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An extension of Segal algebras

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Abstract In this paper we study a class of dense ideals in a locally convex algebra. The results concern the relationship between this concept and abstract Segal algebra, which is introduced by Reiter in 1971. We define generalized Segal algebras and prove that a finite dimensional semisimple Banach algebra cannot be a proper generalized Segal algebra in a Fréchet algebra.

Keywords Fréchet algebra, Segal algebra, multiplicative seminorm, approximate identity, finite dimensional Banach algebra

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1 Introduction

The concept of Segal and abstract Segal algebras were first introduced and studied in [1]. Many authors have considered this notion ever since and investigated several properties of these algebras such as different kinds of amenability, BSE property, Arens regularity and so on (see for instance [2–4]). An abstract Segal algebra is a Banach algebra which is perceived as a certain ideal in another Banach algebra. In this paper we focus our attention on a complete locally m-convex version of this to get some extended results. Our work is motivated by multiplier seminorm defined on an arbitrary Bnach algebra and its essential role for Segal extensions [5].

2 Preliminaries

By a locally m-convex algebra we shall mean an algebra E which is, in particular, a topological vector space whose topology is defined by a family of submultiplicative seminorms. This family forces the algebra to possess a local basis consisting of convex sets which are multiplicative i.e. sets U for them $U.U \subseteq U$. If E is an m-convex algebra, then the ring multiplication of E is jointly continuous by [6, Proposition 1.6].

A given locally m-convex algebra E is said to be complete if the underlying topological vector space of E is complete i.e., every Cauchy filter in E converges.

Let E be a locally m-convex algebra. Then, its completion exists and is a (complete) locally m-convex algebra [6, Lemma 4.1]. A Fréchet algebra is a complete algebra A generated by a sequence $(p_n)_n$ of separating increasing submultiplicative seminorms, i.e., $p_n(xy) \leq p_n(x)p_n(y)$ for all $n \in \mathbb{N}$ and every $x, y \in A$ such that $p_n(x) \leq p_{n+1}(x)$ for all positive integer n and $x \in A$. It is known that if an algebra (not necessarily complete) is equipped with a family of seminorms as above, then its completion is a Fréchet algebra [6, Corollary 4.7]. For any uniform space E in this paper we use $\tilde{E}$ for the completion of E.

A multiplicative linear functional on a Banach algebra A is a non-zero linear functional φ on A such that $\varphi(xy) = \varphi(x)\varphi(y)$ where $x, y \in A$. We denote the set of all such functionals by $\Delta(A)$.

Throughout this paper a Banach algebra is called semisimple if the intersection of the kernels of its multiplicative linear functionals is $\{0\}$.

There are certainly known examples of Fréchet spaces which are not Banach but a remarkable number of results in the area of Banach spaces are also fulfilled for Fréchet spaces. One of them is the closed graph theorem:

Theorem 2.1 [7, Theorem 2.15] *If E and F are Fréchet spaces and $T : E \to F$ is a linear mapping, then T is continuous if its graph $\{(x, Tx);\ x \in E\}$ is closed in $E \times F$.*

As this paper deals with an extended notion of abstract Segal algebra, we need to give the definition of this algebra.

Definition 2.1 Let $(A; \|.\|_A)$ be a Banach algebra. A Banach algebra $(B; \|.\|_B)$ is an abstract Segal algebra with respect to A if
(i) B is a dense left ideal in A;
(ii) there exists $M > 0$ such that $\|b\|_A \leq M\|b\|_B$ for each $b \in B$;
(iii) there exists $C > 0$ such that $\|ab\|_B \leq C\|b\|_B\|a\|_A$ for each $a \in A$ and $b \in B$.

The above version of this notion is given by Reiter [1]. But Barnes proved that the last condition could be obtained by the others. It is also seen by [8, proposition 2.2] that when A is a semisimple Banach algebra, every dense left ideal of A is an abstract Segal algebra.

3 Generalized Segal algebras

In this section the definition of generalized Segal algebras is given and we shall find some relations between the different notions of Segal algebras.

Definition 3.1 Let $(A; \mathcal{P})$ be a complete locally m-convex algebra. A Banach algebra $(B; \|.\|_B)$ is a g-Segal (for generalized Segal) algebra with respect to A if
(i) B is a dense left ideal in A;
(ii) for every $p \in \mathcal{P}$ there exists $M_p > 0$ such that $p(b) \leq M_p\|b\|_B$ for each $b \in B$ and ;
(iii) there exists $C > 0$ such that $\|ab\|_B \leq C\|b\|_B \sup_{p\in\mathcal{P}} p(a)$ for each $a \in A$ and $b \in B$.

