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Direct and inverse theorems of approximation theory in $L^2(\mathbb{R}^d, w_l(x)dx)$

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Abstract In this paper, we prove analogues of direct and some inverse theorems for the Dunkl harmonic analysis, using the function with bounded spectrum and generalized spherical mean operator.

Keywords Generalized continuity modulus; Bernstein theorem; Jackson theorem; best approximation.

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1 Introduction

Yet by the year 1912, S. Bernstein obtained the estimate inverse to Jakson's inequality in the space of continuous functions for some special cases [1], later Stechkin [2], Timan [3], proved such inverse estimates, including the case of the space L^p, $1 < p < \infty$.

Approximation problems for functions in the space $L^2(\mathbb{R}^d, w_l(x)dx)$, where w_l is a weight function invariant under the action of an associated reflection groups, using the function with bounded spectrum, are studied in this paper. Applying the Dunkl transform, Dunkl Laplacian operator and generalized spherical mean operator, we obtain analogs of the Bernstein inequality for function with bounded spectrum, direct and inverse theorem of Jackson type [4],[2],[5], where the modulus of smoothness is constructed on the basis of generalized spherical mean operator.

2 The Dunkl transform and its basic properties

Dunkl [3] defined a family of first-order differential-difference operators related to some reflection groups. These operators generalize in a certain manner the usual differentiation and have gained considerable interest in various fields of mathematics and also in physical applications. The theory of Dunkl operators provides generalizations of various multivariable analytic structures. Among others, we cite the exponential function, the Fourier transform and the translation operator. For more details about these operators see [6], [7], [1] and [8].

Let R be a root system in $\mathbb{R}^d$, W the corresponding reflection group, R_+ a positive subsystem of R and l a non-negative and W-invariant function defined on R. The Dunkl operator is defined for $f \in C^1(\mathbb{R}^d)$ by

$$D_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha\in R_+} l(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, x \in \mathbb{R}^d.$$

Here $\langle , \rangle$ is the usual Euclidean scalar product on $\mathbb{R}^d$ with the associated norm $|.|$ and σ_α the reflection with respect to the hyperplane H_α orthogonal to α. We consider the weight function

$$w_l(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2l(\alpha)},$$

where w_l is W-invariant and homogeneous of degree 2γ where

$$\gamma = \sum_{\alpha \in R_+} l(\alpha).$$

We let η be the normalized surface measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and set

$$d\eta_l(y) = w_l(y) d\eta(y).$$

Then η_l is a W-invariant measure on $\mathbb{S}^{d-1}$, and we let $d_l = \eta_l(\mathbb{S}^{d-1})$.
The Dunkl kernel E_l on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by Dunkl in [9]. For $y \in \mathbb{R}^d$ the function $x \mapsto E(x, y)$ can be viewed as the solution on $\mathbb{R}^d$ of the following initial problem:

$$\begin{cases} D_j u(x,y) = y_j u(x,y), & \text{if } 1 \leq j \leq d, \\ u(0,y) = 0, & \text{for all } y \in \mathbb{R}^d, \end{cases}$$

This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.
Rösler has proved in [8] the following integral representation for the Dunkl kernel,

$$E_l(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\mu_x(y), x \in \mathbb{R}^d, z \in \mathbb{C}^d,$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, |x|)$ of center 0 and radius $|x|$.

Proposition 1 [6] *Let $z, w \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$*
(i) $E_l(z, 0) = 1, E_l(z, w) = E_l(w, z)$, $E_l(\lambda z, w) = E_l(z, \lambda w)$.
(ii) For all $\nu = (\nu_1, ..., \nu_d) \in \mathbb{N}, x \in \mathbb{R}^d, z \in \mathbb{C}^d$, we have

$$|D_z^\nu E_k(x; z)| \leq |x|^{|\nu|} exp(|x||Rez|,$$

where

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial_{z_1}^{|\nu|} \ldots . \partial_{z_d}^{|\nu|}}, |\nu| = \nu_1 + \ldots . + \nu_d.$$

In particular $|D_z^\nu E_l(ix; z)| \leq |x|^{|\nu|}$ for all $x, z \in \mathbb{R}^d$.

