nu > 0 \), let us define the function
\[ P_\nu(f)(x) = \mathcal{F}^{-1}(\hat{f}(\xi)\chi_\nu(\xi)), \]
where \( \chi_\nu(\xi) = 1 \) if \( |\xi| \leq \nu \) and \( \chi_\nu(\xi) = 0 \) if \( |\xi| > \nu \), \( \mathcal{F}^{-1} \) is the inverse Dunkl transform.

One can easily prove that the function \( P_\nu(f)(x) \) is infinitely differentiable and belongs to all classes \( W^k_{2,l} \), \( k = 1, 2, \ldots \).

A function \( f \in L^2_{2}(\mathbb{R}^d) \) is called a function with bounded spectrum of order \( \nu > 0 \) if \( \hat{f}(\xi) = 0 \) for \( |\xi| > \nu \). The set of all such functions is denoted by \( \mathcal{I}_\nu \).

The best approximation of a function \( f \in L^2_{2}(\mathbb{R}^d) \) by functions in \( \mathcal{I}_\nu \) is the quantity

\[
E_\nu(f)_{2,l} := \inf_{g \in \mathcal{I}_\nu} \|f - g\|_{2,l}.
\]

### 3 Bernstein’s inequality and Jackson’s direct theorems

**Bernstein’s Theorem 1**

If \( f \in \mathcal{I}_\nu \), then \( D_l f \in \mathcal{I}_\nu \) and

\[
\|D_l f\|_{2,l} \leq \nu^2 \|f\|_{2,l}.
\]

**Proof**

From equality (1), we have \( D_l f \in \mathcal{I}_\nu \) if \( f \in \mathcal{I}_\nu \).

Formula (1) and Parseval theorem gives

\[
\|D_l f\|_{2,l}^2 = \int_{\mathbb{R}^d} |D_l f(x)|^2 w_l(x) dx
\]

\[
= \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x} \hat{f}(\xi) \right|^2 w_l(\xi) d\xi
\]

\[
= \int_{|\xi| \leq \nu} |\xi| |\hat{f}(\xi)|^2 w_l(\xi) d\xi
\]

\[
\leq \nu^4 \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 w_l(\xi) d\xi
\]

\[
= \nu^4 \int_{\mathbb{R}^d} |f(x)|^2 w_l(x) dx
\]

\[
= \nu^4 \|f\|_{2,l}^2.
\]

Therefore

\[
\|D_l f\|_{2,l} \leq \nu^2 \|f\|_{2,l}.
\]

### Jackson’s Theorem 2

Suppose that \( f \in W^m_{2,l}(m = 1, 2, \ldots) \), then

\[
E_\nu(f)_{2,l} \leq c_2 \nu^{-2m} \omega_k(D_l^m f, 1/\nu)_{2,l},
\]

for all \( \nu > 0 \), where \( c_2 = c_0^{-(k+m)} c_1^m \) is a constant.
Therefore, from (3) and Parseval equality we deduce that
\[
|f - P_\nu(f)|^2 \leq \int_{\mathbb{R}^d} (1 - j_{\gamma+2^{-1}(|\xi|/\nu)})^2 |\hat{f}(\xi)|^2 w_1(\xi) d\xi
\]

By Proposition 2, Parseval equality and formula (1) show that
\[
|1 - j_{\gamma+2^{-1}(|\xi|/\nu)}| \geq c_0
\]
for $|\xi| \geq \nu$.

Therefore, from (3) and Parseval equality we deduce that
\[
\|f - P_\nu(f)\|_{2,l}^2 \leq \frac{1}{c_0^{2(k+m)}} \int_{\mathbb{R}^d} (1 - j_{\gamma+2^{-1}(|\xi|/\nu)})^{2(k+m)} |\hat{f}(\xi)|^2 w_1(\xi) d\xi
\]

Therefore
\[
\|f - P_\nu(f)\|_{2,l} \leq c_0^{-(k+m)} \|I - M_{1/\nu}\|_{k+m} f\|_{2,l}.
\]

Proposition 2, Parseval equality and formula (1) show that
\[
\|(I - M_{1/\nu}) f\|_{2,l}^2 = \int_{\mathbb{R}^d} |(I - M_{1/\nu}) f(x)|^2 w_1(x) dx
\]

\[
\leq c_1^2 \nu^{-4} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 w_1(\xi) d\xi
\]

\[
= c_1^2 \nu^{-4} \int_{\mathbb{R}^d} |\hat{D_1 f}(\xi)|^2 w_1(\xi) d\xi
\]

\[
= c_1^2 \nu^{-4} \int_{\mathbb{R}^d} |D_1 f(x)|^2 w_1(x) dx
\]

\[
= c_1^2 \nu^{-4} \|D_1 f\|_{2,l}^2.
\]
Therefore
\[ \|(I - M_{1/\nu})f\|_{2,l} \leq c_1 \nu^{-2} \|D_I f\|_{2,l}. \] (8)