Remark 3.1 If the second condition in the previous definition is replaced by the following

$$\exists D > 0\ \forall b \in B\ \forall p \in \mathcal{P},\ p(b) \leq D\|b\|,$$

then $\|x\|' := \sup_{p\in\mathcal{P}} p(x)$ defines a norm on B and B is an abstract Segal algebra with respect to $(\widetilde{B, \|.\|'})$ see [5, Proposition 2.6].

Remark 3.2 Note that the locally convex topology generated by the family $\mathcal{P}$ is in general different from the topology induced by the norm $\|.\|'$ (when defined). See for example [9] from which one can construct a uniform algebra A, an ideal M of A and a net $\{e_\alpha\} \subseteq M$ such that for some $m_0 \in M$, $\|e_\alpha m_0 - m_0\|'$ does not converge to zero whereas for each $\varphi \in \Delta(A)$, $\varphi(e_\alpha m_0 - m_0) \to 0$.

As it is pointed out in the introduction, the third condition in Definition 2.1 can be omitted. We have a similar fact when A is a Fréchet algebra.

Theorem 3.1 *Let $(A,\mathcal{P})$ be a Fréchet algebra and B be an ideal in A. Suppose that there exists a positive real number D such that for each $b \in B$ and each $p \in \mathcal{P}$, $p(b) \leq D\|b\|$. Then there exists $M > 0$ with the following inequality*

$$\|ba\|_B, \|ab\|_B \leq M\|b\| \sup_{p\in\mathcal{P}} p(a).$$

Proof Since the closed graph theorem also satisfies for Fréchet spaces (Theorem 2.1), the proof runs along much the same lines as that of Theorem 2.3 of [8]. For every $g \in B$ the operator $f \mapsto fg$ from A into B has closed graph and therefore is continuous. Hence there exists $M_g > 0$ such that

$$\|fg\| \leq M_g\|g\| \sup_{p\in\mathcal{P}} p(f).$$

Let $f \in A$ and L_f be the linear mapping on B given by $L_f(g) = fg$. For $\Gamma = \{L_f \in A;\ \sup_{p\in\mathcal{P}} p(f) \leq 1\}$ and any member g of the unit closed ball of B, we have $\|L_f(g)\|_B \leq M_g$. Therefore since this unit ball is in the second category by the uniform boundedness theorem [7] there exists $M > 0$ such that

$$\|fg\| \leq M \qquad (\|g\| \leq 1, L_f \in \Gamma).$$

For $g = 0$ and $f = 0$ the theorem is obviously satisfied. Let $g \in B \setminus \{0\}$ and $f \in A \setminus \{0\}$. Then $\sup_{p\in\mathcal{P}} p(f) \neq 0$ and we have

$$\begin{aligned} \|fg\| &\leq \sup_{p\in\mathcal{P}} p(f)\|g\|\|\frac{g}{\|g\|}.\frac{f}{\sup_{p\in\mathcal{P}} p(f)}\| \\ &\leq M\|g\| \sup_{p\in\mathcal{P}} p(f). \end{aligned}$$

One can get the other inequality by a similar argument. □

There are non-trivial semisimple Banach algebras with countable structure space (see for example [10]).

Theorem 3.2 *Let $(A, \|.\|_A)$ be a semisimple Banach algebra with countable structure space and $(B, \|.\|_B)$ be a left dense ideal in $(A,\tilde{\mathcal{P}})$, where $\mathcal{P} = \{|\varphi|; \varphi \in \Delta(A)\}$. If the inclusion map from B to $(A, \|.\|_A)$ is continuous, then B is a g-Segal algebra in the Fréchet algebra $(A,\tilde{\mathcal{P}})$.*

Proof By continuity of the inclusion map, there exists a positive real number D such that for every $y \in B$ we have

$$|\phi(y)| \leq \|y\|_A \leq D\|y\|_B.$$

Since $(A,\tilde{\mathcal{P}})$ is a Fréchet algebra the result is obtained by the previous theorem. □

Theorem 3.3 *Let $(A, \|.\|)$ be a Banach algebra and $\mathcal{P}$ be a separating equicontinuous family of seminorms on A. For $a \in A$ define $\|a\|' := \sup_{p\in\mathcal{P}} p(a)$. If A is a g-Segal algebra in $(A,\tilde{\mathcal{P}})$, then A is an abstract Segal algebra in $(A, \tilde{\|.\|'})$.*