We denote by $L_l^2(\mathbb{R}^d) = L^2(\mathbb{R}^d, w_l(x)dx)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{2,l} = \left(\int_{\mathbb{R}^d} |f(x)|^2 w_l(x) dx \right)^{\frac{1}{2}},$$

and D_l the Dunkl Laplacian defined by

$$D_l = \sum_{i=1}^d D_j^2.$$

The scalar product in the Hilbert space $L_l^2(\mathbb{R}^d)$ obeys the formula

$$(f,g) := \int_{\mathbb{R}^d} \overline{f(x)} g(x) w_l(x) dx, \quad f,g \in L_l^2(\mathbb{R}^d).$$

By the partial integration one can verify the correlation

$$(D_l f, g) = (g, D_l f).$$

for any functions $f,g \in \mathcal{D}$ ($\mathcal{D}$ denotes the set of infinitely differentiable functions with a compact support).

As usual, we endow the space $\mathcal{D}$ with a topology; this turns it into a topological vector space [10]. Let $\mathcal{D}'$ stand for the set of generalized functions, i.e., linear continuous functionals on the space $\mathcal{D}$. We denote the value of a functional $f \in \mathcal{D}'$ on a function $\varphi \in \mathcal{D}$ by $\langle f, \varphi \rangle$. The space $L_l^2(\mathbb{R}^d)$ is embedded into $\mathcal{D}'$, provided that for $f \in L_l^2(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}$ we put

$$\langle f, \varphi \rangle := \int_{\mathbb{R}^d} f(x) \varphi(x) w_l(x) dx.$$

One can extend (in a natural way) the action of the Dunkl Laplacian operator D_l onto the space of generalized functions $\mathcal{D}'$, putting

$$\langle D_l f, \varphi \rangle := \langle f, D_l \varphi \rangle, \quad f \in \mathcal{D}', \varphi \in \mathcal{D}.$$

In particular, the action of the operator $D_l f$ is defined for any function $f \in L_l^2(\mathbb{R}^d)$ but, generally speaking, $D_l f$ is a generalized function.

The Dunkl transform is defined for $f \in L_l^1(\mathbb{R}^d) = L^1(\mathbb{R}^d, w_l(x)dx)$

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = c_l^{-1} \int_{\mathbb{R}^d} f(x) E_l(-i\xi, x) w_l(x) dx,$$

where the constant c_l is given by

$$c_l = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2}} w_l(z) dz.$$

The inverse Dunkl transform is defined by the formula

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) E_l(ix, \xi) w_l(\xi) d\xi, x \in \mathbb{R}^d.$$

From [11], we have that if $f \in L_l^2(\mathbb{R}^d)$

$$\widehat{D_l f}(\xi) = -|\xi|^2 \widehat{f}(\xi). \tag{1}$$

The Dunkl transform shares several properties with its counterpart in the classical case. We mention here, in particular that Parseval theorem holds in $L_l^2(\mathbb{R}^d)$. As in the classical case, a generalized translation operator is defined in the Dunkl [5, 12]. Namely, for $f \in L_l^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we define $\tau_x(f)$ to be the unique function in $L_l^2(\mathbb{R}^d)$ satisfying

$$\widehat{\tau_x f}(y) = E_l(ix, y) \widehat{f}(y) \quad \text{a.e} \quad y \in \mathbb{R}^d.$$

Form to Parseval theorem and proposition 1.1, we see that

$$\|\tau_x f\|_{2,l} \leq \|f\|_{2,l} \quad \text{for all} \quad x \in \mathbb{R}^d.$$

For $\alpha > \frac{-1}{2}$, let $j_\alpha(x)$ be a normalized Bessel function of the first kind, i.e.,

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(x)}{x^\alpha},$$

where $J_\alpha(x)$ is a Bessel function of the first kind [13].
The function $J_\alpha(x)$ is infinitely differentiable, $J_\alpha(0) = 1$.

Proposition 2 ([14] *or* [15]) *For* $x \in \mathbb{R}$ *the following inequalities are fulfilled*
(i) $|j_\alpha(x)| \leq 1$.
(ii) $|1 - j_\alpha(x)| \geq c_0$ *with* $|x| \geq 1$, *where* $c_0 > 0$ *is a certain constant which depend only on* α.
(iii) $|1 - j_\alpha(x)| \leq c_1 x^2$, *where* c_1 *is a constant.*

The generalized spherical mean value of $f \in L^2_l(\mathbb{R}^d)$ is defined by

$$M_h f(x) = \frac{1}{d_l} \int_{\mathbb{S}^{d-1}} \tau_x f(hy) d\eta_l(y), x \in \mathbb{R}^d, h > 0.$$

