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Generalized Fourier-Bessel Transform of (η, γ)-Bessel-Lipschitz Functions in the Space $L^p_{\alpha,n}$

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Abstract In this paper, we obtain an analog of Theorem 5.2 in Younis [5] for the generalized Fourier-Bessel transform on the real line for functions satisfying the (η, γ)-Bessel Lipschitz condition in the space $L^p_{\alpha,n}$, $1 < p \leq 2$.

Keywords Singular differential operator; generalized Fourier-Bessel transform; generalized translation operator.

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1 Introduction and Preliminaries

Integral transforms and their inverses the Bessel transform are widely used to solve various problems in calculus, mechanics, mathematical physics, and computational mathematics (see [1],[2]).
Theorem 5.2 of Younis [3] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Dini Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

Theorem 1 [3, Theorem 5.2] *Let $f \in L^2(\mathbb{R})$. Then, the following statements are equivalent*
(a) $\|f(x+h) - f(x)\| = O\left(\frac{h^\eta}{(\log\frac{1}{h})^\gamma}\right)$, *as* $h \to 0, 0 < \eta < 1, \gamma \geq 0,$
(b) $\displaystyle\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right)$, *as* $r \to \infty,$
where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we consider a second-order singular differential operator $\mathcal{B}$ on the half line which generalizes the Bessel operator $\mathcal{B}_\alpha$, we obtain an analog of Theorem 1.1 for the generalized Fourier-Bessel transform associated to $\mathcal{B}$ in $L^p_{\alpha,n}$, $1 < p \leq 2$.

We briefly overview the theory of generalized Fourier-Bessel transform and related harmonic analysis (see[4],[5]).

Consider the second-order singular differential operator on the half line defined by

$$\mathcal{B}f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx} - \frac{4n(\alpha+n)}{x^2}f(x),$$

where $\alpha > \frac{-1}{2}$ and $n = 0, 1, 2, ...$ For $n = 0$, we obtain the classical Bessel operator

$$\mathcal{B}_\alpha f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx}.$$

Let M be the map defined by

$$Mf(x) = x^{2n} f(x), \quad n = 0, 1, ..$$

Let $L^p_{\alpha,n}$, $1 < p \leq 2$, be the class of measurable functions f on $[0, \infty[$ for which

$$\|f\|_{p,\alpha,n} = \|M^{-1} f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx \right)^{1/p}.$$

If $p = 2$, then we have $L^2_{\alpha,n} = L^2([0, \infty[, x^{2\alpha+1} dx)$.

For $\alpha \geq \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(z) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n! \Gamma(n + \alpha + 1)}, \quad z \in \mathbb{C}, \tag{1}$$

where $\Gamma(x)$ is the gamma-function (see [6]). The function $y = j_\alpha(z)$ satisfies the differential equation

$$\mathcal{B}_\alpha y + y = 0,$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$. The function $j_\alpha(z)$ is infinitely differentiable, even, and, moreover entire analytic.

From (1) we see that

$$\lim_{z \to 0} \frac{j_\alpha(z) - 1}{z^2} \neq 0,$$

hence, there exists $c > 0$ and $\nu > 0$ satisfying

$$|z| \leq \nu \Rightarrow |j_\alpha(z) - 1| \geq c|z|^2. \tag{2}$$

For $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$, set

$$\varphi_\lambda(x) = x^{2n} j_{\alpha+2n}(\lambda x). \tag{3}$$

From [4] and [5] recall the following properties:

Proposition 1

(i) φ_λ *satisfies the differential equation*

$$\mathcal{B} \varphi_\lambda = -\lambda^2 \varphi_\lambda.$$

(ii) *For all* $\lambda \in \mathbb{C}$, *and* $x \in \mathbb{R}$

$$|\varphi_\lambda(x)| \leq x^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Bessel transform that we call it the integral transform defined by

$$\mathcal{F}_{\mathcal{B}}f(\lambda)=\int_0^{\infty}f(x)\varphi_\lambda(x)x^{2\alpha+1}dx,\lambda\geq 0,f\in L^1_{\alpha,n},$$

(see [4]).

