MATEMATIKA, 2015, Volume 31, Number 1, 63–75

Stability Analysis of a Three Species Syn-Eco-System with Mortality Rate for Commensal

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Abstract The main objective of the present paper is to examine the stability of three species syn-eco-system. The system comprises of a commensal (S1) common to two hosts S2 and S3 with mortality rate for commensal. Further S2 is a commensal of S3 and S3 is a host of both S1, S2. Here all three species are having limited resources. The mathematical model equations constitute a set of three first order non-linear simultaneous coupled differential equations in the strengths N1, N2, N3 of S1, S2, S3 respectively. Criteria for the asymptotic stability of all the eight equilibrium points are established. Trajectories of the perturbations over the equilibrium points are illustrated. Further the global stability of a three species syn-eco system is established with the aid of suitably constructed Liapunovs function-pair. Also the growth rates of the species are numerically estimated using Runge-Kutta fourth order scheme.

Keywords Commensal; Equilibrium state; Global stability; Host; Mortality rate; Stable; Trajectories.

2010 Mathematics Subject Classification 92D25; 92D40.

1 Introduction

Ecology is a branch of life sciences connected to the existence of diverse species in the same environment and habitat. It is natural that two or more species living in a common habitat interact in different ways. Significant research in the area of theoretical ecology has been thresholded by Lotka [?] and by Volterra [?]. Several mathematicians and ecologists contributed to the growth of this area of knowledge. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and so on.

Mathematical modeling has been playing an important role for the last half a century in explaining several phenomena concerned with individuals and groups of populations in nature. The general concept of modeling has been presented in the treatises of Meyer [?], Kushing [?], Paul [?], Kapur [?]. Srinivas [?] studied competitive ecosystem of two species and three species with limited and unlimited resources. Later, Lakshminarayan [?], Lakshminarayan and Pattabhi Ramacharyulu [?] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by Archana Reddy, Pattabhi Ramacharyulu and Krishna Gandhi [?] and by Bhaskara Rama Sarma and Pattabhi Ramacharyulu [?], while Ravindra Reddy [?] investigated mutualism between two species. Further Phani Kumar [?] studied some mathematical models of ecological commensalism. The present author Hari Prasad [?,?,?,?,?,?,?] discussed on the stability of a three and four species syn-ecosystems.

The present investigation is on an analytical study of three species (S1, S2, S3) syn-eco system with mortality rate for commensal. The system comprises of a commensal (S1), two hosts S2 and S3 ie, S2 and S3 both benefit S1, without getting themselves affected either positively or adversely. Further S2 is a commensal of S3 and S3 is a host of both S1, S2. Commensalism is a symbiotic interaction between two populations where one population (S1) gets benefit from (S2) while the other (S2) is neither harmed nor benefited due to the interaction with (S1). The benefited species (S1) is called the commensal and the other, the helping one (S2) is called the host species. An example is a squirrel in an oak tree gets a place to live and food for its survival, while the tree remains neither benefited nor harmed.

2 Basic Equations of the Model

The model equations for the three species syn ecosystem is given by the following system of first order non-linear ordinary differential equations employing the following notations:

$$
\begin{aligned}
S_1 &: \text{Commensal of } S_2 \text{ and } S_3 \\
S_2 &: \text{Host of } S_1 \text{ and commensal of } S_3 \\
S_3 &: \text{Host of } S_1 \text{ and } S_2 \\
N_i(t) &: \text{The population strength of } S_i \text{ at time } t,\ i = 1, 2, 3 \\
t &: \text{Time instant} \\
d_1 &: \text{Natural death rate of } S_1 \\
a_i &: \text{Natural growth rate of } S_i,\ \ i = 1, 2, 3 \\
a_{ii} &: \text{Self inhibition coefficients of } S_i,\ \ i = 1, 2, 3 \\
a_{12}, a_{13} &: \text{Interaction coefficients of } S_1 \text{ due to } S_2 \text{ and } S_2 \text{ due to } S_3 \\
a_{23} &= \text{Interaction coefficient of } S_2 \text{ due to } S_3 \\
e_1 = \frac{d_1}{a_{11}} &: \text{Extinction coefficient of } S_1 \\
k_i = \frac{a_i}{a_{ii}} &: \text{Carrying capacities of } S_i,\ \ i = 2, 3
\end{aligned}
$$

Further the variables N_1, N_2, N_3 are non-negative and the model parameters d_1, a_2, a_3, a_{11}, $a_{12}, a_{22}, a_{33}, a_{13}, a_{23}$ are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3 are

$$\frac{dN_1}{dt} = -d_1N_1 - a_{11}N_1^2 + a_{12}N_1N_2 + a_{13}\ N_1N_3 \tag{1}$$