We have

$$\|M_h f\|_{2,l} \leq \|f\|_{2,l}. \tag{2}$$

Proposition 3 ([16]) *Let* $f \in L^2_l(\mathbb{R}^d)$ *and fix* $h > 0$. *Then* $M_h f \in L^2_l(\mathbb{R}^d)$ *and*

$$\widehat{M_h f}(\xi) = j_{\gamma+\frac{d}{2}-1}(h|\xi|)\widehat{f}(\xi), \xi \in \mathbb{R}^d. \tag{3}$$

Let the function $f \in L^2_l(\mathbb{R}^d)$. We define differences of the order k ($k = 1, 2, ...$) with a step $h > 0$.

$$\Delta_h^k f(x) = (I - M_h)^k f(x),$$

where I is the unit operator.
For any positive integer k, we define the generalized module of smoothness of the kth order by the formula

$$\omega_k(f, \delta)_{2,l} = \sup_{0<h\leq\delta} \|\Delta_h^k f\|_{2,l}, \delta > 0.$$

Let $W^k_{2,l}$ be the Sobolev space constructed by the operator D_l, i.e.,

$$W^k_{2,l} = \{f \in L^2_l(\mathbb{R}^d) : D^j_l f \in L^2_l(\mathbb{R}^d); j = 1, 2, ..., k\},$$

where $D^0_l f = f, D^j_l f = D_l(D^{j-1}_l f)$.
For any $f \in L^2_l(\mathbb{R}^d)$ and any number $\nu > 0$, let us define the function

$$P_\nu(f)(x) = \mathcal{F}^{-1}(\widehat{f}(\xi)\chi_\nu(\xi)),$$

where $\chi_\nu(\xi) = 1$ if $|\xi| \leq \nu$ and $\chi_\nu(\xi) = 0$ if $|\xi| > \nu$, $\mathcal{F}^{-1}$ is the inverse Dunkl transform. One can easily prove that the function $P_\nu(f)(x)$ is infinitely differentiable and belongs to all classes $W^k_{2,l}$, $k = 1, 2, ...$

A function $f \in L^2_l(\mathbb{R}^d)$ is called a function with bounded spectrum of order $\nu > 0$ if $\widehat{f}(\xi) = 0$ for $|\xi| > \nu$. The set of all such functions is denoted by $\mathcal{I}_\nu$.
The best approximation of a function $f \in L^2_l(\mathbb{R}^d)$ by functions in $\mathcal{I}_\nu$ is the quantity

$$E_\nu(f)_{2,l} := \inf_{g \in \mathcal{I}_\nu} \|f - g\|_{2,l}.$$

3 Bernstein's inequality and Jackson's direct theorems

Bernstein's Theorem 1
If $f \in \mathcal{I}_\nu$, then $D_l f \in \mathcal{I}_\nu$ and

$$\|D_l f\|_{2,l} \leq \nu^2 \|f\|_{2,l}. \tag{4}$$

Proof
From equality (1), we have $D_l f \in \mathcal{I}_\nu$ if $f \in \mathcal{I}_\nu$
Formula (1) and Parseval theorem gives

$$\begin{aligned}
\|D_l f\|^2_{2,l} &= \int_{\mathbb{R}^d} |D_l f(x)|^2 w_l(x) dx \\
&= \int_{\mathbb{R}^d} |\widehat{D_l f}(\xi)|^2 w_l(\xi) d\xi \\
&= \int_{|\xi| \leq \nu} |\xi|^4 |\widehat{f}(\xi)|^2 w_l(\xi) d\xi \\
&\leq \nu^4 \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 w_l(\xi) d\xi \\
&= \nu^4 \int_{\mathbb{R}^d} |f(x)|^2 w_l(x) dx \\
&= \nu^4 \|f\|^2_{2,l}.
\end{aligned}$$

Therefore

$$\|D_l f\|_{2,l} \leq \nu^2 \|f\|_{2,l}. \qquad \square$$

Jackson's Theorem 2
Suppose that $f \in W^m_{2,l} (m = 1, 2, ...)$, then

$$E_\nu(f)_{2,l} \leq c_2 \nu^{-2m} \omega_k(D^m_l f, 1/\nu)_{2,l}, \tag{5}$$

for all $\nu > 0$, where $c_2 = c_0^{-(k+m)} c_1^m$ is a constant.

