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Super-Continuous Maps, Feebly-Regular and Completely Feebly-Regular Spaces

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Abstract In this paper properties of super-continuous maps are further studied. In addition, two new separation axioms for a topological space viz. feebly-regular space and completely feebly-regular spaces are introduced and their basic properties are studied.

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1 Introduction

Several variations of continuity are known in literature. Volicko [1] introduced the concept of super-continuity between topological spaces in which maps are termed as *supper-continuous* maps. Recently, Hawary [2] has given some constructions and characterizations of super-continuity, and also its relation with some other forms of continuity. In this paper some more properties of super-continuity are studied. Motivated by super-continuity, feebly-regular and completely feebly-regular topological spaces are introduced and their basic properties are studied.

2 Super-continuity and its properties

Definition 1 Following Volicko [1] for a subset A of a topological Space X, we define the super-closure of A to be the set A^+ given by:

$$A^+ = \{x \in X : \overline{U} \cap A \neq \phi \text{ for every open set U containing x}\}.$$

Similarly, we define the super-interior of A to be the set A^- given by:

$$A^- = \{x \in X : \overline{U} \subseteq A \text{ for some open set U containing x}\}.$$

Definition 2 A subset A of topological space X is said to be *super-closed* if $A = A^+$ and *super-open* if $A = A^-$.

It immediately follows from this definition that $\overline{A} \subseteq A^+$ and that $A^- \subseteq \mathring{A}$, where $\mathring{A}$ denotes the interior of A and $\overline{A}$ denotes the closure of A. Also it follows immediately that a subset A of X is *closed* whenever it is *super-closed* and *open* whenever it is *super-open*.

Definition 3 A map $f\colon X \longrightarrow Y$, where X and Y are topological spaces is said to be *super-continuous* if $f^{-1}(B)$ is *super-closed* (*super-open*) in X, for every *closed*(resp.*open*) set B in Y.

It is immediate from this definition that super-continuity implies continuity. Hawary [2] has proved that X is *regular* implies that $A^+ = \overline{A}$ and $A^- = \mathring{A}$ for every subset A of X and that continuity coincides with super-continuity. In the following we give examples of a *closed* subset of an infinite T_2 space which is not *super-closed*, and also of a *continuous* map on the same space which is not *super-continuous*.

Example 1 Consider the topology on $\mathbb{R}$ having the usual *open* intervals and the set $\mathbb{Q}$ of rationals as subbasis. Let $\mathbb{I}$ be the set of irrationals. Then $\mathbb{I} = \overline{\mathbb{I}}$ since $\mathbb{Q}$ is *open*. However $\mathbb{I}^+ = \mathbb{R}$. Hence $\mathbb{I}$ is *closed* but not *super-closed*. Then the identity map $i : \mathbb{R} \longrightarrow \mathbb{R}$, which is continuous, is not *super-continuous*, since $i^{-1}(\mathbb{I})$ is not *super-closed*.

The following lemma can be found in [3, Lemma 2].

Lemma 1 Let X be a topological space and let $\mathcal{T}^*$ be the family consisting of all the *super-open* subsets of X. Then $\mathcal{T}^*$ is a topology on X.

Theorem 1 *The projection maps from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$ are super-continuous. Also the addition, subtraction and multiplication operations are super-continuous functions from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$, and the quotient function is a super-continuous function from $\mathbb{R}\times(\mathbb{R}-\{0\})$ to $\mathbb{R}$.*

Proof The stated functions are continuous functions. That these functions are also *super-continuous* follows from [2, Theorem 11] by noting that $\mathbb{R}\times\mathbb{R}$ is *regular*.□

The following result is proved in [3, Theorem 6].

Theorem 2 *Let Y be a topological space and $X = \prod_{\alpha\in\Lambda} X_\alpha$ a product space. A function $f\colon Y \longrightarrow X$ is super-continuous on Y if and only if each coordinate function $\pi_\alpha \circ f$ is super-continuous on Y, where π_α are the canonical projections from $\prod_{\alpha\in\Lambda} X_\alpha$ to X_α.*

The topology on the product space Y above is the *Tychonoff* topology. i.e. the smallest topology on Y such that every projection from Y is continuous. The next theorem, shows that this easily carries over to any space with weak topology.

