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Some aspects of Unitary addition Cayley graph of Gaussian integers modulo n

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Abstract In this paper, the Unitary addition Cayley graph defined by Sinha et al. is extended to the ring of Gaussian integers modulo n. Some graph theoretic properties of unitary addition Cayley graphs of gaussian integers modulo n are discussed here. Moreover, the condition for which the above mentioned graphs are Eulerian and Hamiltonian are also discussed.

Keywords Gaussian Integers, Hamiltonian, Eulerian, Bipartite Graph, Unitary addition Cayley Graph.

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1 Introduction

Let Z_n be the ring of integers modulo n, $n > 1$ and U_n be the set of all units of the ring Z_n. The unitary Cayley graph introduced by Dejter and Giudici [1] is an undirected graph, whose vertex set is the set Z_n and any two vertices a and b of Z_n are adjacent if and only if $a - b \in U_n$. The various structures and properties of unitary Cayley graphs have been studied extensively by Dejter and Gudici[1], Koltz and Sender[2], Akhtar et al.[3] and Boggers et al.[4].

The addition Cayley graph is defined by considering an abelian group Γ and a subset B of Γ. The addition Cayley graph $G' = Cay^+(\Gamma, B)$ is the graph having vertex set $V(G') = \Gamma$ and the edge set $E(G') = \{ab \mid a + b \in B,\ a, b \in \Gamma\}$. Several properties of addition Cayley graphs have been discussed by Grynkiewicz et al.[5] and Grynkiewicz et al.[6].

Sinha, Garg and Singh [7] defined the unitary addition Cayley graph by taking $\Gamma = Z_n$ and $B = U_n$. Several graph theoretic properties of unitary addition Cayley graph have been studied by them.

Our work is a generalisation of the set Z_n to $Z_n[i]$. Every element in $Z_n[i]$ is of the form $a + ib$, where $a, b \in Z_n$. In $Z_n[i]$, a norm is defined as $N(a + ib) = a^2 + b^2$ and an element $c + id$ will be an unit if and only if $gcd(N(c + id), n) = 1$ or simply we can say that $c + id$ will be an unit element in $Z_n[i]$ if and only if $N(c+id)$ is an unit element in Z_n. We denote the Unitary addition Cayley graph of Gaussian integers modulo n by $G_n[i]$ with vertex set $Z_n[i]$ and edge set $U_n[i]$, where $U_n[i]$ is the set of all units in $Z_n[i]$. Any two vertices $a + ib$ and $c + id$ will be adjacent in $G_n[i]$ if and only if $gcd(N((a + c) + i(b + d)), n) = 1$. In this paper we investigate some graph theoretic properties such as degree of a vertex, number of edges, diameter, girth and planarity of the graph $G_n[i]$. We also obtain the condition for which the graph $G_n[i]$ is Eulerian and Hamiltonian.

Before going to our main results we give some definitions and preliminary results which will be used in our main results.

2 Preliminaries and definitions

A *graph* G is a pair $G(V,E)$,where V is a non-empty finite set, and E is a set of unordered pairs of elements of V. The elements of V are called the vertices of G, and the elements of E are the edges of G. The set of vertices and edges of a graph G is denoted by $V(G)$ and $E(G)$ respectively. $|V(G)|$ and $|E(G)|$ denote the cardinality of $V(G)$ and $E(G)$ respectively. The *degree* of a vertex v, denoted by $deg(v)$ in G is the number of edges incident at v. If degree of each vertex is equal, say r in G, then G is called r-regular graph. A *walk* of a *graph* is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, ..., e_n, v_n$ beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. A walk is called a *trail* if all the edges are distinct. If all the vertices and necessarily all the edges of a walk are distinct then it is called a *path*. A closed path(i.e., a path in which $v_0 = v_n$) is called a *cycle*. An *Eulerian trail* is a closed *trail* containing all vertices and edges of a *graph G*. A *graph G* is called an *Eulerian graph* if it contains an *Eulerian trail*. If a *graph G* has a spanning cycle, then G is called *Hamiltonian graph* and the spanning cycle is called a *Hamiltonian cycle*. A *bigraph* or *bipartite graph* G is a graph whose vertex set V can be partitioned in to two subsets V_1 and V_2 such that every edge of V_1 joins with V_2. If G contains every edge joining V_1 and V_2 , then G is called a *complete bipartite graph*. For distinct vertices x and y of a *graph G*, let $d(x,y)$ be the length of a shortest *path* from x to y, the *diameter* of G, denoted by $diam(G) = sup\{d(x,y) : x, y \text{ } are \text{ } vertices \text{ } of \text{ } G\}$. The *girth* of a graph G, denoted by $girth(G)$ is the length of a shortest *cycle* in G ($girth(G) = \infty \text{ } if \text{ } G \text{ } contains \text{ } no \text{ } cycles$). A *graph* that can be drawn in the plane so that edges intersect only at vertices is called *planar*.