Proof
Using the Parseval equality, we have

$$\begin{aligned}\|f-P_\nu(f)\|_{2,l}^2 &= \int_{\mathbb{R}^d}|f(x)-P_\nu(f)(x)|^2w_l(x)dx\\ &= \int_{\mathbb{R}^d}|\widehat{f}(\xi)-\widehat{P_\nu(f)}(\xi)|^2w_l(\xi)d\xi\\ &= \int_{\mathbb{R}^d}|(1-\chi_\nu(\xi))\widehat{f}(\xi)|^2w_l(\xi)d\xi.\end{aligned}$$

Therefore

$$\|f-P_\nu(f)\|_{2,l}^2=\int_{|\xi|>\nu}|\widehat{f}(\xi)|^2w_l(\xi)d\xi. \tag{6}$$

By Proposition 2, we have

$$|1-j_{\gamma+\frac{d}{2}-1}(|\xi|/\nu)|\geq c_0$$

for $|\xi|\geq\nu$.
Therefore, from (3) and Parseval equality we deduce that

$$\begin{aligned}\|f-P_\nu(f)\|_{2,l}^2 &\leq \frac{1}{c_0^{2(k+m)}}\int_{\mathbb{R}^d}(1-j_{\gamma+\frac{d}{2}-1}(|\xi|/\nu))^{2(k+m)}|\widehat{f}(\xi)|^2w_l(\xi)d\xi\\ &= \frac{1}{c_0^{2(k+m)}}\int_{\mathbb{R}^d}|\mathcal{F}((I-M_{1/\nu})^{k+m}f)(\xi)|^2w_l(\xi)d\xi\\ &= \frac{1}{c_0^{2(k+m)}}\int_{\mathbb{R}^d}|(I-M_{1/\nu})^{k+m}f(x)|^2w_l(x)dx\\ &= \frac{1}{c_0^{2(k+m)}}\|(I-M_{1/\nu})^{k+m}f\|_{2,l}^2.\end{aligned}$$

Therefore

$$\|f-P_\nu(f)\|_{2,l}\leq c_0^{-(k+m)}\|(I-M_{1/\nu})^{k+m}f\|_{2,l}. \tag{7}$$

Proposition 2, Parseval equality and formula (1) show that

$$\begin{aligned}\|(I-M_{1/\nu})f\|_{2,l}^2 &= \int_{\mathbb{R}^d}|(I-M_{1/\nu})f(x)|^2w_l(x)dx\\ &= \int_{\mathbb{R}^d}|\mathcal{F}((I-M_{1/\nu})f)(\xi)|^2w_l(\xi)d\xi\\ &= \int_{\mathbb{R}^d}(1-j_{\gamma+\frac{d}{2}-1}(|\xi|/\nu))^2|\widehat{f}(\xi)|^2w_l(\xi)d\xi\\ &\leq c_1^2\nu^{-4}\int_{\mathbb{R}^d}|\xi|^4|\widehat{f}(\xi)|^2w_l(\xi)d\xi\\ &= c_1^2\nu^{-4}\int_{\mathbb{R}^d}|\widehat{D_lf}(\xi)|^2w_l(\xi)d\xi\\ &= c_1^2\nu^{-4}\int_{\mathbb{R}^d}|D_lf(x)|^2w_l(x)dx\\ &= c_1^2\nu^{-4}\|D_lf\|_{2,l}^2.\end{aligned}$$

Therefore

$$\|(I-M_{1/\nu})f\|_{2,l} \le c_1\nu^{-2}\|D_l f\|_{2,l}. \tag{8}$$

Successive applications of (8) to the right-hand side of (7) result in

$$\begin{aligned}\|f-P_\nu(f)\|_{2,l} &\le c_0^{-(k+m)}c_1^m\nu^{-2m}\|(I-T_{1/\nu})^k D_l^m f\|_{2,l}\\ &\le c_2\nu^{-2m}\omega_k(D_l^m f,1/\nu)_{2,l},\end{aligned}$$

where $c_2 = c_0^{-(k+m)}c_1^m$ which implies (5) holds, the theorem is proved. □

Proposition 4 *The modulus of smoothness $\omega_k(f,t)_{2,l}$ has the following properties.*
(i) $\omega_k(f+g,t)_{2,l} \le \omega_k(f,t)_{2,l} + \omega_k(g,t)_{2,l}$.
(ii) $\omega_k(f,t)_{2,l} \le 2^k\|f\|_{2,l}$.
(iii) If $f \in W^k_{2,l}$, then

$$\omega_k(f,t)_{2,l} \le c_3 t^{2k}\|D^k f\|_{2,l},$$

where $c_3 = c_1^k$ is a constant.