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 + a_{23}N_2N_3 \tag{2}$$

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 \tag{3}$$

3 Equilibrium States

The system under investigation has eight equilibrium states given by $\frac{dN_i}{dt} = 0,\ i = 1,\ 2,\ 3$

(i) Fully washed out state.

$$E_1 \,:\, \bar{N}_1 \;=\; 0,\; \bar{N}_2 \;=\; 0,\; \bar{N}_3 \;=\; 0$$

(ii) States in which two of the tree species are washed out and third is not.

$$\begin{aligned} E_2 \,&:\, \bar{N}_1 \;=\; 0,\; \bar{N}_2 \;=\; 0,\; \bar{N}_3 \;=\; k_3 \\ E_3 \,&:\, \bar{N}_1 \;=\; 0,\; \bar{N}_2 \;=\; k_2,\; \bar{N}_3 \;=\; 0 \\ E_4 \,&:\, \bar{N}_1 \;=\; -e_1,\; \bar{N}_2 \;=\; 0,\; \bar{N}_3 \;=\; 0 \end{aligned}$$

(iii) Only one of the three species is washed out while the other two are not.

$$\begin{aligned} E_5 \,&:\, \bar{N}_1 \;=\; 0,\; \bar{N}_2 \;=\; k_2 \;+\; \frac{a_{23}k_3}{a_{22}},\; \bar{N}_3 \;=\; k_3 \\ E_6 \,&:\, \bar{N}_1 \;=\; \frac{a_{13}k_3}{a_{11}} - e_1,\; \bar{N}_2 \;=\; 0,\; \bar{N}_3 \;=\; k_3 \\ E_7 \,&:\, \bar{N}_1 \;=\; \frac{a_{12}k_2}{a_{11}} - e_1,\; \bar{N}_2 \;=\; k_2,\; \bar{N}_3 \;=\; 0 \end{aligned}$$

(iv) The co-existent state or normal steady state.

$$E_8 \,:\, \bar{N}_1 \;=\; \frac{a_{12}}{a_{11}}\left(k_2 + \frac{a_{23}k_3}{a_{22}}\right) \;+\; \frac{a_{13}k_3}{a_{11}} - e_1,\; \bar{N}_2 \;=\; k_2 \;+\; \frac{a_{23}k_3}{a_{11}},\; \bar{N}_3 \;=\; k_3$$

4 Stability of the Equilibrium States

Let

$$N = (N_1, N_2, N_3) = \bar{N} \;+\; U \tag{4}$$

where $U \;=\; (u_1,\; u_2,\; u_3)^T$ is a small perturbation over the equilibrium state

$$\bar{N} = \left(\bar{N}_1,\; \bar{N}_2, \bar{N}_3\right).$$

The basic equations (1), (2) and (3) are quasi-linearized to obtain the equations for the perturbed state as

$$\frac{dU}{dt} \;=\; AU \tag{5}$$

with

$$A \;=\; \begin{bmatrix} -d_1 - 2a_{11}\bar{N}_1 + a_{12}\bar{N}_2 + a_{13}\bar{N}_3 & a_{12}\bar{N}_1 & a_{13}\bar{N}_1 \\ 0 & a_2 - 2a_{22}\bar{N}_2 + a_{23}\bar{N}_3 & a_{23}\bar{N}_2 \\ 0 & 0 & a_3 - 2a_{33}\bar{N}_3 \end{bmatrix} \tag{6}$$

The characteristic equation for the system is det

$$[A - \lambda I] = 0 \tag{7}$$

The equilibrium state is stable, if all the roots of the equation (7) are negative in case they are real or have negative real parts, in case they are complex.

4.1 Fully Washed Out State

In this case we have

$$A = \begin{bmatrix} -d_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \tag{8}$$

The characteristic equation is

$$(\lambda + d_1)(\lambda - a_2)(\lambda - a_3) = 0 \tag{9}$$

The characteristic roots of (9) are $-d_1, a_2, a_3$. Since two of these three are positive. Hence the fully washed out state is unstable and the solutions of the equations (5) are

$$u_1 = u_{10}\, e^{-d_1 t}; u_i = u_{io}\, e^{a_i t}, \quad i = 2,\ 3 \tag{10}$$

where u_{10}, u_{20}, u_{30} are the initial values of u_1, u_2, u_3 respectively.

