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Spectrum of commuting graphs of some classes of finite groups

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Abstract In this paper, we initiate the study of spectrum of the commuting graphs of finite non-abelian groups. We first compute the spectrum of this graph for several classes of finite groups, in particular AC-groups. We show that the commuting graphs of finite non-abelian AC-groups are integral. We also show that the commuting graph of a finite non-abelian group G is integral if G is not isomorphic to the symmetric group of degree 4 and the commuting graph of G is planar. Further, it is shown that the commuting graph of G is integral if its commuting graph is toroidal.

Keywords Commuting graph; Spectrum; Integral Graph; Finite Group.

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1 Introduction

Let G be a finite group with centre $Z(G)$. The commuting graph of a non-abelian group G, denoted by Γ_G, is a simple undirected graph whose vertex set is $G \setminus Z(G)$, and two vertices x and y are adjacent if and only if $xy = yx$. Various aspects of commuting graphs of different finite groups can be found in [1–6]. In [7], the authors have studied the Laplacian spectrum of non-commuting graphs of some classes of finite non-abelian groups. In this paper, we initiate the study of spectrum of commuting graphs of finite non-abelian groups. Recall that the spectrum of a graph $\mathcal{G}$ denoted by $\text{Spec}(\mathcal{G})$ is the set $\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_n^{k_n}\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of $\mathcal{G}$ with multiplicities $k_1, k_2, \ldots, k_n$, respectively. A graph $\mathcal{G}$ is called integral if $\text{Spec}(\mathcal{G})$ contains only integers. It is well known that the complete graph K_n on n vertices is integral. Moreover, if $\mathcal{G}$ is the disjoint union of some complete graphs then also it is integral. The notion of integral graph was introduced by Harary and Schwenk [8] in the year 1974. A very impressive survey on integral graphs can be found in [9].

We observe that the commuting graph of a non abelian finite AC-group is disjoint union of some complete graphs. Therefore, commuting graphs of such groups are integral. In general it is difficult to classify all finite non-abelian groups whose commuting graphs are integral. As applications of our results together with some other known results, in Section 3, we show that the commuting graph of a finite non-abelian group G is integral if G is not isomorphic to S_4, the symmetric group of degree 4, and the commuting graph of G is planar. We also show that the commuting graph of a finite non-abelian group G is integral if the commuting graph of G is toroidal. Recall that the genus of a graph is the smallest non-negative integer n such that the graph can be embedded on the surface obtained by attaching n handles to a sphere. A graph is said to be planar or toroidal if the genus of the graph is zero or one respectively. It is worth mentioning that Afkhami *et al.* [10] and Das *et al.* [11] have classified all finite non-abelian groups whose commuting graphs are planar or toroidal recently.

2 Computing spectrum

It is well known that the complete graph K_n on n vertices is integral and $\mathrm{Spec}(K_n)$ is given by $\{(-1)^{n-1}, (n-1)^1\}$. Further, if $\mathcal{G} = K_{m_1} \sqcup K_{m_2} \sqcup \cdots \sqcup K_{m_l}$, where K_{m_i} are complete graphs on m_i vertices for $1 \leq i \leq l$, then

$$\mathrm{Spec}(\mathcal{G}) = \{(-1)^{\sum_{i=1}^{l} m_i - l}, (m_1-1)^1, (m_2-1)^1, \ldots, (m_l-1)^1\}. \tag{1}$$

If $m_1 = m_2 = \cdots = m_l = m$ then we write $\mathcal{G} = lK_m$ and in that case $\mathrm{Spec}(\mathcal{G}) = \{(-1)^{l(m-1)}, (m-1)^l\}$.

In this section, we compute the spectrum of the commuting graphs of different families of finite non-abelian AC-groups. A group G is called an AC-group if $C_G(x) := \{y \in G : xy = yx\}$ is abelian for all $x \in G \setminus Z(G)$. Various aspects of AC-groups can be found in [11–13]. The following lemma plays an important role in computing spectrum of commuting graphs of AC-groups.

