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On the nilpotency of a pair of Lie algebras

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Abstract In this paper, we introduce the new concepts of nilpotency, upper and lower central series for a pair of Lie algebras (L, N), in which N is an ideal of Lie algebra L. We study the properties of these concepts and prove the analogue of Robinson Theorem for a pair of Lie algebras plays an important role to find the results connecting to the idea of nilpotency in the pair of Lie algebras. In particular we find a criterion such that an extension of pair of Lie algebras can be nilpotent.

Keywords Pair of Lie algebras; nilpotent Lie algebras; central series.

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1 Introduction

Throughout this paper we consider all Lie algebras are over a fixed field F and $[,]$ denotes the Lie bracket. A pair of Lie algebras (L, N) is a Lie algebra L with an ideal N. The pair of lie algebras have been studied by several authors (see for example [1], [2], [3], [7], [8], [9], [10]). In [6], Hassanzadeh *et al.* introduced a notion of nilpotency for pair of groups which has some interesting results in the usual theory of nilpotency. A pair of groups (G, N) is a group G with a normal subgroup N. Here, we introduce a notion of nilpotency for pair of Lie algebras which is between the usual notion of nilpotency for a Lie algebra and its ideal. This concept will be defined in such a way that the nilpotency of L implies that (L, N) and the nilpotency of (L, N) forces N to be nilpotent. Also, we introduce the concepts of central series and lower and upper central series for pair of Lie algebras and study the properties of these concepts. Robinson [11] showed that how the first lower central factor $G_{ab} = \frac{G}{G'}$ exerts a very strong influence on subsequent lower central factors of a group G. In [4], we proved an analogue of Robinson Theorem for Lie algebras and found a criterion such that an extension of Lie algebras could be nilpotent. In the third section, we prove an analogue of Robinson Theorem for a pair of Lie algebras plays an important role to find the results connecting to the idea of nilpotency in the pair of Lie algebras. Using the new notion of nilpotency, we obtain a criterion under which such an extension of pair of Lie algebras can be nilpotent.

2 Preliminaries

In this section we introduce some notations and concepts which shall be used throughout the paper. Using these concepts, some elementary results shall be given. One may consider

$$Z(L, N) = \{n \in N | [n, l] = 0, \forall l \in L\},$$
$$[N, L] = \langle [n, l]; n \in N, l \in L \rangle,$$

as *the center* and *the commutator* of the pair of Lie algebra (L, N), respectively.

Definition 1 A pair of Lie algebras (L, N) is called *nilpotent* if it has a *central series*, that is a series

$$\langle 0\rangle = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_t = N$$

such that N_i is an ideal of L and $\frac{N_{i+1}}{N_i}$ is contained in the center of $(\frac{L}{N_i}, \frac{N}{N_i})$ for all i.(or equivalent $[N_{i+1}, L] \subseteq N_i$, for all i).

The length of a shortest central series of the pair (L, N) is called *nilpotent class* of (L, N).

Definition 2 Let $\gamma_1(L, N) = N$, and $\gamma_{i+1}(L, N) = [\gamma_i(L, N), L]$, for $i \geq 1$. Then the obtained series

$$N = \gamma_1(L, N) \supseteq \gamma_2(L, N) \supseteq \cdots$$

is called the *lower central series* of (L, N).

Definition 3 Let (L, N) be a pair of Lie algebras. Then the series

$$\langle 0\rangle = Z_0(L, N) \subseteq Z_1(L, N) \subseteq \cdots \subseteq Z_n(L, N) \subseteq \cdots$$

is called the *upper central series* of (L, N) if for each i,

$$\frac{Z_{i+1}(L, N)}{Z_i(L, N)} = Z(\frac{L}{Z_i(L, N)}, \frac{N}{Z_i(L, N)}).$$

3 Main results

We begin with a well known property of a nilpotent Lie algebra. A non trivial ideal of a nilpotent Lie algebra L intersects non trivially the center of L. Now , this property can be proved with the weaker condition of nilpotency for a pair Lie algebras (L, N).