4.1.1 Trajectories of Perturbations

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{-\frac{1}{d_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{a_3}} \tag{11}$$

4.2 Equilibrium State $E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3$

In this case we have

$$A = \begin{bmatrix} a_{13}k_3 - d_1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \tag{12}$$

where

$$\mu = a_2 + a_{23}k_3 > 0 \tag{13}$$

The characteristic roots are $a_{13}k_3 - d_1$, μ and $-a_3$. Since one of these three roots is positive, hence the state is unstable. The equations (5) yield the solutions:

$$u_1 = u_{10}e^{(a_{13}k_3 - d_1)t};\ u_2 = u_{20}\, e^{\mu t};\ u_3 = u_{30}e^{-a_3 t} \tag{14}$$

When $d_1 = a_{13}\, k_3$, (14) becomes

$$u_1 = u_{10};\ u_2 = u_{20}\, e^{\mu t};\ u_3 = u_{30}e^{-a_3 t} \tag{15}$$

4.2.1 Trajectories of Perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{a_{13}k_3 - d_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{\mu}} = \left(\frac{u_3}{u_{30}}\right)^{-\frac{1}{a_3}} \tag{16}$$

4.3 Equilibrium state $E_3 : \bar{N}_1 = 0, \bar{N}_2 = k_2, \bar{N}_3 = 0$

In this case we have

$$A = \begin{bmatrix} a_{12}k_2 - d_1 & 0 & 0 \\ 0 & -a_2 & a_{23}k_2 \\ 0 & 0 & a_3 \end{bmatrix} \tag{17}$$

The characteristic roots are $a_{12}k_2 - d_1, \ -a_2, \ a_3$. Since one of these three roots is positive, hence the state is unstable. The equations (5) yield the solutions curves.

$$u_1 = u_{10}e^{(a_{12}k_2 - d_1)t}; \ u_2 = (u_{20} - \alpha) \ e^{-a_2 t} + \alpha e^{a_3 t}; \ u_3 = u_{30}e^{a_3 t} \tag{18}$$

where

$$\alpha = \frac{a_{23}k_3 u_{30}}{a_2 + a_3} > 0 \tag{19}$$

Case I: When $d_1 = a_{12}k_2$ and $\alpha \neq u_{20}$

In this case (18) becomes

$$u_1 = u_{10}; \ u_2 = (u_{20} - \alpha) \ e^{-a_2 t} + \alpha e^{a_3 t}; \ u_3 = u_{30}e^{a_3 t} \tag{20}$$

Case II: When $d_1 \neq a_{12}k_2$ and $\alpha = u_{20}$

In this case (18) becomes

$$u_1 = u_{10}e^{(a_{12}k_2 - d_1)t}; \ u_2 = u_{20}e^{a_3 t}; \ u_3 = u_{30}e^{a_3 t} \tag{21}$$

Case III: When $d_1 = a_{12}k_2$ and $\alpha = u_{20}$

In this case (18) becomes

$$u_1 = u_{10}; \ u_2 = u_{20}e^{a_3 t}; \ u_3 = u_{30}e^{a_3 t} \tag{22}$$

4.3.1 Trajectories of Perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are

$$u_2 = (u_{20} - \alpha) \left(\frac{u_1}{u_{10}} \right)^{\frac{a_2}{d_1 - a_{12}k_2}} + \alpha \left(\frac{u_1}{u_{10}} \right)^{\frac{a_3}{a_{12}k_2 - d_1}} \tag{23}$$

and

$$u_2 = (u_{20} - \alpha) \left(\frac{u_3}{u_{30}} \right)^{\frac{-a_2}{a_3}} + \frac{u_3 \alpha}{u_{30}} \tag{24}$$

respectively.

4.4 Equilibrium State $E_4 : \bar{N}_1 = -e_1, \bar{N}_2 = 0, \bar{N}_3 = 0$

In this case we get

$$A = \begin{bmatrix} d_1 & -a_{12}e_1 & -a_{13}e_1 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \tag{25}$$

The characteristic roots are d_1, a_2, a_3. Since all these three roots are positive, hence the state is unstable. The equations (5) yield the solutions.

$$u_1 = (u_{10} - \beta - \gamma) e^{d_1 t} + \beta e^{a_2 t} + \gamma e^{a_3 t};\ u_2 = u_{20} e^{a_2 t};\ u_3 = u_{30} e^{a_3 t} \tag{26}$$

where

$$u_1 = (u_{10} - \beta - \gamma) e^{d_1 t} + \beta e^{a_2 t} + \gamma e^{a_3 t};\ u_2 = u_{20} e^{a_2 t};\ u_3 = u_{30} e^{a_3 t} \text{ and } \gamma = \frac{a_{13} e_1 u_{30}}{d_1 - a_3} \tag{27}$$

Case I: When $\beta = \gamma$

In this case (26) becomes

$$u_1 = (u_{10} - 2\beta) e^{d_1 t} + \beta \left(e^{a_2 t} + e^{a_3 t}\right);\ u_2 = u_{20} e^{a_2 t};\ u_3 = u_{30} e^{a_3 t} \tag{28}$$