Let Γ be a group and B be a subset of Γ such that B does not contain the identity of Γ. Assume $B^{-1}=\{b^{-1} \mid b \in B\} = B$. The *Cayley graph* $X' = Cay(\Gamma, B)$ is an undirected graph having vertex set $V(X') = \Gamma$ and edge set $E(X') = \{ab \mid ab^{-1} \in B\}$, where $a, b \in \Gamma$. The *Cayley graph* $X' = Cay(\Gamma, B)$ is a regular graph of degree $|B|$.

Gaussian integers contains set of all complex numbers $a+ib$, where a and b are integers. It is denoted by $Z[i]$ and is a *Euclidian domain* under usual complex operations, with norm $N(a+ib) = a^2+b^2$. It is clear that $a+ib$ is a unit in $Z[i]$ if and only if $N(a+ib) = 1$,which implies that $1, -1, i$ and $-i$ are the only units. Let $< n >$ be the principal ideal generated by n in $Z[i]$ where n is a natural number and let $Z_n[i]$ denote the ring of integers modulo n. The result that the factor ring $Z[i]/ < n >$ is isomorphic to $Z_n[i] = \{a + ib \mid a, b \in Z_n\}$ is obtained by Dresden and Dymacek[8] and this result will be used in our paper whenever necessary. Some examples of *unitary addition Cayley graph of Gaussian integers modulo n* are displayed in Figure 1, Figure 2 and Figure 3.

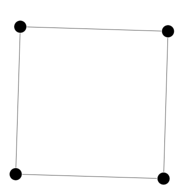

Figure 1: $G_2[i]$

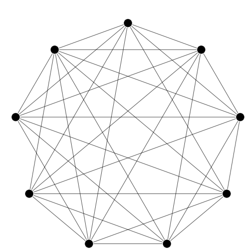

Figure 2: $G_3[i]$

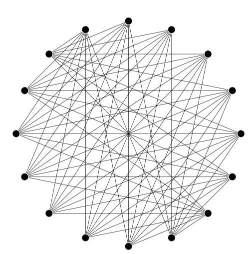

Figure 3: $G_4[i]$

We are now going to investigate the degree of a vertex, number of edges, diameter and girth of the graph $G_n[i]$. But for this investigation we require the following theorem due to R.Boyer[9]. We shall also prove the two lemmas giving the number of elements and number of unit elements $U_n[i]$ in $\mathbb{Z}_n[i]$. First we state the theorem which will be used in proving the second lemma.

Theorem 2.1 [9]*(Chinese Remainder Theorem) Let $A_1, A_2, ..., A_k$ be ideals in R. Consider the mapping*

$$R \longrightarrow R/A_1 \times R/A_2 \times ... \times R/A_k \text{ by } r \longrightarrow (r + A_1, r + A_2, ..., r + A_k)$$

is a ring homomorphism with kernel $A_1 \cap A_2 \cap ... \cap A_k$. If for each $i, j \in \{1, 2, ..., k\}$ with $i \neq j$ the ideals A_i and A_j are comaximal, then the map is surjective and $A_1 \cap A_2 \cap ... \cap A_k = A_1 A_2 ... A_k$. Hence, we have the natural isomorphism $R/A_1 A_2 ... A_k = R/A_1 \cap A_2 \cap ... \cap A_k \cong R/A_1 \times R/A_2 \times ... \times R/A_k$.