Theorem 3 *Let X have weak topology induced by a collection $\{f_\alpha |\, \alpha \in \Lambda\}$ of maps on the topological space X (to topological spaces X_α). Then $f : Y \to X$ is super-continuous iff $f_\alpha \circ f$ is super-continuous, $\forall\, \alpha \in \Lambda$.*

Proof For the necessity part of the proof, Hawary [2] has shown that the composition of two *super-continuous* maps is *super-continuous*. Conversely, assume that $f_\alpha \circ f$ is *super-continuous*, $\forall\alpha \in \Lambda$. For each subbasic *open* set $f_\alpha^{-1}(U_\alpha)$, we see that $f^{-1}(f_\alpha^{-1}(U_\alpha)) = (f_\alpha \circ f)^{-1}(U_\alpha)$ which is *super-open* in Y. Thus, it follows that f is *super-continuous*. □

The following theorem is a particular case of [3, Corollary 1]

Theorem 4 *Let $f, g\colon X \longrightarrow \mathbb{R}$ be super-continuous maps. Define $h\colon X \longrightarrow \mathbb{R}\times\mathbb{R}$ by $h(x) = (f(x), g(x))$, then h is also super-continuous.*

Notation 1 We denote the set of all real-valued *super-continuous* functions on the topological space X by $S(X,\mathbb{R})$. The corresponding set for continuous functions is denoted by $C(X,\mathbb{R})$.

By using the results stated in this section, it is now straightforward to verify that the set $S(X,\mathbb{R})$ is a ring under pointwise addition and pointwise multiplication. However, we shall prove a stronger result in Theorem 4 below. But, first we prove the following lemma.

Lemma 2 Let $f : X \to Y$ be continuous. Then $f^{-1}(V)$ is *super-open* in X for every *super-open* set V in Y.

Proof Let V be *super-open* in Y. So, $f^{-1}(V)$ is *open* in X. Let $x \in f^{-1}(V)$. i.e. $f(x) \in V$. By hypothesis, $f(x) \in W \subset \overline{W} \subset V$ for some *open* subset W of Y. Therefore, $x \in f^{-1}(W) \subset f^{-1}(\overline{W}) \subset f^{-1}(V)$. But $\overline{f^{-1}(W)} \subset f^{-1}(\overline{W})$(by [4, Ch III, 8.3(6)]). Therefore $x \in f^{-1}(W) \subset \overline{f^{-1}(W)} \subset f^{-1}(V)$. Also $f^{-1}(W)$ is *open* in X. Thus $f^{-1}(V)$ is *super-open* in X. □

Theorem 5 *S(X,ℝ)=C(X,ℝ).*

Proof Since every *super-continuous* map is *continuous*, it follows that $S(X, \mathbb{R}) \subset C(X, \mathbb{R})$. Conversely, let $f \in C(X, \mathbb{R})$ and V be open in $\mathbb{R}$. So, $f^{-1}(V)$ is *open* in X. But $\mathbb{R}$ is regular, so V is also *super-open* in $\mathbb{R}$. So by Lemma 2 above, $f^{-1}(V)$ is also *super-open* in X. Thus $f \in S(X, \mathbb{R})$. □

Definition 4 A collection $\{f_\alpha |\, f_\alpha \colon X \to X_\alpha, \alpha \in \Lambda\}$ of functions is said to separate points from *super-closed* sets iff whenever B is *super-closed* and $x \notin B$, then for some $\alpha \in \Lambda$, $f_\alpha(x) \notin \overline{f_\alpha(B)}$.

We conclude this section, with the following result.

Theorem 6 *A collection $\{f_\alpha |\, \alpha \in \Lambda\}$ of super-continuous functions on a topological space X separates points from super-closed sets in X iff the sets $f_\alpha^{-1}(V)$, $\alpha \in \Lambda$, V open in X_α, form a base for the topology $\mathcal{T}^*$ consisting of super-open subsets of X.*

Proof Let $x \in X$ and U be *super-open* in X with $x \in U$. So $B = U^c$ is *super-closed*, and $x \notin B$. So, $\exists \alpha \in \Lambda$ such that $f_\alpha(x) \notin \overline{f_\alpha(B)}$. Now, let $V = \overline{f_\alpha(B)}^c$. So, V is an *open* subset of X_α. Thus $f_\alpha^{-1}(V)$ is *super-open* subset of X containing x. Let $y \in f_\alpha^{-1}(V)$. Therefore, $f_\alpha(y) \in V = \overline{f_\alpha(B)}^c$. So, $f_\alpha(y) \notin f_\alpha(B)$. i.e. $y \notin B$, so that $y \in B^c = U$. Hence $f_\alpha^{-1}(V) \subset U$.