Case II: When $u_{10} = \beta + \gamma$

In this case (26) becomes

$$u_1 = \beta e^{a_2 t} + \gamma e^{a_3 t};\ u_2 = u_{20} e^{a_2 t};\ u_3 = u_{30} e^{a_3 t} \tag{29}$$

4.4.1 Trajectories of Perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_1 = (u_{20} - \beta - \gamma) \left(\frac{u_2}{u_{20}}\right)^{\frac{d_1}{a_2}} + \frac{u_2 \beta}{u_{20}} + \gamma \left(\frac{u_2}{u_{20}}\right)^{\frac{a_3}{a_2}} \text{ and } \left(\frac{u_2}{u_{20}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{a_2} \tag{30}$$

4.5 Equilibrium State $E_5 : \bar{N}_1 = 0, \bar{N}_2 = k_2 + \frac{a_{23}k_3}{a_{22}}, \bar{N}_3 = k_3$

In this case we get

$$A = \begin{bmatrix} \rho - d_1 & 0 & 0 \\ 0 & -\mu & \frac{a_{23}\mu}{a_{22}} \\ 0 & 0 & -a_3 \end{bmatrix} \tag{31}$$

where

$$\rho = \frac{a_{12}\mu}{a_{22}} + a_{13}k_3 > 0 \tag{32}$$

The characteristic roots are $\rho - d_1, -\mu, -a_3$.

Case I: When $\rho < d_1$

In this case all the three roots are negative, hence the state is stable. The equations (5) yield the solutions.

$$u_1 = u_{10}e^{-(d_1+\rho)t};\ u_2 = (u_{20} - \sigma)\, e^{-\mu t} + \sigma e^{-a_3 t};\ u_3 = u_{30}e^{-a_3 t} \tag{33}$$

where

$$\sigma = \frac{a_{23}\mu u_{30}}{a_{22}\,(\mu - a_3)} \tag{34}$$

It can be noticed that $u_1 \to 0, u_2 \to 0$ and $u_3 \to 0$ as $t \to \infty$

Case II: When $\rho > d_1$

In this case one of the three roots is positive, hence the state is unstable. The equations (5) yield the solutions.

$$u_1 = u_{10}e^{(d_1+\rho)t};\ u_2 = (u_{20} - \sigma)\, e^{-\mu t} + \sigma e^{-a_3 t};\ u_3 = u_{30}e^{-a_3 t} \tag{35}$$

Case III: When $\rho = d_1$

In this case the state is neutrally stable and the equations (5) yield the solutions.

$$u_1 = u_{10};\ u_2 = (u_{20} - \sigma)\, e^{-\mu t} + \sigma e^{-a_3 t};\ u_3 = u_{30}e^{-a_3 t} \tag{36}$$

4.5.1 Trajectories of perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_2 = (u_{20} - \sigma)\left(\frac{u_1}{u_{10}}\right)^{\frac{\mu}{d_1-\rho}} + \sigma\left(\frac{u_1}{u_{10}}\right)^{\frac{a_3}{d_1-\rho}} \text{ and } u_2 = (u_{20} - \sigma)\left(\frac{u_3}{u_{30}}\right)^{\frac{\mu}{a_3}} + \frac{u_3\sigma}{u_{30}} \tag{37}$$

4.6 Equilibrium State $E_6 : \bar{N}_1 = \frac{a_{13}k_3}{a_{11}} - e_1, \bar{N}_2 = \bar{N}_3 = k_3$

In this case we get

$$A = \begin{bmatrix} d_1 - a_{13}k_3 & \frac{a_{12}}{a_{11}}(a_{13}k_3 - d_1) & \frac{a_{13}}{a_{11}}(a_{13}k_3 - d_1) \\ 0 & \mu & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \tag{38}$$

The characteristic roots are $d_1 - a_{13}k_3,\ \mu,\ -a_3$. Since one of these three roots is positive, hence the state is unstable. The equations (5) yield the solutions.

$$u_1 = (u_{10} - b - d)\ e^{(d_1 - a_{13}k_3)t} + be^{\mu t} + de^{-a_3 t};\ u_2 = u_{20}e^{\mu t};\ u_3 = u_{30}e^{-a_3 t} \tag{39}$$

where

$$b = \frac{a_{12}\,(d_1 - a_{13}k_3)\, u_{20}}{a_{11}\,(d_1 - a_{13}k_3 - \mu)} \text{ and } d = \frac{a_{13}\,(d_1 - a_{13}k_3)\, u_{30}}{a_{11}\,(d_1 - a_{13}k_3 + a_3)} \tag{40}$$