Lemma 2.1 *Total number of elements in $Z_n[i]$ is n^2 .*

Proof As the elements of $Z_n[i]$ are written as $a+ib$ where, $a, b \in Z_n$. So, the real part can be filled up by ${}^nC_1$ways and the imaginary part can also be filled up by ${}^nC_1$ ways. Hence the total number of elements will be ${}^nC_1 \times^n C_1 = n^2$. □

Remark 2.1 As we are taking vertex set of the graph $G_n[i]$ as $Z_n[i]$. So, the number of vertex of the graph $G_n[i]$ is n^2.

Lemma 2.2 *Total number of unit elements $|U_n[i]|$ in $Z_n[i]$ are given as the following:*

(i) $|U_n[i]| = 2^{2r-1}$*, when* $n = 2^r, r \in \mathbb{N}$.
(ii) $|U_n[i]| = n^2 - 1$*, when* $n \equiv 3(mod\ 4)$.
(iii) $|U_n[i]| = (n-1)^2$*, when* $n \equiv 1(mod\ 4)$.
(iv) $|U_n[i]| = n^2 - n$*, when* $n = k^2$ *and* k *is an odd prime.*
(v) $|U_n[i]| = |U_{n_1}[i]|.|U_{n_2}[i]|$*, for* $n = n_1 n_2$*, where* n_1 *and* n_2 *are distinct primes.*

Proof

(i) Let $a+ib \in Z_n[i]$, where $n = 2^r, r \in \mathbb{N}$. Then the norm of $a+ib$ i.e., $N(a+ib) = a^2+b^2$ is either an even number or an odd number. As there are n^2 elements in $G_n[i]$, so half of the elements in $G_n[i]$ will have the norm as an even number and the remaining half of the numbers will have the norm as an odd number. As n is even therefore, the elements which have the norm as odd number are the unit elements of $G_n[i]$. So,$|U_n[i]| = 2^{2r}/2 = 2^{2r-1}$.
(ii) Let n be an odd prime and $n \equiv 3(mod\ 4)$, then by [10], $Z_n[i]$ forms a field. So, there will be only one zero divisor namely $0 + 0i$. Therefore, $|U_n[i]| = n^2 - 1$.
(iii) Let n be an odd prime and $n \equiv 1(mod\ 4)$, then an element $a + ib$ will be an unit element if and only if $a^2 + b^2 \not\equiv 0(mod\ n)$, where $a, b \in Z_n$ If a is any number other than 0 then we can take the values of a in ${}^{n-1}C_1$ ways. Now for each a there exists an element b for which $a^2 + b^2 \equiv 0(mod\ n)$. If we discard the value of b for which $a^2 + b^2 \equiv 0(mod\ n)$ then for each value of a we can take the value of b in ${}^{n-1}C_1$ ways. Therefore, total number of elements in will be $(n-1)^2$. Hence $|U_n[i]| = (n-1)^2$.

(iv) Let $n = k^2$,where k is an odd prime, $k \equiv 3(mod\ 4)$ or $k \equiv 1(mod\ 4)$ then there are k elements which are not relatively prime to n. Let $a+ib \in Z_n[i]$, then $a^2+b^2 \equiv 0(mod\, n)$ which is possible only when both a and b are multiples of k. So, a can take values in ${}^kC_1$ ways, also b can take values in ${}^kC_1$ ways. Therefore, the total number of ways such that $a^2 + b^2 \equiv 0(mod\ n)$ is ${}^kC_1 \times^k C_1 = k^2 = n$. Hence total number of units in $Z_n[i]$ is $n^2 - n$, that is $|U_n[i]| = n^2 - n$.

