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New Fourth Order Quartic Spline Method for Solving Second Order Boundary Value Problems

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Abstract In this article, a fourth order quartic spline method has been developed to obtain the numerical solution of second order boundary value problem with Dirichlet boundary conditions. The development of the quartic spline method and convergence analysis have been presented. Three test problems have been used for numerical experimentations purposes. Numerical experimentations showed that the quartic spline method generates more accurate numerical results compared with an existing cubic spline method in solving second order boundary value problems.

Keywords Boundary value problem; spline interpolation; quartic spline method; shooting method.

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1 Introduction

Boundary value problems (BVPs) are arising frequently in various fields of sciences and engineering. Generally, it is very difficult to solve these types of problems analytically. Hence, numerous numerical methods have been developed to find the approximate solutions for these problems. One of these methods is called the spline method. The use of spline method for solving BVPs was first discussed by Bickley in 1968, [1]. Following his work, many researchers started using spline methods to approximate BVPs. For instance, Caglar *et al.* [2], Ramadan *et al.* [3], Rashidinia *et al.* [4], Al-Said *et al.* [5], Hamid *et al.* [1, 6] and Fauzi and Sulaiman [7] have used different degrees of splines to approximate second order BVPs. Most of these researchers used their spline methods to approximate special cases of BVPs such as, linear BVPs and BVPs without the presence of the first derivative.

In our work, however, we are considering the general second order BVPs of the form

$$u'' = f\left(x, u, u'\right), \quad a \leq x \leq b, \tag{1}$$

subject to the boundary conditions

$$u\left(a\right) = \alpha, \quad u\left(b\right) = \beta. \tag{2}$$

Keller [8] shown that problem (1) together with the boundary conditions in (2), has a unique solution if $f\left(x, u, u'\right)$ satisfies the following conditions:

(i) $f\left(x, u, u'\right)$ is continuous on a domain Ω, where the domain Ω is defined as $\Omega = \left\{\left(x, u, u'\right) \middle|\, a \leq x \leq b, -\infty < u < \infty, -\infty < u' < \infty\right\}$;

(ii) $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial u'}$ exist and continuous for all $\left(x, u, u'\right) \in \Omega$; and

(iii) $\frac{\partial f}{\partial u} > 0$ and $\left|\frac{\partial f}{\partial u'}\right| \leq W$, for some positive constant W.

However, in the rest of this discussion, we have to assume that $u \in C^5[a,b]$. The main objective of our research is to introduce a new quartic spline method to approximate the second order BVPs as in (1).

This paper is organized as follows. In Section 2, a quartic spline method is constructed. Section 3 discussed the convergence of the proposed method. To show the performance of the proposed method and for comparison purposes, some numerical examples are given in Section 4. Finally, the conclusion is given in Section 5.

2 Quartic Spline

Let P be the partition for the interval $[a,b]$ such that

$$P : a = x_0, x_1, \ldots, x_n = b,$$

where $x_i = a + ih$ and $h = \frac{b-a}{n}$. We assumed that our quartic spline function has to satisfy the following conditions:

(i) $S(x) = s_i(x)$, $x \in [x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$;

(ii) $S(a) = u(a)$, $S(b) = u(b)$; and

(iii) $s_i^{(r)}(x_{i+1}) = s_{i+1}^{(r)}(x_{i+1})$, $r = 0, 1, 2, 3$.

We let $u(x)$ be the exact solution of problem (1) and s_i be the approximate solution to $u_i = u(x_i)$ obtained by the quartic spline $s_i(x)$ on the interval $[x_i, x_{i+1}]$. Since our spline is of degree four, the third derivative is a linear polynomial, which can be written as follows

$$s_i'''(x) = Z_{i+1}\frac{(x - x_i)}{h} + Z_i\frac{(x_{i+1} - x)}{h}, \tag{3}$$

where $Z_i = s_i'''(x)$, $x \in [x_i, x_{i+1}]$. On integrating equation (3) three times, we obtain

$$s_i(x) = Z_{i+1}\frac{(x - x_i)^4}{24h} - Z_i\frac{(x_{i+1} - x)^4}{24h} + A_i(x - x_i)^2 + B_i(x_{i+1} - x) + C_i(x - x_i), \tag{4}$$

where A_i, B_i and C_i, $i = 0, 1, 2, \ldots, n-1$, are coefficients which need to be determined in terms of u_i, u_{i+1}, μ_i and Z_i. In order to derive explicit expressions for the three coefficients of equation (4), we define the following relations:

$$u_i = s_i(x_i), \tag{5}$$

$$u_{i+1} = s_i(x_{i+1}), \tag{6}$$

and

$$\mu_i = s_i''(x_i). \tag{7}$$

From equations (5), (6) and (7), and by using straightforward calculation, we obtain the following equations:

$$A_i = \frac{\mu_i}{2} + \frac{h}{4}Z_i, \tag{8}$$

$$B_i = \frac{u_i}{h} + \frac{h^2}{24} Z_i, \tag{9}$$

and

$$C_i = \frac{u_{i+1}}{h} - \frac{h^2}{4} Z_i - \frac{h^2}{24} Z_{i+1} - \frac{h}{2} \mu_i. \tag{10}$$