Proof

Property (i) follow from the definition of $\omega_k(f,t)_{2,l}$.
Property (ii) follow from the fact that $\|M_h f\|_{2,l} \le \|f\|_{2,l}$.
Assume that $h \in (0,t]$. From formulas (1), (3) and Parseval equality, we have

$$\|\Delta_h^k f\|_{2,l}^2 = \|\widehat{\Delta_h^k f}\|_{2,l}^2 = \int_{\mathbb{R}^d}(1-j_{\gamma-\frac{d}{2}-1}(h|\xi|))^{2k}|\widehat{f}(\xi)|^2 w_l(\xi)d\xi, \tag{9}$$

$$\|D_l^k f\|_{2,l}^2 = \|\widehat{D_l^k f}\|_{2,l}^2 = \int_{\mathbb{R}^d}|\xi|^{4k}|\widehat{f}(\xi)|^2 w_l(\xi)d\xi. \tag{10}$$

Formula (9) implies the equality

$$\|\Delta_h^k f\|_{2,l}^2 = h^{4k}\int_{\mathbb{R}^d}\frac{(1-j_{\gamma-\frac{d}{2}-1}(h|\xi|))^{2k}}{h^{4k}|\xi|^{4k}}|\xi|^{4k}|\widehat{f}(\xi)|^2 w_l(\xi)d\xi.$$

From Proposition 2 and Parseval equality we obtain

$$\begin{aligned}\|\Delta_h^k f\|_{2,l}^2 &\le c_1^{2k}h^{4k}\int_{\mathbb{R}^d}|\xi|^{4k}|\widehat{f}(\xi)|^2 w_l(\xi)d\xi\\ &= c_1^{2k}h^{4k}\|\widehat{D_l^k f}\|_{2,l} = c_1^{2k}h^{4k}\|D_l^k f\|_{2,l}^2.\end{aligned}$$

Therefore

$$\|\Delta_h^k f\|_{2,l} \le c_1^k h^{2k}\|D_l^k f\|_{2,l}.$$

Calculating the supremum with respect to all $h \in (0,t]$, we obtain

$$\omega_k(f,t)_{2,l} \le c_3 t^{2k}\|D_l^k f\|_{2,l},$$

where $c_3 = c_1^k$. □

4 Proofs of the inverse theorems

Proposition 5 *For $j \geq 1$ we have*

$$2^{2k(j-1)}E_{2^j}(f)_{2,l} \leq \sum_{\eta=2^{j-1}+1}^{2^j} \eta^{2k-1}E_\eta(f)_{2,l}.$$

Proof Note that

$$\sum_{\eta=2^{j-1}+1}^{2^j} \eta^{2k-1} \geq (2^{j-1})^{2k-1}2^{j-1} = 2^{2k(j-1)}.$$

Since $E_\eta(f)_{2,l}$ is monotonically decreasing, we conclude that

$$2^{2k(j-1)}E_{2^j}(f)_{2,l} \leq \sum_{\eta=2^{j-1}+1}^{2^j} \eta^{2k-1}E_\eta(f)_{2,l}. \qquad \square$$

Proposition 6 For $n \in \mathbb{N}$ we have

$$2^k E_n(f)_{2,l} \leq \frac{c_4}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1}E_j(f)_{2,l}.$$

Proof Note that

$$\sum_{j=0}^{n}(j+1)^{2k-1} \geq \sum_{j\geq\frac{n}{2}-1}^{n} (j+1)^{2k-1} \geq \left(\frac{n}{2}\right)^{2k-1}\frac{n}{2} = 2^{-2k}n^{2k}.$$

Since $E_j(f)_{2,l}$ is monotonically decreasing, we conclude that

$$2^k E_n(f)_{2,l} \leq \frac{c_4}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1}E_j(f)_{2,l}. \qquad \square$$

Proposition 7. *If $\Phi_\nu \in I_\nu$ such that $\|f-\Phi_\nu\|_{2,l} = E_\nu(f)_{2,l}$ For every $\nu \in \mathbb{N}$, then*

$$\|D_l^k\Phi_{2^{j+1}} - D_l^k\Phi_{2^j}\|_{2,l} \leq 2^{2k(j+1)+1}E_{2^j}(f)_{2,l}.$$