(v) If $n = n_1n_2$, where n_1 and n_2 are distinct primes, then by Theorem 5 [8] and Chinese Remainder Theorem[9] , $Z_n[i] \cong Z_{n_1}[i] \times Z_{n_2}[i]$. Suppose, $|U_{n_1}[i]| = l$ and $|U_{n_2}[i]| = m$. Therefore, number of units in $Z_{n_1}[i] \times Z_{n_2}[i]$ is $l.m$. Thus, $l.m = |U_{n_1}[i]|.|U_{n_2}[i]| = |U_n[i]|$. □

Remark 2.2 If $n = n_1^{\alpha_1}.n_2^{\alpha_2}....n_r^{\alpha_r}$, where $n_1, n_2, ..., , n_r$ are all distinct primes and

$$\alpha_1, \alpha_2, ..., \alpha_r \in \mathbb{N},$$

then $|U_n[i]| = |U_{n_1{}^{\alpha_1}}[i]|.|U_{n_2{}^{\alpha_2}}[i]|...|U_{n_r{}^{\alpha_r}}[i]|$.

Example 2.1 Consider $Z_2[i],\ Z_3[i],\ Z_4[i],\ Z_5[i],\ Z_6[i]$ and $Z_9[i]$ then $|U_2[i]| = 2,\ |U_3[i]| = 8,\ |U_4[i]| = 8,\ |U_5[i]| = 16,\ |U_6[i]| = 16$ and $|U_9[i]| = 72$.

3 Basic invariants of $G_n[i]$

In this section we establish the nature of the degree of each vertex of $G_n[i]$ in theorem 3.1. In theorem 3.2 we find a necessary and sufficient condition for which the graph $G_n[i]$ is complete bipartite. Diameter and girth of $G_n[i]$ are obtained in theorem 3.3 and 3.4 respectively.

Theorem 3.1 *Let $m = a + ib$ be any vertex in $G_n[i]$ then,*

$$deg(m) = \begin{cases} 2^{2r-1}, if n = 2^r, r \in \mathbb{N}. \\ 2^{2r-1}(n_1-1)^2, if\ n = 2^r n_1, r \in \mathbb{N}\ and\ n_1\ \equiv 1(mod\ 4). \\ 2^{2r-1}({n_1}^2-1), if\ n = 2^r n_1, r \in \mathbb{N}\ and\ n_1\ \equiv 3(mod\ 4). \\ (n-1)^2,\ if\ n \equiv 1(mod\ 4)\ and\ gcd(N(m), n) \neq 1. \\ n^2-1,\ if\ n \equiv 3(mod\ 4)\ and\ gcd(N(m), n) \neq 1. \\ n^2-2n,\ if\ n \equiv 1(mod\ 4)\ and\ gcd(N(m), n) = 1. \\ n^2-2,\ if\ n \equiv 3(mod\ 4)\ and\ gcd(N(m), n) = 1. \end{cases}$$

Proof Suppose n is even, and let $a+ib$ be a vertex of the graph $G_n[i]$. Now $0+0i$ will be adjacent to $a+ib$ if and only if $gcd(N((0+a)+i(0+b))) = 1$ that is if $a+ib \in U_n[i]$. It follows that $deg(0+i0) = |U_n[i]|$. Suppose, $c+id \in U_n[i]$ since n is even then $N(c+id)$ is not even. The vertex $a+ib$ will be adjacent to all the vertices of the form $(c-a)+i(d-b)$. But $2(a+ib) \notin U_n[i]$,that means $2(a+ib) \not\equiv (c+id)\ (mod\ n)$, and which implies that $a+ib \not\equiv (c-a)+i(d-b)\ (mod\ n)$. Hence $deg(m) = |U_n[i]|$.

(i) When $n = 2^r, r \in \mathbb{N}$, then $deg(m) = \dfrac{2^r}{2} = 2^{2r-1}$.

(ii) When $n = 2^r n_1, r \in \mathbb{N}$ *and* $n_1 \equiv 1(mod\ 4)$, then
$deg(m) = \dfrac{2^{2r}(n_1-1)^2}{2} = 2^{2r-1}(n_1-1)^2$.