Now, we impose the first and second continuity conditions of quartic spline $s_i(x)$ at the point x_{i+1} i.e. $s_i^{(r)}(x_{i+1}) = s_{i+1}^{(r)}(x_{i+1})$, $r = 1, 2$, and the following relations are obtained

$$\frac{5h^2}{24} Z_i + \frac{6h^2}{24} Z_{i+1} + \frac{h^2}{24} Z_{i+2} + \frac{h}{2} \mu_i + \frac{h}{2} \mu_{i+1} = \frac{u_i - 2u_{i+1} + u_{i+2}}{h}, \tag{11}$$

and

$$\frac{h}{2} Z_i + \frac{h}{2} Z_{i+1} + \mu_i - \mu_{i+1} = 0. \tag{12}$$

Then, we eliminate μ_{i+1} from equations (11) and (12) to obtain

$$\mu_i = \frac{u_i - 2u_{i+1} + u_{i+2}}{h^2} - \frac{11h}{24} Z_i - \frac{12h}{24} Z_{i+1} - \frac{h}{24} Z_{i+2}. \tag{13}$$

On substituting equation (13) into equation (11), we obtain the following main recurrence relation given by

$$Z_i + 11Z_{i+1} + 15Z_{i+2} + Z_{i+3} = \frac{24}{h^3} \left(-u_i + 3u_{i+1} - 3u_{i+2} + u_{i+3}\right), i = 0, 1, 2, \ldots, n-3. \tag{14}$$

Equation (14) forms a system of $n-2$ equations with $n+1$ unknowns, which are the Z_i, $i = 0, 1, 2, \ldots, n$. To solve this system uniquely, we have to add three more conditions at the end points i.e. x_0 and x_n. Hence, we choose $Z_0 = Z_n = \mu_0 = 0$. To obtain the last equation, we substitute $Z_0 = 0$ and $\mu_0 = 0$ in equation (13) to obtain

$$12Z_1 + Z_2 = \frac{24}{h^2} \left(u_0 - 2u_1 + u_2\right). \tag{15}$$

Equations (14) and (15) form a system of $n-1$ equations with $n-1$ unknowns. These unknowns can be solved using the *MATHEMATICA* software. Finally, to construct an algorithm for the quartic spline method, we can use the following steps:

Step 1: Divide the interval $[a, b]$ into $n-1$ subintervals by taking $x_i = a + ih$, where $h = 1/n$ and $i = 0, 1, 2, \ldots, n$.

Step 2: Apply shooting method with the fourth order explicit Runge-Kutta method to problem (1), to obtain the approximate solution u_i at the grid points.

Step 3: Use equations (14) and (15) to form a system of linear equations, and then solve for the values of A_i, B_i and C_i for $i = 0, 1, 2, \ldots, n-1$.

Step 4: Use the values of A_i, B_i, C_i, Z_i and u_i obtained from **Step 2** and **Step 3** to construct the quartic spline solution $s_i(x)$ in equation (4), to approximate the solution of problem (1).

3 Convergence Analysis

Let $s_i(x)$ given by equation (4), denotes the quartic spline using the exact values u_i, μ_i and Z_i. Also, let $\tilde{s}_i(x)$ denotes the quartic spline constructed using $\tilde{u}_i$, $\tilde{\mu}_i$ and $\tilde{Z}_i$, where $\tilde{u}_i$ is the approximate solution of problem (1) obtained by the shooting method with fourth order explicit Runge-Kutta method; while $\tilde{\mu}_i$ and $\tilde{Z}_i$ are the second and the third derivative of the function $s_i(x)$ at the point $(x_i, \tilde{u}_i)$, respectively. Then, $\tilde{s}_i(x)$ is given by

$$\tilde{s}_i(x) = \tilde{Z}_{i+1}\frac{(x-x_i)^4}{24h} - \tilde{Z}_i\frac{(x_{i+1}-x)^4}{24h} + \tilde{A}_i(x-x_i)^2 + \tilde{B}_i(x_{i+1}-x) + \tilde{C}_i(x-x_i), \tag{16}$$