Conversely, let $x \in X$, and B be *super-closed* in X with $x \notin B$. So, $x \in B^c$ which is *super-open*. Thus $x \in f_\alpha^{-1}(V) \subset B^c$, for some $\alpha \in \Lambda$ and some open subset V of X_α, i.e., $f_\alpha(x) \in V$. We conclude that, $V \cap f_\alpha(B) = \phi$. For if $x_\alpha \in f_\alpha(B) \cap V$, then $x_\alpha = f_\alpha(b)$ for some $b \in B$. This then gives $b \in f_\alpha^{-1}(V) \cap B$, which contradicts the fact that $f_\alpha^{-1}(V) \subset B^c$. Hence we conclude $f_\alpha(x) \notin \overline{f_\alpha(B)}$. So, the collection $\{f_\alpha | \alpha \in \Lambda\}$ separates points from *super-closed* sets. □

3 Feebly-regular and completely feebly-regular spaces

In this section we introduce two new separation axioms and study their basic properties.

Definition 5 A topological space X is said to be feebly-regular (f.r in short) *iff* whenever $x \neq y$ in X, there are *super-open* sets U, V such that $x \in U$, $y \in V$, $U \cap V = \phi$.

Definition 6 A topological space X is said to be *completely feebly-regular* (c.f.r in short) *iff* whenever $x \neq y$ in X, there is a *super-continuous* map $f \colon X \longrightarrow \mathbb{R}$ such that $f(x) = 0$, and $f(y) = 1$.

Theorem 7 *For a topological space X, we have the following implications:*

(i) X *f.r* $\implies$ X T_2.
(ii) X *c.f.r* $\implies$ X *f.r.*
(iii) X T_3 $\implies$ X *f.r.*
(iv) X *Tychonoff* $\implies$ X *c.f.r.*

Proof

(i) Since *super-open* sets are *open*, it follows at once that X is T_2 whenever it is f.r.

(ii) Let x, y be distinct points of X. Then $\exists$ *super-continuous* map f such that $f(x) = 0$, $f(y) = 1$. Let U, V be non intersecting basic neighborhoods of $0,\ 1$ respectively. Then $f^{-1}(U)$, $f^{-1}(V)$ are non-intersecting *super-open* sets which contain x, y respectively. Thus X is f.r.

(iii) Let p, q be distinct points of X. Then, $\exists$ disjoint *open* sets U, V which contains p, q respectively. *Regularity* of X forces U, V to be *super-open*. Hence X is f.r.

(iv) Let p, q be distinct points of X. Then, $\exists$ real-valued continuous map f on X which evaluates to 1 at p and 0 at q. *Regularity* of X forces f to be *super-continuous*. Hence X is c.f.r. $\square$

Below, we exhibit an example of a space which is f.r but not *regular*.

Example 2 The space $\mathbb{R}$ in Example 1 above is not *regular* since 1 and $\mathbb{I}$ do not have disjoint neighborhoods. However this space can be easily verified to be f.r.

Notation 2 In the following proposition $\mathrm{cl}_Y V$ denotes the closure of V in Y.

Proposition 1 Let Y be a subspace of X, and U be *super-open* in X, then $U \cap Y$ is *super-open* in Y.

Proof Let $p \in U \cap Y$, So, $p \in U$, super-open in X Thus, $\mathrm{cl}_X B \subset U$ for some open subset B of X containing p. Now, let $V = B \cap Y$ Therefore $V$is open in Y and contains p. Again,

$$\begin{aligned}\mathrm{cl}_Y V &= \mathrm{cl}_Y (B \cap Y)\\ &\subseteq \mathrm{cl}_Y B \cap \mathrm{cl}_Y Y\\ &= (\mathrm{cl}_X B \cap Y) \cap Y\\ &\subset U \cap Y\end{aligned}$$

showing $U \cap Y$ is *super-open* in Y.$\square$

Theorem 8

(i) *Every subspace of a f.r space is f.r.*

(ii) *The product space* $\prod_{\alpha \in \Lambda} X_\alpha$ *is f.r iff* X_α *is f.r* $\forall \alpha \in \Lambda$.

Proof

(i) If X is f.r. and Y is a subspace of X with x, y distinct points of Y, then x, y are distinct points of X. So, $\exists$ disjoint sets U, V both *super-open* in X such that $x \in U$, $y \in V$. Now, $x \in U \cap Y$, $y \in V \cap Y$ and $U \cap Y$, $V \cap Y$ are *super-open* in Y by Proposition 1 above. Hence Y is f.r.