(iii) When $n = 2^r n_1, r \in \mathbb{N}$ *and* $n_1 \equiv 3(mod\ 4)$, then
$deg(m) = \dfrac{2^{2r}(n_1^2-1)}{2} = 2^{2r-1}(n_1^2-1)$.

Suppose, n is odd and $gcd(N(a+ib),n)) \neq 1$. Then $a+ib \notin U_n[i]$, which implies $2(a+ib) \notin U_n[i]$. Thus, $a+ib \not\equiv (c-a)+i(d-b)\ (mod\ n)$. Therefore, $deg(m) = |U_n[i]|$.

Case(i) When $n \equiv 1(mod\ 4)$ then $deg(m) = (n-1)^2$.

Case(ii) When $n \equiv 3(mod\ 4)$ then $deg(m) = (n^2-1)$.

Next we consider n is odd and $gcd(N(a+ib),n)) = 1$. Then $a+ib \in U_n[i]$, which implies $2(a+ib) \in U_n[i]$. Which implies $a+ib \equiv (c-a)+i(d-b)(mod\ n)$. But the vertex cannot be adjacent to itself. Hence $deg(m) = |U_n[i]| - 1$.

Case(i) When $n \equiv 1(mod\ 4)$ then $deg(m) = (n-1)^2 - 1 = n^2 - 2n$.

Case(ii) When $n \equiv 3(mod\ 4)$ then $deg(m) = (n^2-1) - 1 = n^2 - 2$. □

Corollary 3.1 Number of edges q in $G_n[i]$,

$$q = \begin{cases} 2^{2(2r-1)}, if \quad n = 2^r, r \in \mathbb{N}. \\ 2^{2(2r-1)}{n_1}^2(n_1-1)^2, if \quad n = 2^r n_1, r \in \mathbb{N}\ and\ n_1 \equiv 1(mod\ 4). \\ 2^{2(2r-1)}{n_1}^2({n_1}^2-1), if \quad n = 2^r n_1, r \in \mathbb{N}\ and\ n_1 \equiv 3(mod\ 4). \\ \dfrac{(n^2-1)(n-1)^2}{2}, if\ n \equiv 1(mod\ 4). \\ \dfrac{(n^2-1)^2}{2}, if\ n \equiv 3(mod\ 4). \end{cases}$$

Proof First we consider that n is even. We know that the sum of the degrees of the vertices of a graph is twice the number of lines. So, $2q = \sum_{j=1}^{n^2} deg(x_j)$

$\Rightarrow 2q = n^2|U_n[i]|$

$\Rightarrow q = \dfrac{n^2|U_n[i]|}{2}$.

Case(i) When $n = 2^r, r \in \mathbb{N}$, then $q = \dfrac{(2^{2r})(\frac{2^{2r}}{2})}{2}$

$\Rightarrow q = 2^{2(2r-1)}$.

Case(ii) When $n = 2^r n_1, r \in \mathbb{N}$ *and* $n_1 \equiv 1(mod\ 4)$, then $q = \dfrac{(2^{2r}n_1^2)\{\frac{(2^{2r})(n_1-1)^2}{2}\}}{2}$

$\Rightarrow q = 2^{2(2r-1)}{n_1}^2(n_1-1)^2$.

Case(iii) When $n = 2^r n_1, r \in \mathbb{N}$ *and* $n_1 \equiv 3(mod\ 4)$, then $q = \dfrac{(2^{2r})(n_1^2)\{\frac{(2^{2r})(n_1^2-1)}{2}\}}{2}$

$\Rightarrow q = 2^{2(2r-1)}{n_1}^2({n_1}^2-1)$.

Next we consider that n is odd.

Therefore, $2q = (n^2 - |U_n[i]|)|U_n[i]| + |U_n[i]|(|U_n[i]| - 1)$

$\Rightarrow q = \dfrac{(n^2-1)|U_n[i]|}{2}$.