Successive applications of (8) to the right-hand side of (7) result in
\[ \|f - P_\nu(f)\|_{2,l} \leq c_0^{-(k+m)} c_1^m \nu^{-2m} \|(I - T_{1/\nu})^k D_I^m f\|_{2,l} \]
\[ \leq c_2 \nu^{-2m} \omega_k(D_I^m f, 1/\nu)_{2,l}, \]
where \( c_2 = c_0^{-(k+m)} c_1^m \) which implies (5) holds, the theorem is proved. \( \square \)

**Proposition 4** The modulus of smoothness \( \omega_k(f, t)_{2,l} \) has the following properties.
(i) \( \omega_k(f + g, t)_{2,l} \leq \omega_k(f, t)_{2,l} + \omega_k(g, t)_{2,l} \).
(ii) \( \omega_k(f, t)_{2,l} \leq 2^k \|f\|_{2,l} \).
(iii) If \( f \in W_{2,l}^k \), then
\[ \omega_k(f, t)_{2,l} \leq c_3 t^{2k} \|D_k f\|_{2,l}, \]
where \( c_3 = c_1^k \) is a constant.

**Proof**
Property (i) follow from the definition of \( \omega_k(f, t)_{2,l} \).
Property (ii) follow from the fact that \( \|M_h f\|_{2,l} \leq \|f\|_{2,l} \).
Assume that \( h \in (0, t] \). From formulas (1), (3) and Parseval equality, we have
\[ \|\Delta_k f\|_{2,l}^2 = \|\Delta_k f\|_{2,l}^2 = \int_{R^d} (1 - j_{\gamma - \frac{d}{2} - 1}(h|\xi|)) 2^k |\hat{f}(\xi)|^2 w_l(\xi) d\xi, \] (9)
\[ \|D_k f\|_{2,l}^2 = \|D_k f\|_{2,l}^2 = \int_{R^d} |\xi|^{4k} |\hat{f}(\xi)|^2 w_l(\xi) d\xi. \] (10)

Formula (9) implies the equality
\[ \|\Delta_k f\|_{2,l}^2 = h^{4k} \int_{R^d} \frac{(1 - j_{\gamma - \frac{d}{2} - 1}(h|\xi|)) 2^k}{h^{4k} |\xi|^{4k}} |\xi|^{4k} |\hat{f}(\xi)|^2 w_l(\xi) d\xi. \]

From Proposition 2 and Parseval equality we obtain
\[ \|\Delta_k f\|_{2,l}^2 \leq c_1^{2k} h^{4k} \int_{R^d} |\xi|^{4k} |\hat{f}(\xi)|^2 w_l(\xi) d\xi \]
\[ = c_1^{2k} h^{4k} \|\Delta_k f\|_{2,l} \]
\[ = c_1^{2k} h^{4k} \|\Delta_k f\|_{2,l}^2. \]

Therefore
\[ \|\Delta_k f\|_{2,l} \leq c_3 h^{2k} \|D_k f\|_{2,l}. \]
Calculating the supremum with respect to all \( h \in (0, t] \), we obtain
\[ \omega_k(f, t)_{2,l} \leq c_3 t^{2k} \|D_k f\|_{2,l}, \]
where \( c_3 = c_1^k \). \( \square \)
4 Proofs of the inverse theorems

Proposition 5 For \( j \geq 1 \) we have

\[
2^{2k(j-1)} E_{2^j}(f)_{2,l} \leq \sum_{\eta = 2^{j-1}+1}^{2^j} \eta^{2k-1} E_{\eta}(f)_{2,l}.
\]

Proof Note that

\[
\sum_{\eta = 2^{j-1}+1}^{2^j} \eta^{2k-1} \geq (2^{j-1})^{2k-1} = 2^{2k(j-1)}.
\]

Since \( E_{\eta}(f)_{2,l} \) is monotonically decreasing, we conclude that

\[
2^{2k(j-1)} E_{2^j}(f)_{2,l} \leq \sum_{\eta = 2^{j-1}+1}^{2^j} \eta^{2k-1} E_{\eta}(f)_{2,l}.
\]

Proposition 6 For \( n \in \mathbb{N} \) we have

\[
2^k E_n(f)_{2,l} \leq \frac{c_4}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_{2,l}.
\]

