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On Estimates for the Generalized Fourier-Dunkl Transform in the Space $L^2_{\alpha,n}$

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Abstract Two useful estimates are proved for the generalized Fourier-Dunkl transform in the space $L^2_{\alpha,n}$ on certain classes of functions characterized by the generalized continuity modulus.

Keywords Differential-difference operator; Generalized Fourier-Dunkl transform; Generalized translation operator.

Mathematics Subject Classification 42B37, 42B10.

1 Introduction

In [5], Abilov et al. proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator.

In this paper, we consider a first-order singular differential-difference operator Λ on $\mathbb{R}$ which generalizes the Dunkl operator Λ_α, we prove two useful estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the generalized Fourier-Dunkl transform associated to Λ in $L^2_{\alpha,n}$ analogs of the statements proved in [5]. For this purpose, we use a generalized translation operator.

In section 2, we give some definitions and preliminaries concerning the generalized Fourier-Dunkl transform. Two useful estimates are proved in section 3.

2 Preliminaries

In this section, we develop some results from harmonic analysis related to the differential-difference operator Λ. Further details can be found in [1] and [6]. In all what follows assume

where $\alpha > -1/2$ and n a non-negative integer.
Consider the first-order singular differential-difference operator on $\mathbb{R}$ defined by

$$\Lambda f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x} - 2n\frac{f(-x)}{x}.$$

For $n = 0$, we regain the differential-difference operator

$$\Lambda_\alpha f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x},$$

which is referred to as the Dunkl operator of index $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Such operators have been introduced by Dunkl (see [3], [4]) in connection with a generalization of the classical theory of spherical harmonics.
Let M be the map defined by

$$Mf(x) = x^{2n} f(x), \quad n = 0, 1, ..$$

Let $L^p_{\alpha,n}$, $1 \leq p < \infty$, be the class of measurable functions f on $\mathbb{R}$ for which

$$\|f\|_{p,\alpha,n} = \|M^{-1}f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx\right)^{1/p}.$$

If $p = 2$, then we have $L^2_{\alpha,n} = L^2(\mathbb{R}, |x|^{2\alpha+1})$.
The one-dimensional Dunkl kernel is defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz), z \in \mathbb{C}, \tag{1}$$

where

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m!\Gamma(m+\alpha+1)}, z \in \mathbb{C}, \tag{2}$$

is the normalized spherical Bessel function of index α. It is well-known that the functions $e_\alpha(\lambda.)$, $\lambda \in \mathbb{C}$, are solutions of the differential-difference equation

$$\Lambda_\alpha u = \lambda u, u(0) = 1.$$

In the terms of $j_\alpha(x)$, we have (see [2])

$$1 - j_\alpha(x) = O(1), x \geq 1. \tag{3}$$

$$1 - j_\alpha(x) = O(x^2), 0 \leq x \leq 1. \tag{4}$$

$$\sqrt{hx} J_\alpha(hx) = O(1), hx \geq 0, \tag{5}$$

where $J_\alpha(x)$ is Bessel function of the first kind, which is related to $j_\alpha(x)$ by the formula

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1)}{x^\alpha} J_\alpha(x), x \in \mathbb{R}^+. \tag{6}$$

For $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n} e_{\alpha+2n}(i\lambda x),$$

where $e_{\alpha+2n}$ is the Dunkl kernel of index $\alpha + 2n$ given by (1).

Proposition 2.1
(i) φ_λ satisfies the differential equation

$$\Lambda\varphi_\lambda = i\lambda\varphi_\lambda.$$

(ii) For all $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$

$$|\varphi_\lambda(x)| \leq |x|^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Dunkl transform we call the integral transform

$$\mathcal{F}_\Lambda f(\lambda) = \int_{\mathbb{R}} f(x)\varphi_{-\lambda}(x)|x|^{2\alpha+1}dx, \lambda \in \mathbb{R}, f \in L^1_{\alpha,n}.$$

Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_\Lambda(f) \in L^1_{\alpha+2n} = L^1(\mathbb{R}, |x|^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Dunkl transform is given by the formula

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda f(\lambda)\varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n}|\lambda|^{2\alpha+4n+1}d\lambda, \quad a_\alpha = \frac{1}{2^{2\alpha+2}(\Gamma(\alpha+1))^2}.$$

Proposition 2.2
(i) For every $f \in L^2_{\alpha,n}$,

$$\mathcal{F}_\Lambda(\Lambda f)(\lambda) = i\lambda\mathcal{F}_\Lambda(f)(\lambda).$$

(ii) For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2|x|^{2\alpha+1}dx = \int_{\mathbb{R}} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

(iii) The generalized Fourier-Dunkl transform $\mathcal{F}_\Lambda$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2(\mathbb{R}, \mu_{\alpha+2n})$.