In particular

$$\|D_l^k\Phi_1\|_{2,l} = \|D_l^k\Phi_1 - D_l^k\Phi_0\|_{2,l} \leq 2^{4k+1}E_0(f)_{2,l}.$$

Proof By Theorem 1 and the fact that $E_\nu(f)_{2,l}$ is monotone decreasing with respect to ν, we obtain

$$\begin{aligned}
\|D_l^k\Phi_{2^{j+1}} - D_l^k\Phi_{2^j}\|_{2,l} &\leq 2^{2k(j+1)}\|\Phi_{2^{j+1}} - \Phi_{2^j}\|_{2,l} \\
&= 2^{2k(j+1)}\|(f-\Phi_{2^j}) - (f-\Phi_{2^{j+1}})\|_{2,l} \\
&\leq 2^{2k(j+1)}(E_{2^j}(f)_{2,l} + E_{2^{j+1}}(f)_{p,\alpha})_{2,l} \\
&\leq 2^{2k(j+1)+1}E_{2^j}(f)_{2,l}.
\end{aligned}$$

and

$$\begin{aligned}\|D_l^k\Phi_1 - D_l^k\Phi_0\|_{2,l} &\le \|\Phi_1-\Phi_0\|_{2,l} = \|(f-\Phi_1)-(f-\Phi_0)\|_{2,l}\\ &\le E_1(f)_{2,l}+E_0(f)_{2,l}\\ &\le 2E_0(f)_{2,l} \le 2^{4k+1}E_0(f)_{2,l}.\end{aligned}$$ □

The following theorems are analogues of the classical inverse theorems of approximation theory [2,5].

Theorem 3 *For every function $f \in L_l^2(\mathbb{R}^d)$ and every positive integer n we have*

$$\omega_k(f,\frac{1}{n})_{2,l} \le \frac{c}{n^{2k}}\sum_{j=0}^{n}(j+1)^{2k-1}E_j(f)_{2,l},$$

where $c = c(k,\alpha)$ is a positive constant.

Proof

Let $2^m \le n < 2^{m+1}$ for any integer $m \ge 0$.

For every $\nu \ge 0$, let Φ_ν be an element of best approximation to f in the space I_ν, that is, $\Phi_\nu \in I_\nu$ and $\|f-\Phi_\nu\|_{2,l} = E_\nu(f)_{2,l}$. By formulas (i) and (ii) of Proposition 4, we obtain

$$\begin{aligned}\omega_k(f,\frac{1}{n})_{2,l} &\le \omega_k(f-\Phi_{2^{m+1}},\frac{1}{n})_{2,l}+\omega_k(\Phi_{2^{m+1}},\frac{1}{n})_{2,l}\\ &\le 2^k\|f-\Phi_{2^{m+1}}\|_{2,l}+\omega_k(\Phi_{2^{m+1}},\frac{1}{n})_{2,l}.\end{aligned}$$

Therefore

$$\omega_k(f,\frac{1}{n})_{2,l} \le 2^kE_{2^{m+1}}(f)_{2,l}+\omega_k(\Phi_{2^{m+1}},\frac{1}{n})_{2,l} \le 2^kE_n(f)_{2,l}+\omega_k(\Phi_{2^{m+1}},\frac{1}{n})_{2,l}. \qquad (11)$$

Now with the aid of Proposition 5 and 7 and formula (iii) of Proposition 3, we conclude that

$$\begin{aligned}\omega_k(\Phi_{2^{m+1}},\frac{1}{n})_{2,l} &\le \frac{c_3}{n^{2k}}\|B^k\Phi_{2^{m+1}}\|_{2,l}\\ &\le \frac{c_3}{n^{2k}}\left(\|B^k\Phi_1-B^k\Phi_0\|_{2,l}+\sum_{j=0}^{m}\|B^k\Phi_{2^{j+1}}-B^k\Phi_{2^j}\|_{2,l}\right)\\ &\le \frac{c_3}{n^{2k}}\left(2^{4k+1}E_0(f)_{2,l}+\sum_{j=0}^{m}2^{2k(j+1)+1}E_{2^j}(f)_{2,l}\right)\\ &\le \frac{c_3}{n^{2k}}2^{4k+1}\left(E_0(f)_{2,l}+\sum_{j=0}^{m}2^{2k(j-1)}E_{2^j}(f)_{2,l}\right)\\ &\le \frac{c_3}{n^{2k}}2^{4k+1}\left(E_0(f)_{2,l}+E_1(f)_{2,l}+\sum_{j=1}^{m}\sum_{\eta=2^{j-1}+1}^{2^j}\eta^{2k-1}E_\eta(f)_{2,l}\right)\\ &\le \frac{c_3}{n^{2k}}2^{4k+1}\left(E_0(f)_{2,l}+E_1(f)_{2,l}+\sum_{j=2}^{2^m}(j+1)^{2k-1}E_j(f)_{2,l}\right).\end{aligned}$$