Case(i) When $n \equiv 1(mod\ 4)$, then $q = \dfrac{(n^2-1)(n-1)^2}{2}$

$\Rightarrow q = \dfrac{(n^2-1)(n-1)^2}{2}$.

Case(ii) When $n \equiv 3(mod\ 4)$, then $q = \dfrac{(n^2-1)(n^2-1)}{2}$

$\Rightarrow q = \dfrac{(n^2-1)^2}{2}$. □

Lemma 3.1 *If n is an even number then we can partition the vertex set $V(G_n[i])$ in to two subsets $V_1(G_n[i])$ which contains the vertices with even norm and $V_2(G_n[i])$ which contains the vertices with odd norm, also $|V_1(G_n[i])| = |V_2(G_n[i])| = n^2/2$.*

Proof A vertex $a+ib$ in $G_n[i]$ will have even norm if both the values a and b are even or odd. Since n is even so, there will be $n/2$ even numbers and $n/2$ odd numbers. Suppose, both a and b are even then total number of these type of vertices will be ${}^{n/2}C_1 \times {}^{n/2}C_1 = n^2/4$. Similarly, if both a and b are odd then the total number of these type of vertices ${}^{n/2}C_1 \times {}^{n/2}C_1 = n^2/4$. Hence, total number of vertices with even norm $n^2/4 + n^2/4 = n^2/2$. Also the total number of vertices with odd norm will be $n^2 - n^2/2 = n^2/2$. Therefore, $|V_1(G_n[i])| = |V_2(G_n[i])| = n^2/2$. □

Theorem 3.2 *$G_n[i]$ is a complete bipartite graph if and only if $n = 2^r$, $r \in \mathbb{N}$.*

Proof Suppose, $n = 2^r, r \in \mathbb{N}$. Then by Lemma 3.1 we can partition the vertex set into two sets $V_1 =$ containing all the vertices of even norm, $V_2 =$ containing all the vertices with odd norm. Then by Theorem 3.1, $G_n[i]$ is a complete bipartite graph.

Conversely, we suppose that $G_n[i]$ is a complete bipartite graph. We have to show that n is even of the form $2^r, r \in \mathbb{N}$. By Lemma 3.1 it is clear that the vertex set can be partitioned into two sets if n is an even number. Suppose $n = 2^r n_1$, where $n_1 \equiv 3(mod 4)$ then degree of each vertex will be $2^{2r-1}({n_1}^2 - 1)$ which is certainly less than $n^2/2$. We can draw same conclusion when $n = 2^r n_1$, where $n_1 \equiv 1(mod 4)$. Hence in both cases $G_n[i]$ will be a bipartite graph but not complete bipartite graph. Therefore, n is of the form $2^r, r \in \mathbb{N}$. □

Example 3.1 Let $n = 4$ then by Lemma 3.1 $V_1(G_4[i]) = \{0+i0, 0+i2, 1+i, 1+i3, 2+i0, 2+i2, 3+i, 3+i3\}$ and $V_2(G_4[i]) = \{0+i, 0+i3, 1+i0, 1+i2, 2+i, 2+i3, 3+i0, 3+i2\}$ and by Theorem 3.1 degree of each vertex will be 8. Graphical representation of $G_4[i]$ is shown in Figure 4.

Corollary 3.2 $G_n[i]$ is a bipartite graph if and only if $n = 2^r n_1$, $r \in \mathbb{N}$ and $n_1 \equiv 1(mod\ 4)$ or $n_1 \equiv 3(mod\ 4)$.

Proof follows from the theorem 3.2.