The generalized translation operators τ^x, $x \in \mathbb{R}$, tied to Λ are defined by

$$\begin{aligned}\tau^x f(y) &= \frac{(xy)^{2n}}{2}\int_{-1}^{1} \frac{f(\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n}\left(1 + \frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt \\ &+ \frac{(xy)^{2n}}{2}\int_{-1}^{1} \frac{f(-\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n}\left(1 - \frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt,\end{aligned}$$

where

$$A(t) = \frac{\Gamma(\alpha + 2n + 1)}{\sqrt{\pi}\Gamma(\alpha + 2n + 1/2)}(1 + t)(1 - t^2)^{\alpha+2n-1/2}.$$

Proposition 2.3 *Let $x \in \mathbb{R}$ and $f \in L^2_{\alpha,n}$. Then $\tau^x f \in L^2_{\alpha,n}$ and*

$$\|\tau^x f\|_{2,\alpha,n} \leq 2x^{2n}\|f\|_{2,\alpha,n}.$$

Furthermore,

$$\mathcal{F}_\Lambda(\tau^x f)(\lambda) = x^{2n} e_{\alpha+2n}(i\lambda x)\mathcal{F}_\Lambda(f)(\lambda). \tag{7}$$

The generalized modulus of continuity of function $f \in L^2_{\alpha,n}$ is defined as

$$w(f, \delta)_{2,\alpha,n} = \sup_{0<h\leq\delta} \|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha,n}, \delta > 0.$$

Let $W^r_{2,\phi}(\Lambda)$, $r = 0, 1, ...$, denote the class of functions $f \in L^2_{\alpha,n}$ that have generalized derivatives satisfying the estimate

$$\omega(\Lambda^r f, \delta)_{2,\alpha,n} = O(\phi(\delta)), \quad \delta \to 0,$$

where $\phi(x)$ is any nonnegative function given on $[0, \infty)$, and $\Lambda^0 f = f$, $\Lambda^r f = \Lambda(\Lambda^{r-1} f)$, $r = 1, 2, ...$
i.e.,

$$W^r_{2,\phi}(\Lambda) = \{f \in L^2_{\alpha,n}, \Lambda^r f \in L^2_{\alpha,n} \text{and} \quad \omega(\Lambda^r f, \delta)_{2,\alpha,n} = O(\phi(\delta)), \delta \to 0\}.$$

3 Main Results

The goal of this work is to prove two useful estimates for the integral

$$J^2_N(f) = \int_{|\lambda|\geq N} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

in certain classes of functions in $L^2_{\alpha,n}$.

Lemma 3.1 *For $f \in W^r_{2,\phi}(\Lambda)$, we have,*

$$\|\tau^h \Lambda^r f(x) + \tau^{-h}\Lambda^r f(x) - 2h^{2n}\Lambda^r f(x)\|^2_{2,\alpha,n}$$
$$= 4h^{4n} \int_{\mathbb{R}} \lambda^{2r} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

where $r = 0, 1, 2, ...$

Proof From formula (i) of proposition 2.2, we obtain

$$\mathcal{F}_\Lambda(\Lambda^r f)(\lambda) = (i\lambda)^r \mathcal{F}_\Lambda f(\lambda); r = 0, 1, ... \tag{8}$$

By using the formulas (1), (2) and (7), we conclude that

$$\mathcal{F}_\Lambda(\tau^h f + \tau^{-h} f - 2h^{2n} f)(\lambda) = 2h^{2n}(j_{\alpha+2n}(\lambda h) - 1)\mathcal{F}_\Lambda f(\lambda). \tag{9}$$

Now by formulas (8), (9) and Plancherel equality, we have the result. □

Theorem 3.2 *Given* r *and* $f \in W^r_{2,\phi}(\Lambda)$*. Then there exist a constant* $c > 0$ *such that, for all* $N > 0$*,*

$$J_N(f) = O(N^{-r+2n}\phi(c/N)).$$

Proof
Firstly, we have

$$J_N^2(f) \le \int_{|\lambda|\ge N} |j| d\mu + \int_{|\lambda|\ge N} |1-j| d\mu, \tag{10}$$

with $j = j_p(\lambda h)$, $p = \alpha + 2n$ and $d\mu = |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$. The parameter $h > 0$ will be chosen in an instant.
In view of formulas (5) and (6), there exist a constant $c_1 > 0$ such that