Let $f\in L^1_{\alpha,n}$ such that $\mathcal{F}_{\mathcal{B}}(f)\in L^1_{\alpha+2n}=L^1([0,\infty[,x^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Bessel transform is given by the formula (see [1])

$$f(x)=\int_0^{\infty}\mathcal{F}_{\mathcal{B}}f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda)=a_{\alpha+2n}\lambda^{2\alpha+4n+1}d\lambda,\quad a_\alpha=\frac{1}{4^\alpha(\Gamma(\alpha+1))^2}.$$

From [4] and [5] we have the following:

Proposition 2

(i) *For every $f\in L^1_{\alpha,n}\cap L^2_{\alpha,n}$ we have the Plancherel formula*

$$\int_0^{+\infty}|f(x)|^2x^{2\alpha+1}dx=\int_0^{+\infty}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2d\mu_{\alpha+2n}(\lambda).$$

(ii) *The generalized Fourier-Bessel transform $\mathcal{F}_{\mathcal{B}}$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2([0,+\infty[,\mu_{\alpha+2n})$.*

By Plancherel equality and Marcinkiewics interpolation Theorem (see [7]) we get for $f\in L^p_{\alpha,n}$ with $1<p\leq 2$ and q such that $\frac{1}{p}+\frac{1}{q}=1$,

$$\|\mathcal{F}_{\mathcal{B}}(f)\|_{q,\alpha+2n}\leq K\|f\|_{p,\alpha,n}, \tag{4}$$

where K is a positive constant.

Define the generalized translation operator T^h, $h\geq 0$ by the relation

$$T^hf(x)=(xh)^{2n}\tau^h_{\alpha+2n}(M^{-1}f)(x),x\geq 0,$$

where $\tau^h_{\alpha+2n}$ is the Bessel translation operator of order $\alpha+2n$ defined by

$$\tau^h_\alpha f(x)=c_\alpha\int_0^{\pi}f(\sqrt{x^2+h^2-2xh\cos t})\sin^{2\alpha}tdt,$$

where

$$c_\alpha=\left(\int_0^{\pi}\sin^{2\alpha}tdt\right)^{-1}=\frac{\Gamma(\alpha+1)}{\Gamma(\pi)\Gamma(\alpha+\frac{1}{2})}.$$

For $f\in L^p_{\alpha,n}$, we have

$$\mathcal{F}_{\mathcal{B}}(T^hf)(\lambda)=\varphi_\lambda(h)\mathcal{F}_{\mathcal{B}}(f)(\lambda), \tag{5}$$

$$\mathcal{F}_{\mathcal{B}}(\mathcal{B}f)(\lambda)=-\lambda^2\mathcal{F}_{\mathcal{B}}(f)(\lambda). \tag{6}$$

(see [4] and [5] for details).

Denote by $W_p^m(\mathcal{B})$, $1 < p \leq 2$, $m = 0, 1, 2...$, the class of functions $f \in L^p_{\alpha,n}$ that have on $\mathbb{R}^+$ generalized derivatives $f'(x), f''(x), ..., f^{(2m)}(x)$ in the sense of Levi (see [8]) and belong to $L^p_{\alpha,n}$ with $\mathcal{B}^m f \in L^p_{\alpha,n}$, i.e.,

$$W_p^m(\mathcal{B}) = \left\{ f \in L^p_{\alpha,n} / \mathcal{B}^m f \in L^p_{\alpha,n} \right\},$$

where $\mathcal{B}^0 f = f$, $\mathcal{B}^m f = \mathcal{B}(\mathcal{B}^{m-1} f), m = 0, 1, 2....$

2 Main Results

Definition 1 *A function $f \in W_p^m(\mathcal{B})$ is said to be in the (η, γ)-Bessel-Lipschitz class, denoted by Lip(η, γ, p), if*

$$\|(T^h - h^{2n} I)\mathcal{B}^m f(x)\|_{p,\alpha,n} = O\left(\frac{h^{\eta+2n}}{(\log \frac{1}{h})^{\gamma}}\right), \quad as \quad h \to 0, \eta, \gamma \geq 0,$$

where I is the unit operator in $L^p_{\alpha,n}$ and $m = 0, 1, 2, ...$

Lemma 1 *For $f \in W_p^m(\mathcal{B})$, we have*

$$\left(h^{2qn} \int_0^\infty \lambda^{2qm} |j_{\alpha+2n}(\lambda h) - 1|^q |\mathcal{F}_\mathcal{B} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\right)^{\frac{1}{q}} \leq K \|(T^h - h^{2n} I)\mathcal{B}^m f(x)\|_{p,\alpha,n},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m = 0, 1, 2...$

Proof: From formula (6), we obtain

$$\mathcal{F}_\mathcal{B}(\mathcal{B}^m f)(\lambda) = (-1)^m \lambda^{2m} \mathcal{F}_\mathcal{B} f(\lambda); m = 0, 1, ... \tag{7}$$

By using the formulas (3), (5) and (7), we conclude that

$$\mathcal{F}_\mathcal{B}(T^h \mathcal{B}^m f)(\lambda) = (-1)^m h^{2n} j_{\alpha+2n}(\lambda h) \lambda^{2m} \mathcal{F}_\mathcal{B} f(\lambda).$$