Therefore

$$\omega_k(\Phi_{2^{m+1}}, \frac{1}{n})_{2,l} \quad \leq \quad \frac{c_5}{n^{2k}} \sum_{j=0}^{2^m} (j+1)^{2k-1} E_j(f)_{2,l}. \tag{12}$$

Thus from (11) and (12) we derive the estimate

$$\omega_k(f, \frac{1}{n})_{2,l} \leq 2^k E_n(f)_{2,l} + \frac{c_5}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_{2,l}. \tag{13}$$

By Proposition 6 and formula (13), we have

$$\omega_k(f, \frac{1}{n})_{2,l} \leq \frac{c}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2k-1} E_j(f)_{2,l}. \qquad \square$$

Theorem 4 *Suppose that $f \in L_l^2(\mathbb{R}^d)$ and*

$$\sum_{j=1}^{\infty} j^{2m-1} E_j(f)_{2,l} < \infty.$$

Then $f \in W_{2,l}^m$ and, for every positive integer n, we have

$$\omega_k(B^m f, \frac{1}{n})_{2,l} \leq C \left(\frac{1}{n^{2k}} \sum_{j=0}^{n} (j+1)^{2(k+m)-1} E_j(f)_{2,l} + \sum_{j=n+1}^{\infty} j^{2m-1} E_j(f)_{2,l} \right),$$

where $C = c(k, m, \alpha)$ is a positive constant.

Proof.

Let $2^m \leq n < 2^{m+1}$ for any integer $m \geq 0$. For every positive integer $r \leq m$, we consider the series

$$D_l^r \Phi_1 + \sum_{j=0}^{\infty} (D_l^r \Phi_{2^{j+1}} - D_l^r \Phi_{2^j}). \tag{14}$$

It follows from Propositions 7 and 5 that the series (14) converges in the norm of $L_l^2(\mathbb{R}^d)$ because

$$\sum_{j=0}^{\infty} \|D_l^r \Phi_{2^{j+1}} - D_l^r \Phi_{2^j}\|_{2,l} \leq \sum_{j=0}^{\infty} 2^{2r(j+1)+1} E_{2^j}(f)_{2,l}$$

$$= 2^{2r+1} E_1(f)_{2,l} + c_2 2^{4r+1} \sum_{j=1}^{\infty} 2^{2r(j-1)} E_{2^j}(f)_{2,l} \leq 2^{4r+1} \left(E_1(f)_{2,l} + \sum_{j=1}^{\infty} 2^{2r(j-1)} E_{2^j}(f)_{2,l} \right)$$

$$\leq 2^{4r+1} \left(E_1(f)_{2,l} + \sum_{j=1}^{\infty} \sum_{\eta=2^{j-1}+1}^{2^j} \eta^{2r-1} E_\eta(f)_{2,l} \right)$$

$$\leq 2^{4r+1} \sum_{j=1}^{\infty} j^{2r-1} E_j(f)_{2,l} < \infty.$$

Note that

$$f = \Phi_1 + \sum_{j=0}^{\infty} (\Phi_{2^{j+1}} - \Phi_{2^j}). \tag{15}$$

where the series (15) converges in $L_l^2(\mathbb{R}^d)$ and, a fortiori, in the space $\mathcal{D}'$ of distributions. Since the operator D_l is a linear continuous operator on $\mathcal{D}'$, the equality

$$D_l^r f = D_l^r \Phi_1 + \sum_{j=0}^{\infty} (D_l^r \Phi_{2^{j+1}} - D_l^r \Phi_{2^j}), \tag{16}$$

holds in the space $\mathcal{D}'$. Since the right-hand side of (16) belongs to $L_l^2(\mathbb{R}^d)$ for $r \leq m$, we see that f belongs to the Sobolev space $W_{2,l}^m$. In particular, $D_l^m f \in L_l^2(\mathbb{R}^d)$.
By formula (i) of Proposition 4, we obtain

$$\omega_k(D_l^m f, \frac{1}{n})_{2,l} \leq \omega_k(D_l^m f - D_l^m \Phi_{2^{s+1}}, \frac{1}{n})_{2,l} + \omega_k(D_l^m \Phi_{2^{s+1}}, \frac{1}{n})_{2,l}.$$