where

$$\tilde{A}_i = \frac{\tilde{\mu}_i}{2} + \frac{h}{4}\tilde{Z}_i,$$

$$\tilde{B}_i = \frac{\tilde{u}_i}{h} + \frac{h^2}{24}\tilde{Z}_i,$$

and

$$\tilde{C}_i = \frac{\tilde{u}_{i+1}}{h} - \frac{h^2}{4}\tilde{Z}_i - \frac{h^2}{24}\tilde{Z}_{i+1} - \frac{h}{2}\tilde{\mu}_i,$$

for $x \in [x_i, x_{i+1}]$.

Assume that $e(x)$ defines the error between the exact solution $u(x)$ and the spline function $\tilde{s}_i(x)$ for problem (1) given by

$$e(x) = u(x) - \tilde{S}(x), x \in [a,b]. \tag{17}$$

It is easy to verify that we can rewrite the error function $e(x)$ as follows

$$\begin{aligned} e(x) &= [u(x) - S(x)] + \left[S(x) - \tilde{S}(x)\right] \\ e(x) &= e_I(x) + e_D(x), \end{aligned} \tag{18}$$

where $e_I(x)$ is the error caused by spline interpolation and $e_D(x)$ is the error caused by the discretization of problem (1). Now, to estimate $e(x)$, we have to estimate $e_I(x)$ and $e_D(x)$ separately.

Since our spline is a polynomial of degree four, then we can write $e_I(x)$ over the interval $[x_i, x_{i+1}]$ as

$$u(x) - s_i(x) = \frac{u^{(5)}(\zeta_i)}{5!}(x-x_{i-2})(x-x_{i-1})(x-x_i)(x-x_{i+1})(x-x_{i+2}), \tag{19}$$

for some $\zeta_i \in [x_i, x_{i+1}]$. We recalled that every subinterval has length of h, and if we let $t = x - x_i$, then equation (19) can be rewritten as

$$u(x) - s_i(x) = \frac{u^{(5)}(\zeta_i)}{5!}(2h+t)(h+t)(t)(h-t)(2h-t). \tag{20}$$

Calculation on the expression $(2h+t)(h+t)(t)(h-t)(2h-t)$ in equation (20) shows that it has maximum value at $t = -\sqrt{\frac{15+\sqrt{145}}{10}}h$, and it is equal to $3.632h^5$. Then, $u(x) - s_i(x)$ is bounded by

$$\|u(x) - s_i(x)\|_\infty \leq 0.0303h^5\left\|u^{(5)}(\zeta_i)\right\|_\infty. \tag{21}$$

Let $W^5 = \max_{x\in[a,b]} \left\|u^{(5)}(x)\right\|_\infty$. Therefore, it is easy to conclude that

$$\|e_I(x)\|_\infty \le 0.0303 W^5 h^5. \tag{22}$$

In order to estimate the error function $e_D(x)$, we can subtract equation (16) from equation (4) to obtain

$$\begin{aligned} s_i(x) - \tilde{s}_i(x) = \left(Z_{i+1} - \tilde{Z}_{i+1}\right) \tfrac{(x-x_i)^4}{24h} - \left(Z_i - \tilde{Z}_i\right) \tfrac{(x_{i+1}-x)^4}{24h} + \left(A_i - \tilde{A}_i\right)(x-x_i)^2 \\ + \left(B_i - \tilde{B}_i\right)(x_{i+1}-x) + \left(C_i - \tilde{C}_i\right)(x-x_i), \end{aligned} \tag{23}$$

for $x \in [x_i, x_{i+1}]$. Let $X = (x_0, x_1, \ldots, x_n)^{\mathrm{T}}$, $U = (u_0, u_1, \ldots, u_n)^{\mathrm{T}}$, $\tilde{U} = (\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_n)^{\mathrm{T}}$, $\mu = (\mu_0, \mu_1, \ldots, \mu_n)^{\mathrm{T}}$, $\tilde{\mu} = (\tilde{\mu}_0, \tilde{\mu}_1, \ldots, \tilde{\mu}_n)^{\mathrm{T}}$, $Z = (Z_0, Z_1, \ldots, Z_n)^{\mathrm{T}}$ and $\tilde{Z} = \left(\tilde{Z}_0, \tilde{Z}_1, \ldots, \tilde{Z}_n\right)^{\mathrm{T}}$. From equation (23), it is easy to see that

$$\|e_D(x)\|_\infty \le \left\|U - \tilde{U}\right\|_\infty + h^2 \|\mu - \tilde{\mu}\|_\infty + \frac{h^3}{2}\left\|Z - \tilde{Z}\right\|_\infty. \tag{24}$$

