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Supercobalancing numbers

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Abstract A natural number n is called cobalancing number (with cobalancer r) if it satisfies the Diophantine equation $1+2+\cdots+n = (n+1)+(n+2)+\cdots+(n+r)$. However, if for some pair of natural numbers (n, r), $1+2+\cdots+n > (n+1)+(n+2)+\cdots+(n+r)$ and equality is achieved after adding a natural number D to the right hand side then we call n a D-supercobalancing number with D-supercobalaner number r. In this paper, such numbers are studied for certain values of D.

Keywords Balancing numbers, Lucas-balancing numbers, cobalancing numbers, Pell numbers, associated Pell numbers

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1 Introduction

A natural number n is called a balancing number with balancer r [1] if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

The k^{th} balancing number is denoted by B_k and $C_k = \sqrt{8B_k^2 + 1}$ is called the k^{th} Lucas-balancing number [2]. The balancing and Lucas-balancing numbers satisfy the recurrence relations $B_1 = 1, B_2 = 6, B_{n+1} = 6B_n - B_{n-1}$ and $C_1 = 3, C_2 = 17, C_{n+1} = 6C_n - C_{n-1}, n \geq 2$. On other hand, n is called a cobalancing number with cobalancer r [3] if

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r).$$

The n^{th} cobalancing number is denoted by b_n and cobalancing numbers satisfy the nonhomogeneous recurrence $b_1 = 0, b_2 = 2, b_{n+1} = 6b_n - b_{n-1} + 2$. The Binet forms of B_n, C_n and b_n are respectively

$$B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}},\ C_n = \frac{\alpha^{2n} + \beta^{2n}}{2},\ b_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$$

where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$. The k^{th} Pell and associated Pell numbers are denoted by P_k and Q_k respectively and are defined by means of the recurrence relations $P_1 = 1, P_2 = 2, P_{n+1} = 2P_n + P_{n-1}$ and $Q_1 = 1, Q_2 = 3, Q_{n+1} = 2Q_n + Q_{n-1}, n \geq 2$. The Binet forms of P_n, Q_n are

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}},\ Q_n = \frac{\alpha^n + \beta^n}{2}.$$

The following identities will be useful during the proof of some results in the later sections. To prove the following identities, the readers are advised to refer to [4, 5].

(i) $B_{n\pm1} = 3B_n \pm C_n$

(ii) $C_n - 1 = 2B_n + 2b_n$

(iii) $b_{n+1} - b_n = 2B_n$

(iv) $C_n = Q_{2n}$

(v) $2P_{2n}^2 + 1 = Q_{2n}^2$.

Definition 1 For a fixed positive integer D, we call a positive integer n, a D-supercobalancing number if

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) + D \tag{1}$$

for some natural number r, which we call the D-supercobalancer corresponding to n. If D is a negative integer, say $D = -R$, we call n a R-subcobalancing number and r, a R-subcobalancer corresponding to n.

Since, without D, the right hand side of (1) is less than the left hand side, we prefer the name supercobalancing number for n. A similar justification applies when D is negative. Observe that when $D = 0$, the above definition coincides with that of cobalancing numbers and hence, we prefer to exclude the case $D = 0$ from the above definition.

Simplifying the equation (1), we get

$$n(n+1) = \frac{(n+r)(n+r+1)}{2} + D \tag{2}$$

solving the above equation for r, we get

$$r = \frac{1}{2}[-(1+2n) + \sqrt{8n^2 + 8n - 8D + 1}]. \tag{3}$$

Observe that the value of n will generally depend on the choice of D and the existence of n is not ascertained for each value of D, e.g., if $D = 4$, $8n^2 + 8n - 8D + 1 = 8n^2 + 8n - 31$ is not a perfect square for any natural number n. Hence, the choice of D plays a crucial role in the definition of supercobalancing numbers. The definition of balancing numbers ensures a solution to the Diophantine equation (1) when D is restricted to balancing number. It is clear from (3) that when D is a balancing number, then $8n^2+8n-8D+1$ is a perfect square at least for $D = n$. This motivates us to study D-supercobalancing numbers corresponding to $D = B_n$, $n = 1, 2, 3, \cdots$.