Using Propositions 4, 5 and 7 we get

$$\begin{aligned}
\omega_k(D_l^m f - D_l^m \Phi_{2^{s+1}}, \frac{1}{n})_{2,l} &\leq 2^k \|D_l^m f - D_l^m \Phi_{2^{s+1}}\|_{2,l} \\
&\leq 2^k \sum_{j=s+1}^{\infty} \|D_l^m \Phi_{2^{j+1}} - D_l^m \Phi_{2^j}\|_{2,l} \\
&\leq 2^k \sum_{j=s+1}^{\infty} 2^{2m(j+1)+1} E_{2^j}(f)_{2,l} \\
&\leq 2^{k+4m+1} \sum_{j=s+1}^{\infty} 2^{2m(j-1)} E_{2^j}(f)_{2,l} \\
&\leq 2^{k+4m+1} \sum_{j=s+1}^{\infty} \sum_{\eta=2^{j-1}+1}^{2^j} \eta^{2m-1} E_\eta(f)_{2,l} \\
&\leq 2^{k+4m+1} \sum_{j=2^s+1}^{\infty} j^{2m-1} E_j(f)_{2,l}.
\end{aligned}$$

Therefore

$$\omega_k(D_l^m f - D_l^m \Phi_{2^{s+1}}, \frac{1}{n})_{2,l} \quad \leq \quad c_6 \sum_{j=2^s+1}^{\infty} j^{2m-1} E_j(f)_{2,l}. \tag{17}$$

Now with the aid of Propositions 5 and 7 and by formula (iii) of Proposition 4, we conclude that

$$\begin{aligned}
\omega_k(D_l^m\Phi_{2^{s+1}},\frac{1}{n})_{2,l} &\leq \frac{c_3}{n^{2k}}\|D_l^{m+k}\Phi_{2^{s+1}}\|_{2,l}\\
&\leq \frac{c_3}{n^{2k}}\left(\|D_l^{m+k}\Phi_1-D_l^{m+k}\Phi_0\|_{2,l}+\sum_{j=0}^{s}\|D_l^{m+k}\Phi_{2^{j+1}}-D_l^{m+k}\Phi_{2^j}\|_{2,l}\right)\\
&\leq \frac{c_3}{n^{2k}}\left(2^{4(k+m)+1}E_0(f)_{2,l}+\sum_{j=0}^{s}2^{2(k+m)(j+1)+1}E_{2^j}(f)_{2,l}\right)\\
&\leq \frac{c_3}{n^{2k}}2^{4(k+m)+1}\left(E_0(f)_{2,l}+\sum_{j=0}^{s}2^{2(k+m)(j-1)}E_{2^j}(f)_{2,l}\right)\\
&\leq \frac{c_3}{n^{2k}}2^{4(k+m)+1}\left(E_0(f)_{2,l}+E_1(f)_{2,l}+\sum_{j=1}^{s}\sum_{\eta=2^{j-1}+1}^{2^j}\eta^{2(k+m)-1}E_\eta(f)_{2,l}\right)\\
&\leq \frac{c_3}{n^{2k}}2^{4(k+m)+1}\left(E_0(f)_{2,l}+E_1(f)_{2,l}+\sum_{j=2}^{2^s}(j+1)^{2(k+m)-1}E_j(f)_{2,l}\right).
\end{aligned}$$

Therefore

$$\omega_k(D_l^m\Phi_{2^{s+1}},\frac{1}{n})_{2,l} \leq \frac{c_7}{n^{2k}}\sum_{j=0}^{2^s}(j+1)^{2(k+m)-1}E_j(f)_{2,l}. \tag{18}$$

Thus from (17) and (18) we derive the estimate

$$\omega_k(D_l^m f,\frac{1}{n})_{2,l}\leq C\left(\sum_{j=n+1}^{\infty}j^{2m-1}E_j(f)_{2,l}+\frac{1}{n^{2k}}\sum_{j=0}^{n}(j+1)^{2(k+m)-1}E_j(f)_{2,l}\right). \qquad \square$$

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