We first estimate $\|\mu - \tilde{\mu}\|_\infty$. We use equation (13) to obtain

$$\begin{aligned} \mu_i - \tilde{\mu}_i = \tfrac{u_i - \tilde{u}_i}{h^2} - 2\tfrac{u_{i+1} - \tilde{u}_{i+1}}{h^2} + \tfrac{u_{i+2} + \tilde{u}_{i+2}}{h^2} - \tfrac{11h}{24}\left(Z_i - \tilde{Z}_i\right) \\ - \tfrac{12h}{24}\left(Z_{i+1} - \tilde{Z}_{i+1}\right) - \tfrac{h}{24}\left(Z_{i+2} - \tilde{Z}_{i+2}\right). \end{aligned} \tag{25}$$

Therefore, from equation (25), we obtain

$$\|\mu - \tilde{\mu}\|_\infty \le \frac{3}{h^2}\left\|U - \tilde{U}\right\|_\infty + \frac{h}{2}\left\|Z - \tilde{Z}\right\|_\infty. \tag{26}$$

On substituting equation (26) into equation (24), we obtain

$$\|e_D(x)\|_\infty \le 4\left\|U - \tilde{U}\right\|_\infty + h^3\left\|Z - \tilde{Z}\right\|_\infty. \tag{27}$$

Next, to estimate $\left\|Z - \tilde{Z}\right\|_\infty$, we let $Q = (q_{i,j})$ to denote a matrix with

$$\begin{aligned} &q_{1,1} = 12, \\ &q_{1,2} = 1, \\ &q_{i,i} = 15, i = 2, 3, \ldots, n-1, \\ &q_{i,i+1} = q_{i,i-2} = 1, i = 2, 3, \ldots, n-2, \text{and} \\ &q_{i,i-1} = 11, i = 2, 3, \ldots, n-1. \end{aligned}$$

We also let $J = (j_{m,l})$ to denote a matrix with

$$\begin{aligned} &j_{1,1} = -2, \\ &j_{1,2} = 1, \\ &j_{m,m} = -3, m = 2, 3, \ldots, n-1, \\ &j_{m,m+1} = 1, m = 2, 3, \ldots, n-2, \\ &j_{m,m-1} = 3, m = 2, 3, \ldots, n-1, \text{and} \\ &j_{m,m-2} = -1, m = 2, 3, \ldots, n-2. \end{aligned}$$

Let $\psi = \frac{24}{h^3}\left(u_0, -u_0, 0, \ldots, 0, u_n\right)^{\mathrm{T}}$, then the system (14) can be rewritten in a matrix form as

$$QZ = \frac{24}{h^3}JU + \psi. \tag{28}$$

From equation (28), we obtain

$$Q\tilde{Z} = \frac{24}{h^3}J\tilde{U} + \psi + \tau\left(h\right), \tag{29}$$

where $\tau\left(h\right) = \left(\tau_0\left(h\right), \tau_1\left(h\right), \ldots, \tau_n\left(h\right)\right)^{\mathrm{T}}$ is the error in the third derivative due to the discretization. On subtracting equation (29) from equation (28), we obtain

$$Q\left(Z - \tilde{Z}\right) = \frac{24}{h^3}J\left(U - \tilde{U}\right) - \tau\left(h\right). \tag{30}$$

Since $u_0 = \tilde{u}_0$ and $u_n = \tilde{u}_n$, then it is not difficult to show that

$$\tau_0\left(h\right) = \tau_n\left(h\right) = 0, \tag{31}$$

and

$$\tau_i\left(h\right) = \frac{3}{2}hu^{(4)}\left(\zeta_i\right), \tag{32}$$

for $\zeta_i \in \left(x_i, x_{i+1}\right)$. From equations (31) and (32), it follows that

$$\left\|\tau\left(h\right)\right\|_\infty \leq \frac{3c_1h}{2}, \tag{33}$$

where $c_1 = \max\limits_{a\leq\zeta\leq b}\left\|u^{(4)}\left(\zeta\right)\right\|_\infty$. Since Q is strictly diagonally dominant matrix, then Q^{-1} exists, $\left\|Q^{-1}\right\|_\infty \leq \frac{1}{2}$, $\left\|Q\right\|_\infty = 28$ and $\left\|J\right\|_\infty = 8$. Together with equations (30) and (33), we obtain

$$\left\|Z - \tilde{Z}\right\|_\infty \leq \frac{96}{h^3}\left\|U - \tilde{U}\right\|_\infty + \frac{3}{4}c_1h. \tag{34}$$

From equations (27) and (34), we obtain

$$\left\|e_D\left(x\right)\right\|_\infty \leq 100\left\|U - \tilde{U}\right\|_\infty + \frac{3}{4}c_1h^4. \tag{35}$$

In order to estimate $\left\|U - \tilde{U}\right\|_\infty$, we may assume the following result proved by Chawla and Subramanian [9].