2 B_k-Supercobalancing numbers

2.1 Computation of B_1-Supercobalancing numbers

By Definition 1, a natural number n is a B_1-supercobalancing number if

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) + 1 \tag{4}$$

for some natural number r, which is the B_1-supercobalancer corresponding to n. Solving the above equation for r, we get

$$r = \frac{1}{2}[-(1+2n) + \sqrt{8n^2+8n-7}$$
$$= \frac{1}{2}[-(1+2n) + \sqrt{2(2n+1)^2-9}].$$

Example 1 The following examples suggests that $1, 7, 43$ and 253 are B_1-supercobalancing numbers with $0, 3, 18$ and 105 as corresponding B_1-supercobalancers.

(i) $1 = 1$

(ii) $1 + 2 + \cdots + 7 = 8 + 9 + 10 + 1$

(iii) $1 + 2 + \cdots + 43 = 44 + 45 + \cdots + 61 + 1$

(iv) $1 + 2 + \cdots + 253 = 254 + 255 + \cdots + 358 + 1$

In [5], Behera and Panda accepted 1 as a balancing number with balancer 0. In the same way, we accept 1 as a B_1-supercobalancing number with B_1-supercobalancer 0.

A natural number is said to be pronic number if it is of the form $x(x+1)$. Note that, l is a pronic number if and only if $4l+1$ is perfect square. A natural number is said to be triangular number if it is of the form $\frac{y(y+1)}{2}$. Note that m is a triangular number if and only if $8m+1$ is perfect square. We use the above properties of pronic and triangular numbers to explore all B_1-supercobalancing numbers.

Theorem 1 *For $m \geq 0$, $9B_mB_{m+1} + 2 = (3b_{m+1}+1)(3b_{m+1}+2)$.*

Proof Since

$$\begin{aligned} 4[9B_mB_{m+1}+2]+1 &= 36B_mB_{m+1}+9 \\ &= 36B_m(3B_m+C_m)+9 \\ &= 108B_m^2+36B_mC_m+9 \\ &= 36B_m^2+36B_mC_m+9(8B_m^2+1) \\ &= 36B_m^2+36B_mC_m+9C_m^2 \\ &= (6B_m+3C_m)^2, \end{aligned}$$

$9B_mB_{m+1}+2$ is a pronic number. Moreover,

$$\begin{aligned} 9B_mB_{m+1}+2 &= \frac{(6B_m+3C_m)^2-1}{4} \\ &= \left(\frac{6B_m+3C_m-1}{2}\right)\left(\frac{6B_m+3C_m-1}{2}+1\right). \end{aligned}$$

Since, $C_n = 2B_n + 2b_n + 1$ and $b_{n+1} - b_n = 2B_n$ substituting in the above equation, we get

$$9B_mB_{m+1}+2 = (3b_{m+1}+1)(3b_{m+1}+2). \qquad \square$$

Theorem 2 *For* $m \geq 0,\ 9B_mB_{m+1}+1 = \frac{1}{2}(3B_m+3b_{m+1}+1)(3B_m+3b_{m+1}+2)$.

Proof Since

$$\begin{aligned} 8[9B_mB_{m+1}+1]+1 &= 72B_mB_{m+1}+9 \\ &= 72B_m(3B_m+C_m)+9 \\ &= 216B_m^2+72B_mC_m+9 \\ &= 144B_m^2+72B_mC_m+9(8B_m^2+1) \\ &= 144B_m^2+72B_mC_m+9C_m^2 \\ &= (12B_m+3C_m)^2, \end{aligned}$$

$9B_mB_{m+1}+1$ is a triangular number. Moreover,

$$9B_mB_{m+1}+1 = \frac{(12B_m+3C_m)^2-1}{8} = \frac{1}{2}\left(\frac{12B_m+3C_m-1}{2}\right)\left(\frac{12B_m+3C_m-1}{2}+1\right).$$

Since, $C_n = 2B_n+2b_n+1$ and $b_{n+1}-b_n = 2B_n$, we have

$$9B_mB_{m+1}+1 = \frac{1}{2}(3B_m+3b_{m+1}+1)(3B_m+3b_{m+1}+2). \qquad \square$$

It follows from Theorems 1 and 2 that for each natural number m, $9B_mB_{m+1}+1$ is a triangular number while $9B_mB_{m+1}+2$ is a pronic number. Hence we have the following theorem.