Proof Let Definition 3.1 satisfies for A and $\widetilde{(A, \mathcal{P})}$. Since $\mathcal{P}$ is equicontinuous, there is a positive real number M such that for every $a \in A$ and every $p \in \mathcal{P}$ we have $p(a) \leq M\|a\|$. By the definition of g-Segal algebra it is enough only to prove that A is an ideal in $\widetilde{(A, \|.\|')}$. Suppose that $x \in \widetilde{(A, \|.\|')}$ and $y \in A$. Then x can be regarded as the equivalence class of a Cauchy net $(x_\alpha)_\alpha$ in $(A, \|.\|')$. On the other hand

$$\|x_\alpha y - x_\beta y\| \leq C\|y\| \sup_{p \in \mathcal{P}} p(x_\alpha - x_\beta).$$

Therefore $(x_\alpha y)_\alpha$ is a Cauchy net in $(A, \|.\|)$ and must converge to a point in A, y_0, say. But for each $p \in \mathcal{P}$, $p(x_\alpha y - y_0) \leq M\|x_\alpha y - y_0\|$. This along with the separability of $\mathcal{P}$ means that the equivalence classes of $(x_\alpha y)$ and (y_0) are the same. In other words, $xy = y_0$. □

Let A be an involutive Banach algebra. A representation of A is a $*$-homomorphism π of A into the C^*-algebra $B(H)$ of all bounded operators on a Hilbert space H. If $\pi(x) \neq 0$ for every nonzero $x \in A$, then π is called faithful.

Example 3.1 Suppose that A is an A^*-algebra i.e., A is an involutive Banach algebra with at least a faithful $*$-representation. Denote by $\sum$ the set of $*$-representations of A. By [11, Proposition 5.2] for each $\pi \in \sum$ we have $\|\pi\| \leq 1$. Then the set of functions $p_\pi(x) := \|\pi(x)\|_{op}$ is an equicontinuous separating family of seminorms on A, when $\|.\|_{op}$ is the operator norm in the related operator algebra. If A is a g-Segal algebra in $\widetilde{(A, \mathcal{P})}$, then A is an abstract Segal algebra in $\widetilde{(A, \|.\|')}$, where $\|a\|' = \sup_{\pi \in \sum} p_\pi(a)$. In particular for a locally compact group G, since $L^1(G)$ is not in general an ideal in the group C^*-algebra $C^*(G)$, the group algebra $L^1(G)$ is not a g-Segal algebra in $\widetilde{(L^1(G), \mathcal{P})}$.

Lemma 3.1 *Let B be a g-Segal algebra with a bounded approximate identity in a Fréchet algebra $(A, \mathcal{P})$ and $\mathcal{P}$ is equicontinuous on B. Then $B = \widetilde{(B, \|.\|')}$.*

Proof By Theorem 3.3 B is an abstract Segal algebra in $\widetilde{(B, \|.\|')}$. Now by [12] $B = \widetilde{(B, \|.\|')}$ as Banach algebras. □

Theorem 3.4 *Let $(A, \mathcal{P})$ be a Fréchet algebra. Let also there exists a finite dimensional Banach algebra $(B, \|.\|)$ with a bounded approximate identity, which is a g-Segal algebra in A. If $\mathcal{P}$ is equicontinuous on B, then $B = A$ and A is a Banach algebra.*

Proof Thanks to the open mapping theorem, it is only enough to show that B is of the second category in A, see [7, Chapter 2]. Since B is finite dimensional, it is $\sigma-$compact. In other words, there is a countable collection $\{B_i\}_{i=1}^\infty$ of first category in A, then there exists a countable collection $\{D_j\}_{j=1}^\infty$ of nowhere dense subsets of A such that $B = \cup D_j$. Put $E_{i,j} = B_i \cap D_j$. Then $B = \cup E_{i,j}$ and each $E_{i,j}$ is nowhere dense in A. We prove that two topologies induced by the family of seminorms $\mathcal{P}$ and $\|.\|'$ are the same on B_i. Since each B_i is compact, by an undergraduate argument, the functional sequence $\{p_n\}$ converges uniformly to $\|.\|'$ on each B_i. Then $\mathcal{P}$ and $\|.\|'$ induce the same topology on each B_i. Hence by Lemma 3.1 these two topologies agree on each $E_{i,j}$ with the original norm topology of B. In particular each $E_{i,j}$ is a nowhere dense subset in $(B, \|.\|)$. This means that B

is of the first category in itself, which would lead to contradiction with Baire's category theorem [7, Theorem 2.2]. □

It is shown that a finite dimensional Banach algebra is amenable if and only if it is semisimple. So we have the following corollary.

Corollary 3.1 *A finite dimensional semisimple Banach algebra is a proper g-Segal algebra in no Fréchet algebra.*

4 Conclusion

It is seen in this paper that based on different locally convex structures on a given algebra, the notions of abstract or g-Segal algebras are comparable. According to the topics of Section 3, there are examples of known algebras with subalgebras which may or may not be g-Segal algebra in them.

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