Theorem 1 *Assume that $u\left(x\right)$ is sufficiently smooth. Then there exist a constant c independent of h such that*

$$\left\|U - \tilde{U}\right\|_\infty \leq ch^4.$$

Therefore, from equation (34) and Theorem 1, we arrived at

$$\left\|e_D\left(x\right)\right\|_\infty \leq c_2h^4, \tag{36}$$

where $c_2 = 100c + \frac{3c_1}{4}$. Finally, from equation (18) together with our findings from equations (22) and (36), we obtain the following result.

Theorem 2 *With the assumptions of Theorem 1, our proposed quartic spline method $\tilde{S}(x)$ as described in Section 2, provides order h^4 uniformly convergent approximations for the solution $u(x)$ of problem (1), that is*

$$\|e(x)\|_\infty \leq \|e_I(x)\|_\infty + \|e_D(x)\|_\infty \leq c_3 h^4,$$

where $c_3 = \frac{0.0303(b-a)W^5}{5} + c_2$.

4 Numerical Experiments

In this section, we implemented the proposed method on three examples of the second order BVPs. We denote $\max\limits_{0<i<n-1} \left|u\left(x_{i+1/2}\right) - s_i\left(x_{i+1/2}\right)\right|$ as the maximum absolute errors between the nodal points that are tabulated in Table 1 for step size equal to 0.1. We also compared our results with those obtained by the cubic spline method developed by Chawla and Subramanian [9].

Problem 1 [10]
Consider the following linear second order BVPs

$$u''(x) = \frac{-2}{x}u'(x) + \frac{2}{x^2}u(x) + \frac{\sin(\ln x)}{x^2}, u(1) = 1, u(2) = 2, 1 \leq x \leq 2.$$

The exact solution for *Problem 1* is given by $u(x) = c_1 x + \frac{c_2}{x^2} - \frac{3}{10}\sin(\ln x) - \frac{1}{10}\cos(\ln x)$, where $c_2 = \frac{1}{70}(8 - 12\sin(\ln 2) - 4\cos(\ln 2))$ and $c_1 = \frac{11}{10} - c_2$.

Problem 2 [10]
Consider the following nonlinear second order BVPs

$$u''(x) = 2u(x)^3, u(-1) = \frac{1}{2}, u(0) = \frac{1}{3}, -1 \leq x \leq 0.$$

The exact solution for *Problem 2* is given by $u(x) = \frac{1}{x+3}$.

Problem 3 [11]
Consider the following second order Bratu type equation

$$u''(x) + 2e^{u(x)} = 0, u(0) = u(1) = 0, 0 \leq x \leq 1.$$

The exact solution for *Problem 3* is given by

$$u(x) = -2\ln\left(\frac{\cosh(1.17878(x-0.5))}{\cosh(0.589388)}\right).$$

From Table 1, we observed that our proposed method and the existing cubic spline method by Chawla and Subramanian [9] are found to have comparable accuracy in solving *Problem 1* and *Problem 3*. However, our proposed method is more accurate than the existing cubic spline method in solving *Problem 2*.

Table 1: Maximum Absolute Errors for *Problem 1*, *Problem 2* and *Problem 3*

Problem	Methods	Maximum absolute errors
1	Chawla and Subramanian [9]	9.82365×10^{-6}
	The proposed method	6.75736×10^{-6}
2	Chawla and Subramanian [9]	1.68860×10^{-5}
	The proposed method	4.67347×10^{-6}
3	Chawla and Subramanian [9]	6.26403×10^{-4}
	The proposed method	1.09834×10^{-4}

5 Conclusion

In this article, we have presented a new quartic spline method for the numerical solution of second order BVPs in (1). An algorithm to apply the new method is presented as well. Convergence analysis showed that the order of convergence of the new method is 4. We have chosen three test problems to evaluate the effectiveness of the proposed method; and compared with an existing method by Chawla and Subramanian [9] in terms of numerical accuracy. Numerical experimentations seemed to indicate that the new proposed method is reliable and may generate more accurate numerical results in solving second order BVPs in (1).

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