Proof Note that

\[
\sum_{j=0}^{n} (j+1)^{2k-1} \geq \sum_{j \geq \frac{n}{2}-1} (j+1)^{2k-1} \geq \left(\frac{n}{2}\right)^{2k-1} \frac{n}{2} = 2^{-2k} n^{2k}.
\]

Since \( E_j(f)_{2,l} \) is monotonically decreasing, we conclude that

\[
2^k E_n(f)_{2,l} \leq \frac{c_4}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_{2,l}.
\]

Proposition 7. If \( \Phi_\nu \in I_\nu \) such that \( \| f - \Phi_\nu \|_{2,l} = E_\nu(f)_{2,l} \) For every \( \nu \in \mathbb{N} \), then

\[
\| D^k_1 \Phi_{2^j+1} - D^k_1 \Phi_2 \|_{2,l} \leq 2^{2k(j+1)+1} E_{2^j}(f)_{2,l}.
\]

In particular

\[
\| D^k_1 \Phi_1 \|_{2,l} = \| D^k_1 \Phi - D^k_1 \Phi_0 \|_{2,l} \leq 2^{4k} E_0(f)_{2,l}.
\]

Proof By Theorem 1 and the fact that \( E_\nu(f)_{2,l} \) is monotone decreasing with respect to \( \nu \), we obtain

\[
\| D^k_1 \Phi_{2^j+1} - D^k_1 \Phi_2 \|_{2,l} \leq 2^{2k(j+1)} \| \Phi_{2^j+1} - \Phi_2 \|_{2,l} = 2^{2k(j+1)} \| (f - \Phi_2) - (f - \Phi_{2^j+1}) \|_{2,l} \leq 2^{2k(j+1)} (E_{2^j}(f)_{2,l} + E_{2^j+1}(f)_{p,\alpha})_{2,l} \leq 2^{2k(j+1)+1} E_{2^j}(f)_{2,l}.
\]
and
\[
\|D^k\Phi_1 - D^k\Phi_0\|_{2,t} \leq \|\Phi_1 - \Phi_0\|_{2,t} = \|(f - \Phi_1) - (f - \Phi_0)\|_{2,t}
\leq E_1(f)_{2,t} + E_0(f)_{2,t}
\leq 2E_0(f)_{2,t} \leq 2^{4k+1}E_0(f)_{2,t}.
\]

The following theorems are analogues of the classical inverse theorems of approximation theory [2,5].

**Theorem 3** For every function \( f \in L^2_\omega(\mathbb{R}^d) \) and every positive integer \( n \) we have
\[
\omega_k(f, \frac{1}{n})_{2,t} \leq \frac{c}{n^k} \sum_{j=0}^n (j + 1)^{2k-1}E_j(f)_{2,t},
\]
where \( c = c(k, \alpha) \) is a positive constant.

**Proof**

Let \( 2^m \leq n < 2^{m+1} \) for any integer \( m \geq 0 \).

For every \( \nu \geq 0 \), let \( \Phi_\nu \) be an element of best approximation to \( f \) in the space \( I_\nu \), that is, \( \Phi_\nu \in I_\nu \) and \( \|f - \Phi_\nu\|_{2,t} = E_\nu(f)_{2,t} \). By formulas (i) and (ii) of Proposition 4, we obtain
\[
\omega_k(f, \frac{1}{n})_{2,t} \leq \omega_k(f - \Phi_{2^m+1}, \frac{1}{n})_{2,t} + \omega_k(\Phi_{2^m+1}, \frac{1}{n})_{2,t}
\leq 2^k\|f - \Phi_{2^m+1}\|_{2,t} + \omega_k(\Phi_{2^m+1}, \frac{1}{n})_{2,t}.
\]

Therefore
\[
\omega_k(f, \frac{1}{n})_{2,t} \leq 2^kE_{2^m+1}(f)_{2,t} + \omega_k(\Phi_{2^m+1}, \frac{1}{n})_{2,t} \leq 2^kE_n(f)_{2,t} + \omega_k(\Phi_{2^m+1}, \frac{1}{n})_{2,t}.
\]