Hence

$$\mathcal{F}_\mathcal{B}((T^h - h^{2n} I)\mathcal{B}^m f)(\lambda) = (-1)^m h^{2n} (j_{\alpha+2n}(\lambda h) - 1) \lambda^{2m} \mathcal{F}_\mathcal{B} f(\lambda).$$

Now by formula (4), we have the result. □

Theorem 2 *Let f belong to Lip(η, γ, p). Then*

$$\int_r^\infty |\mathcal{F}_\mathcal{B} f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2qm-q\eta}}{(\log r)^{q\gamma}}\right), \quad as \quad r \to \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m = 0, 1, 2, ...$

Proof: Let $f \in Lip(\eta,\gamma,p)$. Then

$$\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|_{p,\alpha,n} = O\left(\frac{h^{\eta+2n}}{(\log\frac{1}{h})^{\gamma}}\right), \quad as \quad h \to 0.$$

From Lemma 2, we have

$$\int_0^{\infty} \lambda^{2qm}|j_{\alpha+2n}(\lambda h)-1|^q|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \le \frac{K^q}{h^{2qn}}\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|^q_{p,\alpha,n}.$$

By formula (2), we get

$$\int_{\frac{\nu}{2h}}^{\frac{\nu}{h}} \lambda^{2qm}|j_{\alpha+2n}(\lambda h)-1|^q|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \ge \frac{c^q\nu^{2q}}{2^{2q}}\int_{\frac{\nu}{2h}}^{\frac{\nu}{h}} \lambda^{2qm}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda).$$

Note that there exists then a positive constant C such that

$$\begin{aligned}\int_{\frac{\nu}{2h}}^{\frac{\nu}{h}} \lambda^{2qm}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) &\le C\int_{\frac{\nu}{2h}}^{\frac{\nu}{h}} \lambda^{2qm}|j_{\alpha+2n}(\lambda h)-1|^q|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\\ &\le \frac{CK^q}{h^{2qn}}\|(T^h - h^{2n}I)\mathcal{B}^m f(x)\|^q_{p,\alpha,n}\\ &= O\left(\frac{h^{q\eta}}{(\log\frac{1}{h})^{q\gamma}}\right).\end{aligned}$$

So we obtain

$$\int_r^{2r} \lambda^{2qm}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \le C'\frac{r^{-q\eta}}{(\log r)^{q\gamma}}.$$

where C' is a positive constant. Now, we have

$$\begin{aligned}\int_r^{\infty} \lambda^{2qm}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) &= \sum_{i=0}^{\infty}\int_{2^i r}^{2^{i+1}r} \lambda^{2qm}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)\\ &\le C'\left(\frac{r^{-q\eta}}{(\log r)^{q\gamma}} + \frac{(2r)^{-q\eta}}{(\log 2r)^{q\gamma}} + \frac{(4r)^{-q\eta}}{(\log 4r)^{q\gamma}} + \cdots\right)\\ &\le C'\left(\frac{r^{-q\eta}}{(\log r)^{q\gamma}} + \frac{(2r)^{-q\eta}}{(\log r)^{q\gamma}} + \frac{(4r)^{-q\eta}}{(\log r)^{q\gamma}} + \cdots\right)\\ &\le C'\frac{r^{-q\eta}}{(\log r)^{q\gamma}}\left(1 + 2^{-q\eta} + (2^{-q\eta})^2 + (2^{-q\eta})^3 + \cdots\right)\\ &\le K_\eta\frac{r^{-q\eta}}{(\log r)^{q\gamma}},\end{aligned}$$

where $K_\eta = C'(1-2^{-q\eta})^{-1}$ since $2^{-q\eta} < 1$.
Consequently

$$\int_r^{\infty} \lambda^{2qm}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-q\eta}}{(\log r)^{q\gamma}}\right), \quad as \quad r \to \infty,$$

i.e.,

$$\int_r^{\infty} |\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2qm-q\eta}}{(\log r)^{q\gamma}}\right), \quad as \quad r \to \infty.\Box$$

3 Conclusion

In this work we have succeded to generalise the theorem in [3] for the generalized Fourier-Bessel transform in the space $W_p^m(\mathcal{B})$ constructed by the singular differential operator $\mathcal{B}$. We proved that $f(x)$ belong to $Lip(\eta,\gamma,p)$. Then

$$\int_r^\infty |\mathcal{F}_{\mathcal{B}}f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2qm-q\eta}}{(\log r)^{q\gamma}}\right), \quad as \quad r\to\infty,$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $m=0,1,2,....$

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