Theorem 3 *For each* $m \geq 0,\ (x,y) = (3b_{m+1}+1, 3B_m+3b_{m+1}+1)$ *satisfies the Diophantine equation* $x(x+1) = \frac{y(y+1)}{2}+1$.

Indeed, $\{3b_{m+1}+1, 3B_m+3b_{m+1}+1 : m \geq 0\}$ is the complete solution set of the Diophantine equation $x(x+1) = \frac{y(y+1)}{2}+1$. The interested readers can verify this claim by writting $x(x+1) = \frac{y(y+1)}{2}+1$ as $y^2-2(2x+1)^2 = -9$ and solving the later as a generalized Pell's equation.

In view of Theorem 3 and equations (2) and (4), one can conclude that for $m \geq 0$, the numbers of the form $3b_{m+1}+1$ and $3B_m$ are B_1-supercobalancing numbers and corresponding B_1-supercobalancers respectively.

2.2 Computation of B_2-supercobalancing numbers

By Definition 1, a natural number n is a B_2-supercobalancing number if

$$1+2+\cdots+n = (n+1)+(n+2)+\cdots+(n+r)+6 \tag{5}$$

for some natural number r, which is a B_2-supercobalancer corresponding to n.

Example 2 The following examples suggests that $3, 6$ and 8 are B_2-supercobalancing numbers with $0, 2$ and 3 as corresponding B_2-supercobalancers.

(i) $1+2+3=6$

(ii) $1+2+\cdots+6=7+8+6$

(iii) $1+2+\cdots+8=9+10+11+6$

It follows from equations (3) and (5) that if n is a B_2-supercobalancing number then the corresponding B_2-supercobalancer is

$$r=\frac{-(2n+1)+\sqrt{8n^2+8n-47}}{2}.$$

We thus conclude that, if n is a B_2-supercobalancing number then $2x^2-49$ is a perfect square, where $x=2n+1$. One can easily check that there are three classes of solutions corresponding to the positive values of x satisfying the equation $2x^2-49=y^2$. One class of solution corresponds to the case $x\equiv 0 \pmod 7$ and then, of course, $y\equiv 0 \pmod 7$ and the equation $2x^2-49=y^2$ can be written as

$$\left(\frac{y}{7}\right)^2-2\left(\frac{x}{7}\right)^2=-1,$$

which is a Pell's equation. It's solutions are given by $(y,x)=(7Q_{2l-1},7P_{2l-1}), l\geq 1$. Hence the set

$$\left\{\frac{7P_{2l-1}-1}{2} : \ l=1,2,\ldots\right\} \tag{6}$$

lists a class of B_2-supercobalancing numbers. For finding the other two classes of solutions, we consider the congruence

$$x^2\equiv 25(2x^2-49)(\text{mod } 49)$$

which implies

$$x\equiv \pm 5\sqrt{2x^2-49}(\text{mod } 49).$$

Thus,

$$\frac{x+5\sqrt{2x^2-49}}{49} \text{ or } \frac{x-5\sqrt{2x^2-49}}{49}$$

is a natural number. Since

$$2\Big[\frac{x\pm 5\sqrt{2x^2-49}}{49}\Big]^2+1=\Big[\frac{10x\pm\sqrt{2x^2-49}}{49}\Big]^2,$$

it follows that either

$$\frac{10x+\sqrt{2x^2-49}}{49} \text{ or } \frac{10x-\sqrt{2x^2-49}}{49}$$

is an even ordered associated Pell number. Since $C_n=Q_{2n}$, letting

$$C=\frac{10x\pm\sqrt{2x^2-49}}{49},$$

we obtain

$$(49C-10x)^2=2x^2-49,$$

which leads to the quadratic equation

$$2x^2 - 20Cx + 49C^2 + 1 = 0$$

whose solutions are $x = 5C \pm 2B$, (C is the Lucas-balancing number associated with B). We further observe that

$$2(5C \pm 2B)^2 - 49 = (C \pm 20B)^2.$$

Thus, the B_2-supercobalancing numbers are of the form $\frac{1}{2}[5C \pm 2B - 1]$. Hence the set

$$\left\{\frac{7P_{2l-1} - 1}{2}, \frac{5C_l + 2B_l - 1}{2}, \frac{5C_l - 2B_l - 1}{2} : l = 1, 2, \ldots\right\}$$

lists all the B_2-supercobalancing numbers.