When $d_1 = a_{13}k_3$, (39) becomes

$$u_1 = u_{10};\ u_2 = u_{20}e^{\mu t};\ u_3 = u_{30}e^{-a_3 t} \tag{41}$$

4.6.1 Trajectories of Perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are

$$u_1 = (u_{10} - b - d)\left(\frac{u_2}{u_{20}}\right)^{\frac{d_1 - a_{13}k_3}{\mu}} + \frac{u_2 b}{u_{20}} + d\left(\frac{u_2}{u_{20}}\right)^{\frac{-a_3}{\mu}} \tag{42}$$

and

$$u_1 = (u_{10} - b - d)\left(\frac{u_3}{u_{30}}\right)^{\frac{a_{13}k_3 - d_1}{a_3}} + b\left(\frac{u_3}{u_{30}}\right)^{\frac{-\mu}{a_3}} + \frac{u_3 d}{u_{30}} \tag{43}$$

respectively.

4.7 Equilibrium State $E_7 : \bar{N}_1 = \frac{a_{12}k_2}{a_{11}} - d_1, \bar{N}_2 = k_2,\ \bar{N}_3 = 0$

In this case we get

$$A = \begin{bmatrix} d_1 - a_{12}k_2 & \frac{a_{12}}{a_{11}}(a_{12}k_2 - d_1) & \frac{a_{13}}{a_{11}}(a_{12}k_2 - d_1) \\ 0 & -a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \tag{44}$$

The characteristic roots are $d_1 - a_{12}k_2, -a_2, a_3$. Since one of these three roots is positive, hence the state is unstable. The equations (5) yield the solutions.

$$u_1 = (u_{10} - m - n)\, e^{(d_1 - a_{12}k_2)t} + me^{-a_2 t} + ne^{a_3 t};\ u_2 = u_{20}e^{-a_2 t}\ ;\ u_3 = u_{30}e^{a_3 t} \tag{45}$$

where

$$m = \frac{a_{12}(a_{12}k_2 - d_1)\, u_{20}}{a_{11}(a_2 + d_1 - a_{12}k_2)} \text{ and } n = \frac{a_{13}(d_1 - a_{12}k_2)\, u_{30}}{a_{11}(a_3 - d_1 + a_{12}k_2)} \tag{46}$$

When $d_1 = a_{12}k_2$ (45) becomes

$$u_1 = u_{10};\ u_2 = u_{20}e^{-a_2 t}\ ;\ u_3 = u_{30}e^{a_3 t} \tag{47}$$

4.7.1 Trajectories of Perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_1 = (u_{10} - m - n)\left(\frac{u_2}{u_{20}}\right)^{\frac{a_{13}k_3 - d_1}{a_2}} + \frac{mu_2}{u_{20}} + n\left(\frac{u_2}{u_{20}}\right)^{\frac{-a_3}{a_2}} \text{ and } \left(\frac{u_2}{u_{20}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{-a_2} \tag{48}$$

4.8 The Normal Steady State: $E_8\,(\bar{N}_1,\ \bar{N}_2,\ \bar{N}_3)$

In this case we get

$$A = \begin{bmatrix} \alpha_1 & a_{12}\gamma & a_{13}\gamma \\ 0 & -\mu & \frac{a_{23}\mu}{a_{22}} \\ 0 & 0 & -a_3 \end{bmatrix} \tag{49}$$

where

$$\alpha_1 = d_1 + a_{12}k_2 - \frac{a_{12}\mu}{a_{22}} \text{and} \gamma = \frac{1}{a_{11}}\left[\frac{a_{12}\mu}{a_{22}} + a_{13}k_3 - d_1\right] \tag{50}$$

The characteristic roots are $\alpha_1,\ -\mu,\ -a_3$.

Case I: When $\alpha_1 > 0,\ i.e,\ d_1 + a_{12}k_2 > \frac{a_{12}\mu}{a_{22}}$

In this case one of the three roots is positive, hence the state is unstable. The equations (5) yield the solutions

$$u_1 = [u_{10} - (\eta - \xi)]\ e^{\alpha_1 t} + \eta e^{-\mu t} - \xi e^{-a_3 t};\ u_2 = (u_{20} - \psi)\ e^{-\mu t} + \psi e^{-a_3 t};\ u_3 = u_{30}e^{-a_3} \tag{51}$$

where

$$\psi = \frac{a_{23}\mu u_{30}}{a_{22}(\mu - a_3)}, \eta = \frac{a_{12}\gamma(u_{20} - \psi)}{\alpha_1 + \mu},\ \xi = \frac{\gamma(a_{12}\psi + a_{13}u_{30})}{\alpha_1 + a_3} \tag{52}$$