Lemma 1 *If (L, N) is a nilpotent pair of Lie algebras and M is a non trivial ideal of L contained in N, then $M \cap Z(L, N) \neq \langle 0\rangle$.*

Proof Since (L, N) is nilpotent, then there exists a natural number s such that $Z_s(L, N) = N$. Put

$$A = \{k | k \geq 1, M \cap Z_k(L, N) \neq \langle 0\rangle\}.$$

Clearly $s \in A$, hence $A \neq \emptyset$. If $m = \min A$, then $M \cap Z_{m-1}(L, N) = \langle 0\rangle$. Now, $[M \cap Z_m(L, N), L] \subseteq [Z_m(L, N), L] \subseteq Z_{m-1}(L, N)$ and $[M \cap Z_m(L, N), L] \subseteq M \cap Z_m(L, N) \subseteq M$. Hence,

$$[M \cap Z_m(L, N), L] \subseteq M \cap Z_{m-1}(L, N) = \langle 0\rangle.$$

This implies that $M \cap Z_m(L, N) \subseteq M \cap Z(L, N)$ and lead to the result. □

The following corollary is immediate result of the above lemma.

Corollary 1 *If (L,N) is a nilpotent pair of Lie algebras, then $Z(L,N) \neq \langle 0 \rangle$.*

The interesting point in the Lemma 1 and in the Corollary 1 is that, with a weaker condition we reach to a sharper conclusion. This happens of lying the center of a pair in the center of the Lie algebra itself.
In following theorem we prove an analogue of Robinson Theorem for pair of Lie algebras.

Remark 1 Note that in the under theorem $F_i = \frac{\gamma_i(L,N)}{\gamma_{i+1}(L,N)}$ and $\frac{L}{[N,L]}$ act trivially on each other. Hence the Lie tensor product $F_i \otimes \frac{L}{[N,L]}$ is isomorphic to the usual tensor product of F-modules. (see [5] for more information).

Theorem 1 *Let (L,N) be a pair of Lie algebras and $F_i = \frac{\gamma_i(L,N)}{\gamma_{i+1}(L,N)}$ for $i \geq 1$. Then the map*

$$F_i \otimes \frac{L}{[N,L]} \to F_{i+1}$$
$$(x + \gamma_{i+1}(L,N)) \otimes (l + [N,L]) \mapsto [x,l] + \gamma_{i+2}(L,N)$$

is a well-defined epimorphism.

Proof We define a function of the Lie algebra $F_i \times \frac{L}{[N,L]}$ to the Lie algebra F_{i+1}, given by $(x + \gamma_{i+1}(L,N), l + [N,L]) \mapsto [x,l] + \gamma_{i+2}(L,N)$ for all $x \in \gamma_i(L,N)$ and $l \in L$. Hence, by Remark 1 there exists an induced homomorphism of the Lie algebra $F_i \otimes \frac{L}{[N,L]}$ to the Lie algebra F_{i+1}, given by $(x+\gamma_{i+1}(L,N)) \otimes (l+[N,L]) \mapsto [x,l]+\gamma_{i+2}(L,N)$ for all $x \in \gamma_i(L,N)$ and $l \in L$. It can be easily seen that the induced homomorphism is onto, and the proof is complete. □

In the following corollary, we intend to give sufficient conditions under which the second term of a pair of Lie algebras can be finite-dimensional.

Corollary 2 *If (L,N) is a nilpotent Lie algebra such that $\frac{L}{[N,L]}$ is finite-dimensional, then N is finite-dimensional.*

Proof Since $\frac{L}{[N,L]}$ is finite-dimensional, then $\frac{N}{[N,L]}$ is also finite-dimensional. Now, let $F_i = \frac{\gamma_i(L,N)}{\gamma_{i+1}(L,N)}$ be finite-dimensional. Then by Remark 1 and Theorem 1, F_{i+1} is also finite-dimensional, since the finite-dimensional property is inherited by images of tensor products. Hence, by induction on i every lower central factor is finite-dimensional. Since (L,N) is nilpotent, then $\gamma_{c+1}(L,N) = \langle 0 \rangle$, for some non-negative integer c. As the finite-dimensional property is closed under forming extensions, thus N is finite-dimensional. □

The following definition is very useful in our further investigations.

Definition 4 Let L and K be Lie algebras such that L acts on K. Then we say L *acts nilpotently* on K if K having the series

$$\langle 0 \rangle = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_t = K$$

of ideals of K, where ${}^{l}k_{i+1} \in K_i, (i \geq 0)$ for all $l \in L$ and $k_{i+1} \in K_{i+1}$.