2.3 Computation of B_4-supercobalancing numbers

In view of the Definition 1, a natural number n is a B_4-supercobalancing number if

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r) + 204 \qquad (7)$$

for some natural number r, which is the B_4-supercobalancer corresponding to n.

Example 3 The following examples suggests that $29, 36, 50$ and 63 are B_4-supercobalancing numbers with $7, 11, 18$ and 24 as corresponding B_4-supercobalancers.

(i) $1 + 2 + \cdots + 29 = 30 + 31 + \cdots + 36 + 204$

(ii) $1 + 2 + \cdots + 36 = 37 + 38 + \cdots + 47 + 204$

(iii) $1 + 2 + \cdots + 50 = 51 + 52 + \cdots + 68 + 204$

(iv) $1 + 2 + \cdots + 63 = 64 + 65 + \cdots + 87 + 204$

It is easy to see that if n is a B_4-supercobalancing number then the corresponding B_4-supercobalancer is

$$r = \frac{-(2n + 1) + \sqrt{8n^2 + 8n - 1631}}{2}.$$

Thus, if n is a B_4-supercobalancing number then $2x^2 - 1633$ is a perfect square, where $x = 2n+1$. Therefore, computation of B_4-supercobalancing numbers reduces to solving the Diophantine equation

$$2x^2 - 1633 = y^2. \qquad (8)$$

To find all the B_4-supercobalancing numbers one needs to solve the generalized Pell's equation $y^2 - 2x^2 = -1633$. The bounds for x corresponding to the fundamental solutions are given by $\sqrt{1633/2} \le x \le \sqrt{1633}$, that is $28 < x < 40$, [see [6]]. Thus, we need to find those integers x in the interval $(28, 40)$ such that $2x^2 - 1633$ is a perfect square. This happens for $(x, y) = (29, \pm 7)$ and $(31, \pm 17)$ from which it is easy to see that there are four fundamental solutions $-7 + 29\sqrt{2}$, $7 + 29\sqrt{2}$, $-17 + 31\sqrt{2}$ and $17 + 31\sqrt{2}$ respectively.

Further, Corresponding to each fundamental class there is a class of solutions for x and one can easily get the solutions as

$$\{29C_l + 14B_l,\ 29C_l - 14B_l,\ 31C_l + 34B_l,\ 31C_l - 34B_l\}$$

where $l \geq 1$. Hence the following set

$$\left\{\frac{29C_l + 14B_l - 1}{2}, \frac{29C_l - 14B_l - 1}{2}, \frac{31C_l + 34B_l - 1}{2}, \frac{31C_l - 34B_l - 1}{2}\right\}$$

lists all the B_4-supercobalancing numbers.

We will provide a different method which uses modular arithmetic for obtaining these classes of solutions. During the process we use the fact that $ax \equiv \pm b (\text{mod } m)$ implies $a^2x^2 \equiv b^2 (\text{mod } m)$ for any positive integer m. Thus, any solution of the congruence $ax \equiv \pm b (\text{mod } m)$ is also a solution of the congruence $a^2x^2 \equiv b^2 (\text{mod } m)$.

Since $2x^2 - 1633$ is a perfect square, the congruence

$$(7x)^2 \equiv 29^2(2x^2 - 1633)(\text{mod } 1633)$$

holds and is implied by the pair of congruences (but need not imply since 1633 is not a prime)

$$7x \equiv \pm 29\sqrt{2x^2 - 1633}(\text{mod } 1633). \tag{9}$$

Thus, if x is any solution of (8) then $2x^2 - 1633$ is a perfect square. In view of (9), either

$$\frac{7x + 29\sqrt{2x^2 - 1633}}{1633} \quad \text{or} \quad \frac{7x - 29\sqrt{2x^2 - 1633}}{1633}$$

is a natural number. Since

$$2\left[\frac{7x \pm 29\sqrt{2x^2 - 1633}}{1633}\right]^2 + 1 = \left[\frac{58x \pm 7\sqrt{2x^2 - 1633}}{1633}\right]^2,$$