Case II: When $\alpha_1 > 0,\ i.e,\ d_1 + a_{12}k_2 > \frac{a_{12}\mu}{a_{22}}$

In this case all the three roots are negative, hence the state is stable. The equations (5) yield the solutions

$$u_1 = [u_{10} - (\eta - \xi)]\ e^{-\alpha_1 t} + \eta e^{-\mu t} - \xi e^{-a_3 t};\ u_2 = (u_{20} - \psi)\ e^{-\mu t} + \psi e^{-a_3 t};\ u_3 = u_{30}e^{-a_3} \tag{53}$$

It can be noticed that $u_1 \to 0, u_2 \to 0$ and $u_3 \to 0$ as $t \to \infty$

Case III: When $\alpha_1 = 0,\ i.e,\ d_1 + a_{12}k_2 = \frac{a_{12}\mu}{a_{22}}$

In this case the state is neutrally stable and the equations (5) yield the solutions.

$$u_1 = u_{10} - (\eta - \xi) + \eta e^{-\mu t} - \xi e^{-a_3 t};\ u_2 = (u_{20} - \psi)\ e^{-\mu t} + \psi e^{-a_3 t};\ u_3 = u_{30}e^{-a_3} \tag{54}$$

4.8.1 Trajectories of Perturbations

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$u_1 = [u_{10} - (\eta - \xi)]\left(\frac{u_3}{u_{30}}\right)^{\frac{-\alpha_1}{a_3}} + \eta\left(\frac{u_3}{u_{30}}\right)^{\frac{\mu}{a_3}} - \frac{u_3\mu}{u_{30}} \text{and} u_2 = (u_{20} - \psi)\left(\frac{u_3}{u_{30}}\right)^{\frac{\mu}{a_3}} + \frac{u_3\psi}{u_{30}} \tag{55}$$

5 Liapunovs Function for Global Stability

In section 4 we discussed the local stability of all eight equilibrium states. From which only two states $E_5\left(0, \bar{N}_2, \bar{N}_3\right)$ and $E_8\left(\bar{N}_1, \bar{N}_2, \bar{N}_3\right)$ are stable and rest of them are unstable. We now examine the global stability of dynamical system (1), (2) and (3) at these two states by suitable Liapunovs functions.

Theorem 1 *The equilibrium state $E_5\left(0, \bar{N}_2, \bar{N}_3\right)$ is globally asymptotically stable.*

Proof Let us consider the following Liapunovs function

$$V(N_2, N_3) = N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right) + l_1\left[N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right)\right] \tag{56}$$

where l_1 is a suitable constant to be determined as in the subsequent steps. Now, the time derivative of V, along with solutions of (2) and (3) can be written as

$$\begin{aligned}\frac{dV}{dt} &= \left(\frac{N_2 - \bar{N}_2}{N_2}\right)\frac{dN_2}{dt} + l_1\left(\frac{N_3 - \bar{N}_3}{N_3}\right)\frac{dN_3}{dt} \\ &= \left(N_2 - \bar{N}_2\right)(a_2 - a_{22}N_2 + a_{23}N_3) + l_1\left(N_3 - \bar{N}_3\right)(a_3 - a_{33}N_3) \\ &= -a_{22}\left(N_2 - \bar{N}_2\right)^2 + a_{23}\left(N_2 - \bar{N}_2\right)\left(N_3 - \bar{N}_3\right) + l_1\left[-a_{33}\left(N_3 - \bar{N}_3\right)^2\right] \end{aligned} \tag{57}$$

Hence

$$\begin{aligned}\frac{dV}{dt} = &-\left[\sqrt{a_{22}}\left(N_2 - \bar{N}_2\right) + \sqrt{l_1 a_{33}}\left(N_3 - \bar{N}_3\right)\right]^2 \\ &-\left(2\sqrt{l_1 a_{22} a_{33}} - a_{23}\right)\left(N_2 - \bar{N}_2\right)\left(N_3 - \bar{N}_3\right)\end{aligned} \tag{58}$$

The positive constant l_1 as so chosen that, the coefficient of $\left(N_2 - \bar{N}_2\right)\left(N_3 - \bar{N}_3\right)$ in (58) vanish. Then we have $l_1 = \frac{a_{23}^2}{4a_{22}a_{33}} > 0$ and, with this choice of the constant l_1

$$V(N_2, N_3) = N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right) + \frac{a_{23}^2}{4a_{22}a_{33}}\left[N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right)\right] \tag{59}$$

$$\frac{dV}{dt} = -\left[\sqrt{a_{22}}\left(N_2 - \bar{N}_2\right) - \frac{a_{23}}{2\sqrt{a_{22}}}\left(N_3 - \bar{N}_3\right)\right]^2 \tag{60}$$

which is negative definite. Hence, the steady state is globally asymptotically stable. □