Lemma 2.1 *Let G be a finite non-abelian AC-group. Then the commuting graph of G is given by*

$$\Gamma_G = \bigsqcup_{i=1}^{n} K_{|X_i|-|Z(G)|}$$

where $X_1, \ldots, X_n$ are the distinct centralizers of non-central elements of G.

Proof Let G be a finite non-abelian AC-group and $X_1, \ldots, X_n$ be the distinct centralizers of non-central elements of G. Let $X_i = C_G(x_i)$ where $x_i \in G \setminus Z(G)$ and $1 \leq i \leq n$. Let $x, y \in X_i \setminus Z(G)$ for some i and $x \neq y$ then, since G an AC-group, there is an edge between x and y in the commuting graph of G. Suppose that $x \in (X_i \cap X_j) \setminus Z(G)$ for some $1 \leq i \neq j \leq n$. Then $[x, x_i] = 1$ and $[x, x_j] = 1$. Hence, by Lemma 3.6 of [12] we have $C_G(x) = C_G(x_i) = C_G(x_j)$, a contradiction. Therefore, $X_i \cap X_j = Z(G)$ for any $1 \leq i \neq j \leq n$. This shows that $\Gamma_G = \bigsqcup_{i=1}^{n} K_{|X_i|-|Z(G)|}$. □

Theorem 2.1 *Let G be a finite non-abelian AC-group. Then the spectrum of the commuting graph of G is given by*

$$\{(-1)^{\sum_{i=1}^{n} |X_i| - n(|Z(G)|+1)}, (|X_1| - |Z(G)| - 1)^1, \ldots, (|X_n| - |Z(G)| - 1)^1\}$$

where $X_1, \ldots, X_n$ are the distinct centralizers of non-central elements of G.

Proof The proof follows from Lemma 2.1 and (1). □

Corollary 2.1 *Let G be a finite non-abelian AC-group and A be any finite abelian group. Then the spectrum of the commuting graph of $G \times A$ is given by*

$$\{(-1)^{|A|\sum_{i=1}^{n}(|X_i| - n|Z(G)|) - n}, (|A|(|X_1| - |Z(G)|) - 1))^1, \ldots, (|A|(|X_n| - |Z(G)|) - 1))^1\}$$

where $X_1, \ldots, X_n$ are the distinct centralizers of non-central elements of G.

Proof It is easy to see that $Z(G \times A) = Z(G) \times A$ and $X_1 \times A, X_2 \times A, \ldots, X_n \times A$ are the distinct centralizers of non-central elements of $G \times A$. Therefore, if G is an AC-group then $G \times A$ is also an AC-group. Hence, the result follows from Theorem 2.1. □

Now we compute the spectrum of the commuting graphs of some particular families of AC-groups. We begin with the well-known family of quasidihedral groups.

Proposition 2.1 *The spectrum of the commuting graph of the quasidihedral group* $QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1}\rangle$, *where* $n \geq 4$, *is given by*

$$\operatorname{Spec}(\Gamma_{QD_{2^n}}) = \{(-1)^{2^n-2^{n-2}-3}, 1^{2^{n-2}}, (2^{n-1}-3)^1\}.$$

Proof It is well-known that $Z(QD_{2^n}) = \{1, a^{2^{n-2}}\}$. Also

$$C_{QD_{2^n}}(a) = C_{QD_{2^n}}(a^i) = \langle a\rangle \text{ for } 1 \leq i \leq 2^{n-1}-1, i \neq 2^{n-2}$$

and

$$C_{QD_{2^n}}(a^j b) = \{1, a^{2^{n-2}}, a^i b, a^{i+2^{n-2}} b\} \text{ for } 1 \leq j \leq 2^{n-2}$$

are the only centralizers of non-central elements of QD_{2^n}. Note that these centralizers are abelian subgroups of QD_{2^n}. Therefore, by Lemma 2.1

$$\Gamma_{QD_{2^n}} = K_{|C_{QD_{2^n}}(a)\setminus Z(QD_{2^n})|} \sqcup (\bigsqcup_{j=1}^{2^{n-2}} K_{|C_{QD_{2^n}}(a^j b)\setminus Z(QD_{2^n})|}).$$