Now with the aid of Proposition 5 and 7 and formula (iii) of Proposition 3, we conclude that
\[
\omega_k(\Phi_{2^m+1}, \frac{1}{n})_{2,t} \leq \frac{c_1}{n^{2k}}\|B^k\Phi_{2^m+1}\|_{2,t}
\leq \frac{c_1}{n^{2k}} \left( \|B^k\Phi_1 - B^k\Phi_0\|_{2,t} + \sum_{j=0}^m \|B^k\Phi_{2^j+1} - B^k\Phi_{2^j}\|_{2,t} \right)
\leq \frac{c_1}{n^{2k}} \left( 2^{4k+1}E_0(f)_{2,t} + \sum_{j=0}^m 2^{2k(j+1)}E_2^j(f)_{2,t} \right)
\leq \frac{c_1}{n^{2k}} 2^{4k+1} \left( E_0(f)_{2,t} + \sum_{j=0}^m 2^{2k(j-1)}E_2^j(f)_{2,t} \right)
\leq \frac{c_1}{n^{2k}} 2^{4k+1} \left( E_0(f)_{2,t} + E_1(f)_{2,t} + \sum_{j=1}^m \sum_{\eta=2^{j-1}+1}^{2^j} \eta^{2k-1}E_\eta(f)_{2,t} \right)
\leq \frac{c_1}{n^{2k}} 2^{4k+1} \left( E_0(f)_{2,t} + E_1(f)_{2,t} + \sum_{j=2}^m (j+1)^{2k-1}E_j(f)_{2,t} \right).
\]
Therefore
\[
\omega_k(\Phi_{2m+1}, \frac{1}{n})_{2,l} \leq \frac{c_5}{n^{2k}} \sum_{j=0}^{2^m} (j+1)^{2k-1} E_j(f)_{2,l}.
\]

(12)

Thus from (11) and (12) we derive the estimate
\[
\omega_k(f, \frac{1}{n})_{2,l} \leq 2^k E_n(f)_{2,l} + \frac{c_5}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_{2,l}.
\]

(13)

By Proposition 6 and formula (13), we have
\[
\omega_k(f, \frac{1}{n})_{2,l} \leq \frac{c}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_{2,l}.
\]

(14)

Theorem 4 Suppose that \(f \in L^2_2(\mathbb{R}^d)\) and
\[
\sum_{j=1}^{\infty} j^{2m-1} E_j(f)_{2,l} < \infty.
\]

Then \(f \in W^m_{2,l}\) and, for every positive integer \(n\), we have
\[
\omega_k(B^n f, \frac{1}{n})_{2,l} \leq C \left( \frac{1}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2(k+m)-1} E_j(f)_{2,l} + \sum_{j=n+1}^{\infty} j^{2m-1} E_j(f)_{2,l} \right),
\]
where \(C = c(k, m, \alpha)\) is a positive constant.

Proof. Let \(2^m \leq n < 2^{m+1}\) for any integer \(m \geq 0\). For every positive integer \(r \leq m\), we consider the series
\[
D^r f_1 + \sum_{j=0}^{\infty} (D^r f_{2j+1} - D^r f_{2j}).
\]

(14)

It follows from Propositions 7 and 5 that the series (14) converges in the norm of \(L^2_2(\mathbb{R}^d)\) because
\[
\sum_{j=0}^{\infty} \|D^r f_{2j+1} - D^r f_{2j}\|_{2,l} \leq \sum_{j=0}^{\infty} 2^{2r(j+1)} E_{2j}(f)_{2,l}
\]
\[
= 2^{2r+1} E_1(f)_{2,l} + c_2 2^{4r+1} \sum_{j=1}^{\infty} 2^{2r(j-1)} E_{2j}(f)_{2,l} \leq 2^{4r+1} \left( E_1(f)_{2,l} + \sum_{j=1}^{\infty} 2^{2r(j-1)} E_{2j}(f)_{2,l} \right)
\]
\[
\leq 2^{4r+1} \left( E_1(f)_{2,l} + \sum_{j=1}^{\infty} \sum_{j=2^{j-1}+1}^{2^j} \eta^{2r-1} E_{\eta}(f)_{2,l} \right)
\]
\[
\leq 2^{4r+1} \sum_{j=1}^{\infty} j^{2r-1} E_j(f)_{2,l} < \infty.
\]
Note that

\[ f = \Phi_1 + \sum_{j=0}^{\infty} (\Phi_{2^{j+1}} - \Phi_{2^j}). \quad (15) \]

where the series (15) converges in \( L_2^r(\mathbb{R}^d) \) and, a fortiori, in the space \( D' \) of distributions. Since the operator \( D_l \) is a linear continuous operator on \( D' \), the equality

\[ D_l^r f = D_l^r \Phi_1 + \sum_{j=0}^{\infty} (D_l^r \Phi_{2^{j+1}} - D_l^r \Phi_{2^j}), \quad (16) \]

holds in the space \( D' \). Since the right-hand side of (16) belongs to \( L_2^r(\mathbb{R}^d) \) for \( r \leq m \), we see that \( f \) belongs to the Sobolev space \( W_{m,2} \). In particular, \( D_l^m f \in L_2^r(\mathbb{R}^d) \).