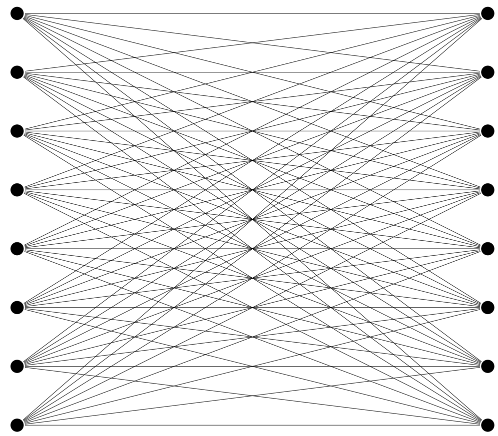

Figure 4: $G_4[i]$

Example 3.2 Let $n = 6$ then by Lemma 3.1 $V_1(G_6[i]) = \{0+i0, 0+i2, 0+i4, 1+i, 1+i3, 1+i5, 2+i0, 2+i2, 2+i4, 3+i, 3+i3, 3+i5, 4+i0, 4+i2, 4+i4, 5+i, 5+i3, 5+i5\}$ and $V_2(G_6[i]) = \{0+i, 0+i3, 0+i5, 1+i0, 1+i2, 1+i4, 2+i, 2+i3, 2+i5, 3+i0, 3+i2, 3+i4, 4+i, 4+i3, 4+i5, 5+i0, 5+i2, 5+i4\}$ containing 18 vertices in each set. By Theorem 3.1 degree of each vertex will be 16. Graphical representation of $G_6[i]$ is shown in Figure 5.

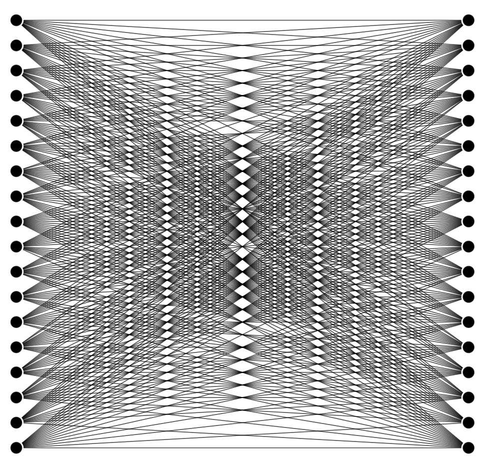

Figure 5: $G_6[i]$

Theorem 3.3 *For* $n \geqslant 2$, $diam(G_n[i])= \begin{cases} \textit{2, if } n \textit{ is even or odd.} \\ \textit{3, if } n = n_1 n_2 \textit{ if } n_1 \textit{ is even} \\ \quad \textit{and } n_2 \textit{ is an odd prime.} \end{cases}$

Proof Suppose n is even then by Theorem 3.2, $G_n[i]$ is a complete bipartite graph and hence $diam(G_n[i]) \doteq 2$.

Suppose n is odd and consider the vertices $(n-1)+in$ and $1+in$, they are not adjacent but $0+i0$ is adjacent to both of them. Thus there will be a path $(n-1)+in-0+i0-1+in$. Hence, $diam(G_n[i]) \leqslant 2$. Theorem 3.1 suggests that $G_n[i]$ will never be a complete graph. So, $diam(G_n[i]) \geqslant 2$. So, $diam(G_n[i])$=2.

Suppose $n = n_1 n_2$, where n_1 is an even number and n_2 is any odd prime. Let us consider the vertices $0+i0$ and n_2+in then they will never have a common vertex adjacent to them. So, $diam(G_n[i]) > 2$. Next we consider the vertices $0+i0$, $(n-1)+in$, $1+i(n-1)$ and n_2+in, then there will be a path $0+i0 - (n-1)+in - 1+i(n-1) - n_2+in$. Therefore, $diam(G_n[i]) \leqslant 3$. Hence, $diam(G_n[i])$=3. □

Theorem 3.4 *For* $n \geqslant 2$, $girth(G_n[i])= \begin{cases} \textit{3, if } n \textit{ is odd.} \\ \textit{4, if } n \textit{ is even.} \end{cases}$

Proof Suppose, n is even then by Corollary 3.1.1, $G_n[i]$ is a bipartite graph and hence $girth(G_n[i]) \geqslant 4$. Now let us consider the vertices $a = 0+i0$, $b = 1+in$, $c = 2+in$ and $d = 3+in$. We can easily see that $gcd(N(a+b), n) = 1$, $gcd(N(b+c), n) = 1$, $gcd(N(c+d), n) = 1$ and $gcd(N(d+a), n) = 1$. But $gcd(N(a+c), n) \neq 1$, $gcd(N(b+d), n) \neq 1$. Thus, we get a closed path $0+i0 - 1+in - 2+in - 3+in - 0+i0$ of length 4. Hence $girth(G_n[i]) = 4$.