$$|j| \le c_1(|\lambda| h)^{-p-\frac{1}{2}}.$$

Then

$$\int_{|\lambda|\ge N} |j| d\mu \le c_1 (hN)^{-p-\frac{1}{2}} J_N^2(f).$$

Choose a constant c_2 such that the number $c_3 = 1 - c_1 c_2^{-p-\frac{1}{2}}$ is positif.
Setting $h = c_2/N$ in the inequality (10), we have

$$C_3 J_N^2(f) \le \int_{|\lambda|\ge N} |1-j| d\mu. \tag{11}$$

By Hölder inequality the second term in (11) satisfies

$$\begin{aligned}
\int_{|\lambda|\ge N} |1-j| d\mu &= \int_{|\lambda|\ge N} |1-j|.1.d\mu \\
&\le \left(\int_{|\lambda|\ge N} |1-j|^2 d\mu\right)^{1/2} \left(\int_{|\lambda|\ge N} d\mu\right)^{1/2} \\
&\le \left(\int_{|\lambda|\ge N} \lambda^{-2r}|1-j|^2 \lambda^{2r} d\mu\right)^{1/2} J_N(f) \\
&\le N^{-r}\left(\int_{|\lambda|\ge N} |1-j|^2 \lambda^{2r} d\mu\right)^{1/2} J_N(f).
\end{aligned}$$

From Lemma 3.1, we conclude that

$$\int_{|\lambda|\ge N} |1-j|^2 \lambda^{2r} d\mu \le h^{-4n} \|\tau^h \Lambda^r f(x) + \tau^{-h}\Lambda^r f(x) - 2h^{2n}\Lambda^r f(x)\|^2_{2,\alpha,n}.$$

Therefore

$$\int_{|\lambda|\geq N} |1-j| d\mu \leq N^{-r} h^{-2n} \|\tau^h \Lambda^r f(x) + \tau^{-h} \Lambda^r f(x) - 2h^{2n} \Lambda^r f(x)\|_{2,\alpha,n} J_N(f).$$

For $f \in W^r_{2,\phi}(\Lambda)$ there exist a constant $c_4 > 0$ such that

$$\|\tau^h \Lambda^r f(x) + \tau^{-h} \Lambda^r f(x) - 2h^{2n} \Lambda^r f(x)\| \leq c_4 \phi(h).$$

For $h = c_2/N$, we obtain

$$c_3 J_N^2(f) \leq c_2^{-2n} N^{2n-r} c_4 \phi(c_2/N) J_N(f).$$

Consequently

$$c_2^{2n} c_3 J_N(f) \leq c_4 N^{-r+2n} \phi(c_2/N).$$

for all $N > 0$. The theorem is proved with $c = c_2$. □

Theorem 3.3 *Let* $\phi(t) = t^\nu$, *then*

$$J_N(f) = O(N^{-r-\nu+2n}) \Leftrightarrow f \in W^r_{2,\phi}(\Lambda),$$

where, $r = 0, 1, ...; 0 < \nu < 2.$

Proof We prove sufficiency by using Theorem 3.2 let $f \in W^r_{2,\phi}(\Lambda)$ then

$$J_N(f) = O(N^{-r-\nu+2n}).$$

To prove necessity let

$$J_N(f) = O(N^{-r-\nu+2n})$$

i.e.

$$\int_{|\lambda|\geq N} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(N^{-2r-2\nu+4n})$$

It is easy to show, that there exists a function $f \in L^2_{\alpha,n}$ such that $\Lambda^r f \in L^2_{\alpha,n}$ and

$$\Lambda^r f(x) = i^r \int_{\mathbb{R}} \lambda^r \mathcal{F}_\Lambda f(\lambda) \varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda). \tag{12}$$

From formula (12) and Plancherel equality, we have

$$\begin{aligned}\|\tau^h \Lambda^r f(x) &+ \tau^{-h} \Lambda^r f(x) - 2h^{2n} \Lambda^r f(x)\|^2_{2,\alpha,n} \\ &= 4h^{4n} \int_{\mathbb{R}} \lambda^{2r} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).\end{aligned}$$