By formula \((i)\) of Proposition 4, we obtain

\[ \omega_k(D_l^m f - D_l^m \Phi_{2^{s+1}}, \frac{1}{n})_{2,l} \leq \omega_k(D_l^m f - D_l^m \Phi_{2^{s+1}}, \frac{1}{n})_{2,l} + \omega_k(D_l^m \Phi_{2^{s+1}}, \frac{1}{n})_{2,l}. \]

Using Propositions 4, 5 and 7 we get

\[ \omega_k(D_l^m f - D_l^m \Phi_{2^{s+1}}, \frac{1}{n})_{2,l} \leq 2^{k} \| D_l^m f - D_l^m \Phi_{2^{s+1}} \|_{2,l} \]
\[ \leq 2^{k} \sum_{j=s+1}^{\infty} \| D_l^m \Phi_{2^{j+1}} - D_l^m \Phi_{2^j} \|_{2,l} \]
\[ \leq 2^{k} \sum_{j=s+1}^{\infty} 2^{2m(j+1)-1} E_{2^j}(f)_{2,l} \]
\[ \leq 2^{k+4m+1} \sum_{j=s+1}^{\infty} 2^{2m(j-1)} E_{2^j}(f)_{2,l} \]
\[ \leq 2^{k+4m+1} \sum_{j=s+1}^{\infty} \sum_{\eta=2^{j-1}+1}^{2^j} \eta^{2m-1} E_\eta(f)_{2,l} \]
\[ \leq 2^{k+4m+1} \sum_{j=2^{s+1}}^{\infty} j^{2m-1} E_j(f)_{2,l}. \]

Therefore

\[ \omega_k(D_l^m f - D_l^m \Phi_{2^{s+1}}, \frac{1}{n})_{2,l} \leq c_6 \sum_{j=2^{s+1}}^{\infty} j^{2m-1} E_j(f)_{2,l}. \quad (17) \]

Now with the aid of Propositions 5 and 7 and by formula \((iii)\) of Proposition 4, we conclude
that

\[
\omega_k(D^m_l \Phi_{2^s+1}, \frac{1}{n})_{2,l} \leq \frac{c_3}{n^{2k}} \|D^{m+k}_l \Phi_{2^s+1}\|_{2,l} \\
\leq \frac{c_3}{n^{2k}} \left( \|D^{m+k}_l \Phi_1 - D^{m+k}_l \Phi_0\|_{2,l} + \sum_{j=0}^{s} \|D^{m+k}_l \Phi_{2^j+1} - D^{m+k}_l \Phi_{2^j}\|_{2,l} \right) \\
\leq \frac{c_3}{n^{2k}} 2^{4(k+m)+1} E_0(f)_{2,l} + \sum_{j=0}^{s} 2^{2(k+m)(j+1)+1} E_{2^j}(f)_{2,l} \\
\leq \frac{c_3}{n^{2k}} 2^{4(k+m)+1} \left( E_0(f)_{2,l} + \sum_{j=0}^{s} 2^{2(k+m)(j-1)} E_{2^j}(f)_{2,l} \right) \\
\leq \frac{c_3}{n^{2k}} 2^{4(k+m)+1} \left( E_0(f)_{2,l} + E_1(f)_{2,l} + \sum_{j=1}^{s} \sum_{\eta=2^{-j}+1}^{2^j} \eta^{2(k+m)-1} E_{\eta}(f)_{2,l} \right) \\
\leq \frac{c_3}{n^{2k}} 2^{4(k+m)+1} \left( E_0(f)_{2,l} + E_1(f)_{2,l} + \sum_{j=2}^{2^s} (j+1)^2^{(k+m)-1} E_{j}(f)_{2,l} \right). 
\]

Therefore

\[
\omega_k(D^m_l \Phi_{2^s+1}, \frac{1}{n})_{2,l} \leq \frac{c_7}{n^{2k}} \sum_{j=0}^{2^s} (j+1)^2^{(k+m)-1} E_{j}(f)_{2,l}. 
\]

(18)

Thus from (17) and (18) we derive the estimate

\[
\omega_k(D^m_l f, \frac{1}{n})_{2,l} \leq C \left( \sum_{j=n+1}^{\infty} j^{2m-1} E_j(f)_{2,l} + \frac{1}{n^{2k}} \sum_{j=0}^{n} (j+1)^2^{(k+m)-1} E_j(f)_{2,l} \right). 
\]

\[\square\]

Acknowledgement

The authors would like to thank the referee for his valuable comments and suggestions.

References