it follows that either

$$\frac{58x + 7\sqrt{2x^2 - 1633}}{1633} \quad \text{or} \quad \frac{58x - 7\sqrt{2x^2 - 1633}}{1633}$$

is an even ordered associated Pell number. Since $C_n = Q_{2n}$, letting

$$C = \frac{58x \pm 7\sqrt{2x^2 - 1633}}{1633},$$

we obtain

$$(1633C - 58x)^2 = 49(2x^2 - 1633),$$

which leads to the quadratic equation

$$2x^2 - 116Cx + 1633C^2 + 49 = 0$$

whose solutions are $x = 29C \pm 14B$. We further observe that

$$2(29C \pm 14B)^2 - 1633 = (7C \pm 116B)^2.$$

Thus, the numbers of the form $\frac{1}{2}[29C \pm 14B - 1]$ constitutes two classes of B_4-supercobalancing numbers.

The other two classes of solutions are obtained by using a similar modular arithmetic technique. Since $2x^2 - 1633$ is a perfect square, the congruence

$$(17x)^2 \equiv 31^2(2x^2 - 1633)(\text{mod } 1633)$$

holds and is implied by the pair of congruences

$$17x \equiv \pm 31\sqrt{2x^2 - 1633}(\text{mod } 1633).$$

Thus, either

$$\frac{17x + 31\sqrt{2x^2 - 1633}}{1633} \quad \text{or} \quad \frac{17x - 31\sqrt{2x^2 - 1633}}{1633}$$

is a natural number. Since

$$2\left[\frac{17x \pm 31\sqrt{2x^2 - 1633}}{1633}\right]^2 + 1 = \left[\frac{62x \pm 17\sqrt{2x^2 - 1633}}{1633}\right]^2,$$

it follows that either

$$\frac{62x + 17\sqrt{2x^2 - 1633}}{1633} \quad \text{or} \quad \frac{62x - 17\sqrt{2x^2 - 1633}}{1633}$$

is a even ordered associated Pell number. Since $C_n = Q_{2n}$, letting

$$C = \frac{62x \pm 17\sqrt{2x^2 - 1633}}{1633},$$

leads to

$$(1633C - 62x)^2 = 289(2x^2 - 1633),$$

which can be rearranged to form the quadratic equation

$$2x^2 - 124Bx + 1633C^2 + 289 = 0$$

whose solutions are $x = 31C \pm 34B$. We further observe that

$$2(31C \pm 34B)^2 - 1633 = (17C \pm 124B)^2.$$

Thus, these numbers of the form $\frac{1}{2}[31C \pm 34B - 1]$ constitute the other two classes of B_4-supercobalancing numbers. Hence the set

$$\left\{\frac{29C_l + 14B_l - 1}{2}, \frac{29C_l - 14B_l - 1}{2}, \frac{31C_l + 34B_l - 1}{2}, \frac{31C_l - 34B_l - 1}{2}\right\}$$

where $l \geq 1$, lists all the B_4-supercobalancing numbers.

From the above discussion, it is clear that, for some values of k, there can be more than two classes of B_k-supercobalancing numbers. Obtaining all such classes for an arbitrary k is a difficult task. However, for all values of k, we manage to explore two classes of B_k-supercobalancing numbers using modular arithmetic. The following lemma will be useful while proving subsequent theorems.

Lemma 1 For $m, l \in \mathbb{Z}$

(i) $(B_{m+1} - B_m)C_l + 2(B_{m-1} + B_m)B_l - 1 = 2[B_{l+m} - B_{l-m} + b_{l+m}]$

(ii) $(B_{m+1} - B_m)C_l - 2(B_{m-1} + B_m)B_l - 1 = 2[B_{l+m} + B_{l-m} + b_{l-m}]$

(iii) $(2B_m + C_{m-1})C_l + 2(C_m - 4B_{m-1})B_l - 1 = 2[B_{l+m} + B_{l-m+1} + b_{l-m+1}]$

(iv) $(2B_m + C_{m-1})C_l - 2(C_m - 4B_{m-1})B_l - 1 = 2[B_{l+m-1} - B_{l-m} + b_{l+m-1}]$