This integral is divided into two

$$\int_{\mathbb{R}} = \int_{|\lambda|\leq N} + \int_{|\lambda|\geq N} = I_1 + I_2,$$

where $N=[h^{-1}]$, We estimate them separately.
From (3), we have the estimate

$$\begin{aligned} I_2 &\leq c_5\int_{|\lambda|\geq N}\lambda^{2r}|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\ &= c_5\sum_{l=0}^{\infty}\int_{N+l\leq|\lambda|\leq N+l+1}\lambda^{2r}|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\\ &\leq c_5\sum_{l=0}^{\infty}a_l(u_l-u_{l+1}), \end{aligned}$$

with $a_l=(N+l+1)^{2r}$ and $u_l=\displaystyle\int_{|\lambda|\geq N+l}|\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$.
For all integers $m\geq 1$, the Abel transformation shows

$$\begin{aligned} \sum_{l=0}^{m}a_l(u_l-u_{l+1}) &= a_0u_0+\sum_{l=1}^{m}(a_l-a_{l-1})u_l-a_mu_{m+1}\\ &\leq a_0u_0+\sum_{l=1}^{n}\left(a_l-a_{l-1}\right)u_l, \end{aligned}$$

because $a_mu_{m+1}\geq 0$. Moreover by the finite increments theorem, we have

$$a_l-a_{l-1}\leq 2r(N+l+1)^{2r-1}$$

Furthermore by the hypothesis of f there exists $c_6>0$ such that, for all $N>0$

$$J_N^2(f)\leq c_6N^{-2r-2\nu+4n},$$

For $N\geq 1$, we have

$$\begin{aligned} \sum_{l=1}^{m}\left(a_l-a_{l-1}\right)u_l &\leq c_6\left(1+\frac{1}{N}\right)^{2r}N^{-2\nu+4n}+2rc_6\sum_{l=1}^{m}\left(1+\frac{1}{N+l}\right)^{2r-1}(N+l)^{-1-2\nu+4n}\\ &\leq 2^{2r}c_6N^{-2\nu+4n}+2r2^{2r-1}c_6\sum_{l=1}^{m}(N+l)^{-1-2\nu+4n}. \end{aligned}$$

Finally, by the integral comparison test we have

$$\sum_{l=1}^{m}(N+l)^{-1-2\nu+4n}\leq\int_N^{\infty}x^{-1-2\nu+4n}dx=\frac{1}{2\nu-4n}N^{-2\nu+4n}.$$

Letting $m\to\infty$ we see that, for $r\geq 0$ and $\nu>0$, there exists a constant c_7 such that, for all $N\geq 1$ and for $h>0$,

$$I_2\leq c_7N^{-2\nu+4n}.$$

Now, we estimate I_1. From formula (4), we have

$$\begin{aligned} I_1 &\leq c_8 h^4 \int_{|\lambda|\leq N} \lambda^{2r+4} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\ &= c_8 h^4 \sum_{l=0}^{N-1} \int_{l\leq|\lambda|\leq l+1} \lambda^{2r+4} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\ &\leq c_8 h^4 \sum_{l=0}^{N-1} (l+1)^{2r+4}(v_l - v_{l+1}), \end{aligned}$$

with $v_l = \displaystyle\int_{|\lambda|\geq l} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$.
Using an Abel transformation and proceeding as with I_2 we obtain

$$\begin{aligned} I_1 &\leq c_8 h^4 \left(v_0 + \sum_{l=1}^{N-1} ((l+1)^{2r+4} - l^{2r+4}) v_l \right) \\ &\leq c_8 h^4 \left(v_0 + (2r+4) c_6 \sum_{l=1}^{N-1} (l+1)^{2r+3} l^{-2r-2\nu+4n} \right), \end{aligned}$$

since $v_l \leq c_6 l^{-2r-2\nu+4n}$ by hypothesis. From the inequality $l+1 \leq 2l$ we conclude

$$I_1 \leq c_8 h^4 \left(v_0 + c_9 \sum_{l=1}^{N-1} l^{3-2\nu+4n} \right).$$

As a consequence of a series comparison for $\mu \geq 1$ and $\mu < 1$ we have the inequality,

$$\mu \sum_{l=1}^{N-1} l^{\mu-1} < N^\mu, \text{for} \quad \mu > 0 \quad \text{and} \quad N \geq 2.$$

If $\mu = 4 - 2\nu + 4n > 0$ for $\nu < 2$ then we obtain

$$I_1 \leq c_8 h^4 \left(v_0 + c_{10} N^{4-2\nu+4n} \right) \leq c_8 h^4 \left(v_0 + c_{10} h^{-4+2\nu-4n} \right),$$

since $N \leq 1/h$. If h is sufficiently small then $v_0 \leq c_{10} h^{-4+2\nu-4n}$. Then we have

$$I_1 \leq c_{11} h^{2\nu-4n}$$

Combining the estimates for I_1 and I_2 gives

$$\|\tau^h \Lambda^r f(x) + \tau^{-h} \Lambda^r f(x) - 2h^{2n} \Lambda^r f(x)\|_{2,\alpha,n} = O(h^\nu),$$

The necessity is proved. □

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