Theorem 2 *The equilibrium state $E_8\left(\bar{N}_1, \bar{N}_2, \bar{N}_3\right)$ is globally asymptotically stable.*

Proof Let us consider the following Liapunovs function

$$\begin{aligned}V(N_1, N_2, N_3) = &N_1 - \bar{N}_1 - \bar{N}_1 \ln\left(\frac{N_1}{\bar{N}_1}\right) + l_1\left[N_2 - \bar{N}_2 - \bar{N}_2 \ln\left(\frac{N_2}{\bar{N}_2}\right)\right] \\ &+ l_2\left[N_3 - \bar{N}_3 - \bar{N}_3 \ln\left(\frac{N_3}{\bar{N}_3}\right)\right]\end{aligned} \tag{61}$$

where l_1 and l_2 are suitable constants to be determined as in the subsequent steps.

Now, the time derivative of V, along with solutions of (1), (2) and (3) can be written as

$$\begin{aligned}\frac{dV}{dt} &= \left(\frac{N_1 - \bar{N}_1}{N_1}\right)\frac{dN_1}{dt} + l_1\left(\frac{N_2 - \bar{N}_2}{N_2}\right)\frac{dN_2}{dt} + l_2\left(\frac{N_3 - \bar{N}_3}{N_3}\right)\frac{dN_3}{dt} \\ &= \left(N_1 - \bar{N}_1\right)(-d_1 - a_{11}N_1 + a_{12}N_2 + a_{13}N_3) + l_1\left(N_2 - \bar{N}_2\right)(a_2 - a_{22}N_2 + a_{23}N_3) \\ &\quad + l_2\left(N_3 - \bar{N}_3\right)(a_3 - a_{33}N_3) \\ &= -a_{11}\left(N_1 - \bar{N}_1\right)^2 + a_{12}\left(N_1 - \bar{N}_1\right)\left(N_2 - \bar{N}_2\right) + a_{13}\left(N_1 - \bar{N}_1\right)\left(N_3 - \bar{N}_3\right) \\ &\quad + l_1\left[-a_{22}\left(N_2 - \bar{N}_2\right)^2 + a_{23}\left(N_2 - \bar{N}_2\right)\left(N_3 - \bar{N}_3\right)\right] + l_2\left[-a_{33}\left(N_3 - \bar{N}_3\right)^2\right]\end{aligned} \tag{62}$$

Hence

$$\begin{aligned}\frac{dV}{dt} = &-\left[\sqrt{a_{11}}\left(N_1-\bar{N}_1\right)+\sqrt{l_1 a_{22}}\left(N_2-\bar{N}_2\right)+\sqrt{l_2 a_{33}}\left(N_3-\bar{N}_3\right)\right]^2 \\ &+\left(2\sqrt{l_1 a_{11} a_{22}}+a_{12}\right)\left(N_1-\bar{N}_1\right)\left(N_2-\bar{N}_2\right) \\ &+\left(2\sqrt{l_2 a_{11} a_{33}}+a_{13}\right)\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right) \\ &+\left(2\sqrt{l_1 l_2 a_{22} a_{33}}+l_1 a_{23}\right)\left(N_2-\bar{N}_2\right)\left(N_3-\bar{N}_3\right)\end{aligned} \tag{63}$$

The positive constants l_1 and l_2 as so chosen that, the coefficients of $\left(N_1-\bar{N}_1\right)\left(N_2-\bar{N}_2\right)$, $\left(N_1-\bar{N}_1\right)\left(N_3-\bar{N}_3\right)$ and $\left(N_2-\bar{N}_2\right)\left(N_3-\bar{N}_3\right)$ in (63) vanish. Then we have $l_1 = \frac{a_{12}^2}{4a_{11}a_{12}} > 0$ and $l_2 = \frac{a_{13}^2}{4a_{11}a_{33}} > 0$, with this choice of the constants l_1 and l_2.

$$\frac{dV}{dt} = -\sqrt{a_{11}}\left[\left(N_1-\bar{N}_1\right)+\frac{a_{12}}{2a_{11}}\left(N_2-\bar{N}_2\right)+\frac{a_{13}}{2a_{11}}\left(N_3-\bar{N}_3\right)\right]^2 \tag{64}$$

which is negative definite, when $2a_{13}a_{22} = a_{12}a_{23}$. Hence, the normal steady state is globally asymptotically stable. □

6 Numerical Examples

Figure **??** shows the numerical example for the first species has the least initial value and the interaction coefficients a_{13} and a_{23} are almost equal. Further, the first species is dominated by the third which itself dominated by the second as shown in . This is a numerical example for unstability case.