Next we consider that n is odd. Let us take the vertices $a' = 0+i0$, $b' = 1+i0$ and $c' = 0+i$. We observe that $gcd(N(a'+b'), n) = 1$, $gcd(N(b'+c'), n) = 1$ and $gcd(N(c'+a'), n) = 1$. Thus, we get a closed path $0+i0 - 1+i0 - 0+i - 0+i0$ of length 3. Hence, $girth(G_n[i]) = 3$. □

4 Traversability and planarity of $G_n[i]$

Theorem 4.1 *The unitary addition Cayley graph of Gaussian integers modulo* n, $G_n[i]$ *is Hamiltonian.*

Proof Suppose, n is even. We can construct a cycle $C = 0+i0 - 1+in - 2+in - ... - (n-1)+in - 1+i(n-1) - 2+i(n-1) - 3+i(n-1) - ... - (n-1)+i(n-1) - ... - n+i(n-1) - n+i(n-2) - ... - n+i3 - n+i2 - n+i - 0+i0$ Since the cycle contains all the vertices of $G_n[i]$ exactly once. Thus, C is one of the Hamiltonian cycles of $G_n[i]$. Hence $G_n[i]$ is a Hamiltonian graph.

Next we consider that n is odd. We can construct a cycle $C' = 1+i0 - 3+i0 - 5+i0 - 7+i0 - ...(n-4)+i0 - (n-2)+i0 - 0+i0 - 2+in - 4+in - 6+in - 8+in - (n-5)+in - (n-3)+in - (n-1)+in - 1+i(n-1) - 2+i(n-1) - (n-1)+i(n-1) - 1+i(n-2) - 2+i(n-2) - 3+i(n-2) - (n-2)+i - (n-1)+i - 1+i0$. In this case also C' contains all the vertices of $G_n[i]$ exactly once. Thus, C' is one of the Hamiltonian cycles of $G_n[i]$. Hence $G_n[i]$ is a Hamiltonian graph. □

Remark 4.1 As we get a Hamiltonian cycle when n is even as well as when n is odd. We can say that in both cases the graph $G_n[i]$ is a connected graph.

Theorem 4.2 *If n is an even number then unitary addition Cayley graph of Gaussian integers modulo n, $G_n[i]$ is Eulerian.*

Proof Suppose $n = 2^r, r \in \mathbb{N}$, then by the Theorem 3.1, the degree of each vertex will be $|U_n[i]| = 2^{2(2r-1)}$. Again when $n = 2^r n_1, r \in \mathbb{N}$ and $n_1 \equiv 1(mod\ 4)$ *or* $n_1 \equiv 3(mod\ 4)$, then the degree of each vertex will be $2^{2(2r-1)}\ (n_1)^2\ (n_1 - 1)^2$ or $2^{2(2r-1)}(n_1)^2({n_1}^2 - 1)$, which are even numbers. Again a graph G is Eulerian if and only if G is connected and all its vertices have even degrees. By the remark above, the graph $G_n[i]$ is connected and, the degree of all the vertices of the graph $G_n[i]$ is even. Therefore, the graph $G_n[i]$ is Eulerian. □

Theorem 4.3 *$G_n[i]$ is planer if and only if $n = 1, 2$.*

Proof When $n = 2$ then $G_n[i]$ is a cycle of length 4. Hence $G_n[i]$ is planar for $n = 1, 2$. Again we know that a simple planar connected graph has a vertex of degree less than six. But when $n = 3$ then the degree of unit elements of $G_3[i]$ is seven and the degree of zero divisor is eight. Therefore, $G_n[i]$ is nonplaner for $n \geqslant 3$. □

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