That is, $\Gamma_{QD_{2^n}} = K_{2^{n-1}-2} \sqcup 2^{n-2}K_2$, since $|C_{QD_{2^n}}(a)| = 2^{n-1}, |C_{QD_{2^n}}(a^j b)| = 4$ for $1 \leq j \leq 2^{n-2}$ and $|Z(QD_{2^n})| = 2$. Hence, the result follows from (1). □

Proposition 2.2 *The spectrum of the commuting graph of the projective special linear group* $PSL(2, 2^k)$, *where* $k \geq 2$, *is given by*

$$\{(-1)^{2^{3k}-2^{2k}-2^{k+1}-2}, (2^k-1)^{2^{k-1}(2^k-1)}, (2^k-2)^{2^k+1}, (2^k-3)^{2^{k-1}(2^k+1)}\}.$$

Proof We know that $PSL(2, 2^k)$ is a non-abelian group of order $2^k(2^{2k}-1)$ with trivial center. By Proposition 3.21 of [12], the set of centralizers of non-trivial elements of $PSL(2, 2^k)$ is given by

$$\{xPx^{-1}, xAx^{-1}, xBx^{-1} : x \in PSL(2, 2^k)\}$$

where P is an elementary abelian 2-subgroup and A, B are cyclic subgroups of $PSL(2, 2^k)$ having order $2^k, 2^k-1$ and 2^k+1 respectively. Also the number of conjugates of P, A and B in $PSL(2, 2^k)$ are $2^k+1, 2^{k-1}(2^k+1)$ and $2^{k-1}(2^k-1)$ respectively. Note that $PSL(2, 2^k)$ is a AC-group and so, by Lemma 2.1, the commuting graph of $PSL(2, 2^k)$ is given by

$$(2^k+1)K_{|xPx^{-1}|-1} \sqcup 2^{k-1}(2^k+1)K_{|xAx^{-1}|-1} \sqcup 2^{k-1}(2^k-1)K_{|xBx^{-1}|-1}.$$

That is, $\Gamma_{PSL(2,2^k)} = (2^k+1)K_{2^k-1} \sqcup 2^{k-1}(2^k+1)K_{2^k-2} \sqcup 2^{k-1}(2^k-1)K_{2^k}$. Hence, the result follows from (1). □

Proposition 2.3 *The spectrum of the commuting graph of the general linear group* $GL(2, q)$, *where* $q = p^n > 2$ *and* p *is a prime integer, is given by*

$$\{(-1)^{q^4-q^3-2q^2-q}, (q^2-3q+1)^{q(q+1)/2}, (q^2-q-1)^{q(q-1)/2}, (q^2-2q)^{q+1}\}.$$

Proof We have $|GL(2,q)| = (q^2-1)(q^2-q)$ and $|Z(GL(2,q))| = q-1$. By Proposition 3.26 of [12], the set of centralizers of non-central elements of $GL(2,q)$ is given by

$$\{xDx^{-1}, xIx^{-1}, xPZ(GL(2,q))x^{-1} : x \in GL(2,q)\}$$

where D is the subgroup of $GL(2,q)$ consisting of all diagonal matrices, I is a cyclic subgroup of $GL(2,q)$ having order q^2-1 and P is the Sylow p-subgroup of $GL(2,q)$ consisting of all upper triangular matrices with 1 in the diagonal. The orders of D and $PZ(GL(2,q))$ are $(q-1)^2$ and $q(q-1)$ respectively. Also the number of conjugates of D, I and $PZ(GL(2,q))$ in $GL(2,q)$ are $q(q+1)/2, q(q-1)/2$ and $q+1$ respectively. Since $GL(2,q)$ is an AC-group (see Lemma 3.5 of [12]), by Lemma 2.1 we have $\Gamma_{GL(2,q)} =$

$$\frac{q(q+1)}{2}K_{|xDx^{-1}|-q+1} \sqcup \frac{q(q-1)}{2}K_{|xIx^{-1}|-q+1} \sqcup (q+1)K_{|xPZ(GL(2,q))x^{-1}|-q+1}.$$