Proof We will prove (i) only. Other proofs are similar.
Since,

$$\begin{aligned}(B_{m+1} - B_m)C_l + 2(B_{m-1} + B_m)B_l &= (2B_m + C_m)C_l + 2(4B_m - C_m)B_l \\ &= -2(B_lC_m - C_lB_m) + (C_lC_m + 8B_lB_m) \\ &= -2B_{l-m} + C_{l+m} \\ &= -2B_{l-m} + 2B_{l+m} + 2b_{l+m} + 1,\end{aligned}$$

the proof of (i) follows. □

Theorem 4 *For $m > 0$, the values of n satisfying the Diophantine equation*

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) + B_{2m}$$

for suitable natural numbers r may result in multiple classes. Two such classes are $B_{l+m} - B_{l-m} + b_{l+m}$ and $B_{l+m} + B_{l-m} + b_{l-m}$ for $l \geqslant 1$.

Proof In view of (3), $2x^2 - 8B_{2m} - 1$ is perfect square, where $x = 2n + 1$. Since,

$$(B_{m-1} + B_m)^2 - 2(B_{m+1} - B_m)^2 = -[8B_{2m} + 1]$$

we have,

$$(B_{m-1} + B_m)^2x^2 \equiv (B_{m+1} - B_m)^2(2x^2 - 8B_{2m} - 1) \pmod{8B_{2m} + 1}. \tag{10}$$

Hence the values of x satisfying the following congruences

$$(B_{m-1} + B_m)\, x \equiv \pm(B_{m+1} - B_m)\sqrt{2x^2 - 8B_{2m} - 1} \pmod{8B_{2m} + 1} \tag{11}$$

is also a solution of the congruence (10). To obtain two classes of B_{2m}-supercobalancing number we solve the congruences (11) which are also solutions of the congruence (10). It is clear from (11) that either

$$\frac{(B_{m-1}+B_m)x+(B_{m+1}-B_m)\sqrt{2x^2-8B_{2m}-1}}{8B_{2m}+1}$$

or

$$\frac{(B_{m-1}+B_m)x-(B_{m+1}-B_m)\sqrt{2x^2-8B_{2m}-1}}{8B_{2m}+1}$$

is a natural number. Since

$$2\left[\frac{(B_{m-1}+B_m)x\pm(B_{m+1}-B_m)\sqrt{2x^2-8B_{2m}-1}}{8B_{2m}+1}\right]^2+1$$
$$=\left[\frac{2(B_{m+1}-B_m)x\pm(B_{m-1}+B_m)\sqrt{2x^2-8B_{2m}-1}}{8B_{2m}+1}\right]^2$$

it follows that either

$$\frac{2(B_{m+1}-B_m)x+(B_{m-1}+B_m)\sqrt{2x^2-8B_{2m}-1}}{8B_{2m}+1}$$

or

$$\frac{2(B_{m+1}-B_m)x-(B_{m-1}+B_m)\sqrt{2x^2-8B_{2m}-1}}{8B_{2m}+1}$$

is an even ordered associated Pell number. Since $C_n = Q_{2n}$, letting

$$C=\frac{2(B_{m+1}-B_m)x\pm(B_{m-1}+B_m)\sqrt{2x^2-8B_{2m}-1}}{8B_{2m}+1},$$

we get

$$[2(B_{m+1}-B_m)x-(8B_{2m}+1)C]^2=(B_{m-1}+B_m)^2(2x^2-8B_{2m}-1)$$

which can be transformed to the quadratic equation

$$2x^2-4(B_{m+1}-B_m)Cx+(8B_{2m}+1)C^2+(B_{m-1}+B_m)^2=0$$

whose solutions are $x=(B_{m+1}-B_m)C\pm 2(B_{m-1}+B_m)B$. We further observe that

$$2[(B_{m+1}-B_m)C\pm 2(B_{m-1}+B_m)B]^2-8B_{2m}-1=[(B_{m-1}+B_m)C\pm 4(B_{m-1}-B_m)B]^2.$$

Thus two classes of B_{2m}-supercobalancing numbers are

$$\frac{1}{2}[(B_{m+1}-B_m)C_l+2(B_{m-1}+B_m)B_l-1],\ \frac{1}{2}[(B_{m+1}-B_m)C_l-2(B_{m-1}+B_m)B_l-1]$$

for $l \geq 1$. In view of Lemma 1 (i) and (ii), the conclusion of the theorem follows. □

Theorem 5 *For $m > 1$, the values of n satisfying the Diophantine equation*

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r)+B_{2m-1}$$

for some natural number r may result in multiple classes. Two such classes are $B_{l+m}+B_{l-m+1}+b_{l-m+1}$ and $B_{l+m-1}-B_{l-m}+b_{l+m-1}$ for $l \geqslant 1$.