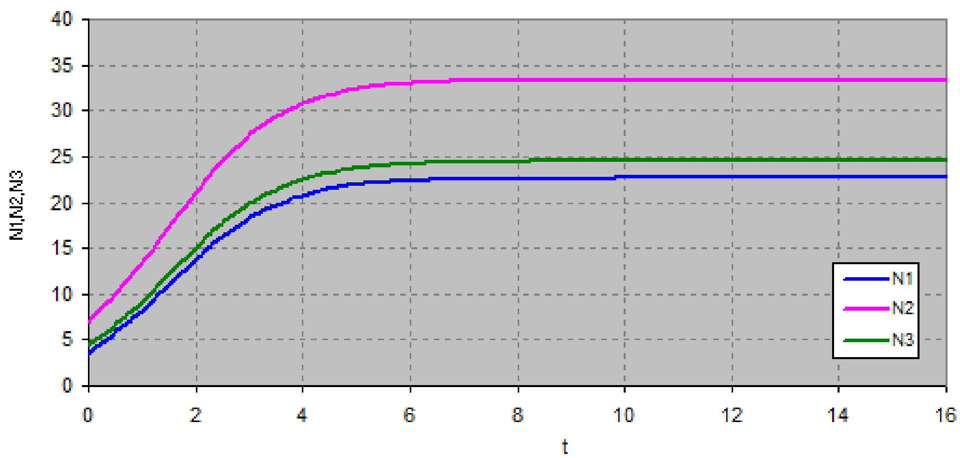

Figure 1: Variation of N_1, N_2, N_3 against time (t) for $d_1 = 12.98, a_2 = 7.82, a_3 = 0.98, a_{11} = 11.86, a_{22} = 4.8, a_{33} = 0.04, a_{12} = 6.16, a_{13} = 3.1, a_{23} = 6.22, N_{10} = 3.5, N_{20} = 7, N_{30} = 4.5$

Figure ?? shows the numerical example for the initial values of S_1, S_2, S_3 are in ascending order. The self inhibition coefficient of the first species is least. Initially the third species dominates over the second till the time instant $t^* = 0.83$ and thereafter the dominance is reversed. Further it is evident that all the three species asymptotically converge to the equilibrium point as illustrated in Figure ??. This is an example for stability case.

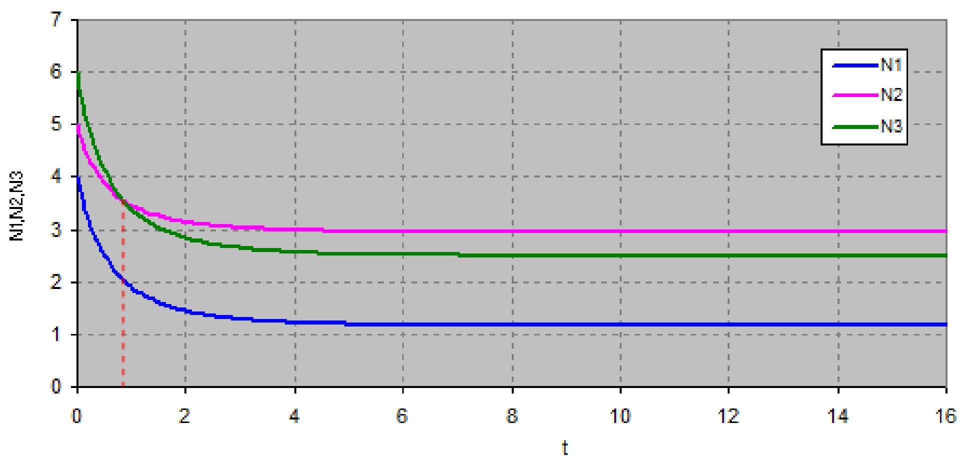

Figure 2: Variation of N_1, N_2, N_3 against time (t) for $d_1 = 18.88, a_2 = 12.04, a_3 = 0.8, a_{11} = 11.16, a_{22} = 7.72, a_{33} = 0.32, a_{12} = 6.64, a_{13} = 4.94, a_{23} = 4.28, N_{10} = 4, N_{20} = 5, N_{30} = 6$

7 Conclusion

The present paper deals with an investigation on the stability of a three species syn ecosystem with mortality rate for commensal. The system comprises of a commensal (S_1), two hosts S_2 and S_3 ie., S_2 and S_3 both benefit S_1, without getting themselves effected either positively or adversely. It is observed that, in all eight equilibrium states, only two states E_5 and E_8 are locally stable. Further the global stability is established with the aid of suitably constructed Liapunovs function-pair and the growth rates of the species are numerically estimated using Runge-Kutta fourth order scheme.

Acknowledgments

I thank to Professor (Retd), N.Ch.Pattabhi Ramacharyulu, Department of Mathematics, NIT, Warangal (A.P.), India for his valuable suggestions and encouragement.

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