Remark 2 Let M and N be ideals of some Lie algebra L such that $N \subseteq M$. Then the Lie algebra L acts on $\frac{M}{N}$ by the following defined action:

$$^{l}(m+N) = [l,m] + N \text{ for } l \in L \text{ and } m \in M.$$

Hence, L acts nilpotently on $\frac{M}{N}$, if $\frac{M}{N}$ having the series

$$\langle 0 \rangle = \frac{M_0}{N} \subseteq \frac{M_1}{N} \subseteq \cdots \subseteq \frac{M_t}{N} = \frac{M}{N}$$

where $[L, M_{i+1}] \subseteq M_i$ for all $i \geq 0$.

Lemma 2 *([4]) Let A, B, C and D be ideals of Lie algebra L such that $B \subseteq A$ and $D \subseteq C$. Also let $\frac{A}{B}$ and $\frac{C}{D}$ act on each other. If L acts nilpotently on $\frac{A}{B}$ and $\frac{C}{D}$, then L acts nilpotently on $\frac{A}{B} \otimes \frac{C}{D}$.*

It is known that an extension of a nilpotent Lie algebra by another nilpotent Lie algebra may not be nilpotent in general. Using the new notion of nilpotency, we obtain a criterion under which such an extension of pair of Lie algebras can be nilpotent.

Theorem 2 *Let M be an ideal of Lie algebra L contained in N. If (N, M) and $(\frac{L}{[M,N]}, \frac{N}{[M,N]})$ are nilpotent Lie algebras, then (L, N) is nilpotent.*

Proof Since $(\frac{L}{[M,N]}, \frac{N}{[M,N]})$ is nilpotent, then it has a central series as follows:

$$\langle 0 \rangle = \frac{N_0}{[M,N]} \subseteq \frac{N_1}{[M,N]} \subseteq \cdots \subseteq \frac{N_t}{[M,N]} = \frac{N}{[M,N]}. \quad (1)$$

Hence, L acts nilpotently on $\frac{N}{[M,N]}$. Now we construct the following series for $\frac{M}{[M,N]}$:

$$\langle 0 \rangle = \frac{M_0}{[M,N]} \subseteq \frac{M_1}{[M,N]} \subseteq \cdots \subseteq \frac{M_t}{[M,N]} = \frac{M}{[M,N]},$$

where $M_j = N_j \cap M$, $(0 \leq j \leq t)$. The Lie algebra L acts on $\frac{M}{[M,N]}$ by the following defined action:

$$^{l}(m + [M,N]) = [l,m] + [M,N] \text{ for } l \in L \text{ and } m \in M.$$

Hence, by the above central series, the action of L on $\frac{M}{[M,N]}$ is nilpotent. Put $F_i = \frac{\gamma_i(N,M)}{\gamma_{i+1}(N,M)}$ for $i \geq 1$. Then L acts nilpotently on F_1. Suppose that L acts nilpotently on F_i, then by Lemma 2, L acts nilpotently on $F_i \otimes \frac{N}{[M,N]}$. By Theorem 1, F_{i+1} is an image of $F_i \otimes \frac{N}{[M,N]}$ and hence, L acts nilpotently on F_{i+1}. Therefore by induction on i, L acts nilpotently on every lower central factor of (N, M). Since (N, M) is nilpotent, then there exists a non-negative integer c such that $\gamma_{c+1}(N, M) = \langle 0 \rangle$. Now, combining the lower central series of (N, M) and (1) we obtain

$$\langle 0 \rangle = \gamma_{c+1}(N,M) \subseteq \cdots \subseteq \gamma_2(N,M) = [M,N] = N_0 \subseteq \cdots \subseteq N_t = N.$$

By the fact that L acts nilpotently on F_i, there is a series

$$\langle 0 \rangle = \frac{K_{i_1}}{\gamma_{i+1}(N,M)} \subseteq \cdots \subseteq \frac{K_{i_r}}{\gamma_{i+1}(N,M)} = F_i$$

such that $[L, K_{i_{j+1}}] \subseteq K_{i_j}$. Now we obtain a central series of (L, N) which provides the nilpotency of (L, N), as required. □

The following corollary is an immediate result of the above theorem.

Corollary 3 *Let (L, N) and $\frac{L}{[N,L]}$ be nilpotent. Then so is L. In particular, L is nilpotent if and only if $(L, [L, L])$ is nilpotent, where $[L, L]$ is the derived subalgebra of L.*

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