(ii) If the product space is f.r, then by part (i) above, each factor of the product is also f.r. Conversely, let x, y be distinct points of $\prod_{\alpha \in \Lambda} X_\alpha$. So, for at least one α, $x_\alpha \neq y_\alpha$. Fix one such α. Now, X_α is f.r, so $\exists$ *super-open* sets U_α,V_α such that $U_\alpha \cap V_\alpha = \phi$, $x_\alpha \in U_\alpha$, $y_\alpha \in V_\alpha$. We have,

$$\prod_{\beta \in \Lambda} U_\beta \text{ (where } U_\beta = X_\beta,\ \forall \beta \text{ except for } \beta = \alpha),$$
$$\prod_{\beta \in \Lambda} V_\beta \text{ (where } V_\beta = X_\beta,\ \forall \beta \text{ except for } \beta = \alpha),$$

are disjoint neighborhoods of x,y respectively. To complete our proof, we claim that these are also *super-open* sets. It is enough to show that $\prod_{\beta\in\Lambda} U_\beta$ is *super-open*.

Let $(z_\beta) \in \prod_{\beta\in\Lambda} U_\beta$. So $z_\alpha \in W_\alpha \subset \overline{W_\alpha} \subset U_\alpha$, for some open subset W_α of X_α. Now consider the product $\prod_{\beta\in\Lambda} W_\beta$ where $W_\beta = X_\beta$, $\forall\beta$ except for $\beta = \alpha$. We have,

$$\overline{\prod_{\beta\in\Lambda} W_\beta} = \prod_{\beta\in\Lambda} \overline{W_\beta} \subset \prod_{\beta\in\Lambda} U_\beta.$$

But, $\prod_{\beta\in\Lambda} W_\beta$ is a neighborhood of z. Hence, $\prod_{\beta\in\Lambda} U_\beta$ is *super-open.*□

Remark 1 Quotient of a f.r space need not be f.r. The space X in [5, eg 14.11(b)] is *Tychonoff* and hence f.r. But its continuous open image Y is not T_2 and hence not f.r.

Theorem 9

(i) *Every subspace of a c.f.r space is c.f.r.*

(ii) *The product space* $\prod_{\alpha\in\Lambda} X_\alpha$ *is c.f.r iff* X_α *is c.f.r* $\forall\alpha \in \Lambda$.

Proof

(i) If X is c.f.r and Y is a subspace of X with x, y distinct points of Y, then x, y are distinct points of X. So, $\exists$ *super-continuous* map $f : X \longrightarrow \mathbb{R}$ satisfying $f(x) = 0$ and $f(y) = 1$. Now $f_{|Y} : Y \rightarrow \mathbb{R}$ is also *super-continuous* since for any *open* subset U of $\mathbb{R}$, $(f_{|Y})^{-1}(U) = f^{-1}(U) \cap Y$, which is *super-open* in Y by Proposition 1. Hence Y is c.f.r.

(ii) If the product space is c.f.r, then the fact that each factor is c.f.r follows at once from part (i) above. Conversely, suppose that each factor space X_α is c.f.r. Let x, y be distinct points of $\prod_{\alpha\in\Lambda} X_\alpha$. So $x_\alpha \neq y_\alpha$ in at least one coordinate α. Now, $\exists$ *super-continuous* map $f_\alpha : X_\alpha \rightarrow \mathbb{R}$ such that $f_\alpha(x_\alpha) = 0$, $f_\alpha(y_\alpha) = 1$.

Define $g : \prod_{\alpha\in\Lambda} X_\alpha \rightarrow \mathbb{R}$ by $g(w) = (f_\alpha \circ \pi_\alpha)(w)$. Let U be open in $\mathbb{R}$. Therefore $g^{-1}(U) = (f_\alpha \circ \pi_\alpha)^{-1}(U) = \pi_\alpha^{-1} f_\alpha^{-1}(U)$. Again, $f_\alpha^{-1}(U)$ is *super-open* in X_α, and π_α is continuous, so by Lemma 2 above, $g^{-1}(U)$ is also *super-open*. Thus g is also *super-continuous*. Moreover, $g(x) = 0$ and $g(y) = 1$. □

Remark 2 Quotient of a c.f.r space need not be c.f.r. The space X in [5, eg 14.11(b)] is Tychonoff and hence c.f.r. But its continuous open image is not T_2 and hence not c.f.r.

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