Proof In view of (3), $2x^2 - 8B_{2m-1} - 1$ is perfect square (where $x = 2n + 1$) and hence the congruence

$$(C_m - 4B_{m-1})^2 x^2 \equiv (2B_m + C_{m-1})^2 (2x^2 - 8B_{2m-1} - 1) \pmod{8B_{2m-1} + 1},$$

holds and is implied by the pair of congruences

$$(C_m - 4B_{m-1})\, x \equiv \pm(2B_m + C_{m-1})\sqrt{2x^2 - 8B_{2m-1} - 1} \pmod{8B_{2m-1} + 1}.$$

Any solution of the latter congruence is a solution of the former. In view of the latter congruence

$$\frac{(C_m - 4B_{m-1})x + (2B_m + C_{m-1})\sqrt{2x^2 - 8B_{2m-1} - 1}}{8B_{2m-1} + 1}$$

or

$$\frac{(C_m - 4B_{m-1})x - (2B_m + C_{m-1})\sqrt{2x^2 - 8B_{2m-1} - 1}}{8B_{2m-1} + 1}$$

is a natural number. Since

$$2\left[\frac{(C_m - 4B_{m-1})x \pm (2B_m + C_{m-1})\sqrt{2x^2 - 8B_{2m-1} - 1}}{8B_{2m-1} + 1}\right]^2 + 1$$
$$= \left[\frac{2(2B_m + C_{m-1})x \pm (C_m - 4B_{m-1})\sqrt{2x^2 - 8B_{2m-1} - 1}}{8B_{2m-1} + 1}\right]^2$$

it follows that either

$$\frac{2(2B_m + C_{m-1})x + (C_m - 4B_{m-1})\sqrt{2x^2 - 8B_{2m-1} - 1}}{8B_{2m-1} + 1}$$

or

$$\frac{2(2B_m + C_{m-1})x - (C_m - 4B_{m-1})\sqrt{2x^2 - 8B_{2m-1} - 1}}{8B_{2m-1} + 1}$$

is an even ordered associated-pell number. Since $C_n = Q_{2n}$, letting

$$C = \frac{2(2B_m + C_{m-1})x \pm (C_m - 4B_{m-1})\sqrt{2x^2 - 8B_{2m-1} - 1}}{8B_{2m-1} + 1}$$

we get

$$[2(2B_m + C_{m-1})x - (8B_{2m-1} + 1)C]^2 = (C_m - 4B_{m-1})^2 (2x^2 - 8B_{2m-1} - 1)$$

which can be transformed to the quadratic equation

$$2x^2 - 4(2B_m + C_{m-1})Cx + (8B_{2m-1} + 1)C^2 + (C_m - 4B_{m-1})^2 = 0$$

whose solutions are $x = (2B_m + C_{m-1})C \pm 2(C_m - 4B_{m-1})B$. We further observe that

$$2[(2B_m + C_{m-1})C \pm 2(C_m - 4B_{m-1})B]^2 - 8B_{2m-1} - 1 = [(C_m - 4B_{m-1})C \pm 4(2B_m + C_{m-1})]^2.$$

Thus two classes of B_{2m-1}-supercobalancing numbers are

$$\frac{1}{2}[(2B_m + C_{m-1})C_l + 2(C_m - 4B_{m-1})B_l - 1],\ \frac{1}{2}[(2B_m + C_{m-1})C_l - 2(C_m - 4B_{m-1})B_l - 1]$$

for $l \geq 1$. In view of Lemma 1 (iii) and (iv), the conclusion of the theorem follows. □

3 Conclusion

In this paper, we have defined D-supercobalancing numbers n and D-supercobalancer numbers r as solutions of the Diophantine equation $1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n+r)+D$. Since there are infinitely many choices of D, one has ample scope for exploring D-supercobalancing numbers for many other values of D.

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