That is, $\Gamma_{GL(2,q)} = \frac{q(q+1)}{2}K_{q^2-3q+2} \sqcup \frac{q(q-1)}{2}K_{q^2-q} \sqcup (q+1)K_{q^2-2q+1}$. Hence, the result follows from (1). □

Theorem 2.2 *Let G be a finite group and $\frac{G}{Z(G)} \cong Sz(2)$, where $Sz(2)$ is the Suzuki group presented by $\langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2\rangle$. Then*

$$\text{Spec}(\Gamma_G) = \{(-1)^{19|Z(G)|-6}, (4|Z(G)|-1)^1, (3|Z(G)|-1)^5\}.$$

Proof We have

$$\frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^5Z(G) = b^4Z(G) = Z(G), b^{-1}abZ(G) = a^2Z(G)\rangle.$$

Observe that

$$\begin{array}{ll}
C_G(a) & = Z(G) \sqcup aZ(G) \sqcup a^2Z(G) \sqcup a^3Z(G) \sqcup a^4Z(G), \\
C_G(ab) & = Z(G) \sqcup abZ(G) \sqcup a^4b^2Z(G) \sqcup a^3b^3Z(G), \\
C_G(a^2b) & = Z(G) \sqcup a^2bZ(G) \sqcup a^3b^2Z(G) \sqcup ab^3Z(G), \\
C_G(a^2b^3) & = Z(G) \sqcup a^2b^3Z(G) \sqcup ab^2Z(G) \sqcup a^4bZ(G), \\
C_G(b) & = Z(G) \sqcup bZ(G) \sqcup b^2Z(G) \sqcup b^3Z(G) \quad \text{and} \\
C_G(a^3b) & = Z(G) \sqcup a^3bZ(G) \sqcup a^2b^2Z(G) \sqcup a^4b^3Z(G)
\end{array}$$

are the only centralizers of non-central elements of G. Also note that these centralizers are abelian subgroups of G. Thus G is an AC-group. By Lemma 2.1, we have

$$\Gamma_G = K_{4|Z(G)|} \sqcup 5K_{3|Z(G)|}$$

since $|C_G(a)| = 5|Z(G)|$ and

$$|C_G(ab)| = |C_G(a^2b)| = |C_G(a^2b^3)| = |C_G(b)| = |C_G(a^3b)| = 4|Z(G)|.$$

Therefore, by (1), the result follows. □

Proposition 2.4 *Let $F = GF(2^n), n \geq 2$ and ϑ be the Frobenius automorphism of F, i. e., $\vartheta(x) = x^2$ for all $x \in F$. Then the spectrum of the commuting graph of the group*

$$A(n,\vartheta) = \left\{ U(a,b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \vartheta(a) & 1 \end{bmatrix} : a, b \in F \right\}.$$

under matrix multiplication given by $U(a,b)U(a',b') = U(a+a', b+b'+a'\vartheta(a))$ is

$$\mathrm{Spec}(\Gamma_{A(n,\vartheta)}) = \{(-1)^{(2^n-1)^2}, (2^n-1)^{2^n-1}\}.$$

Proof Note that $Z(A(n,\vartheta)) = \{U(0,b) : b \in F\}$ and so $|Z(A(n,\vartheta))| = 2^n - 1$. Let $U(a,b)$ be a non-central element of $A(n,\vartheta)$. It can be seen that the centralizer of $U(a,b)$ in $A(n,\vartheta)$ is $Z(A(n,\vartheta)) \sqcup U(a,0)Z(A(n,\vartheta))$. Clearly $A(n,\vartheta)$ is an AC-group and so by Lemma 2.1 we have $\Gamma_{A(n,\vartheta)} = (2^n-1)K_{2^n}$. Hence the result follows by (1). □

Proposition 2.5 *Let $F = GF(p^n)$, p be a prime. Then the spectrum of the commuting graph of the group*

$$A(n,p) = \left\{ V(a,b,c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a, b, c \in F \right\}.$$

under matrix multiplication $V(a,b,c)V(a',b',c') = V(a+a', b+b'+ca', c+c')$ is

$$\mathrm{Spec}(\Gamma_{A(n,p)}) = \{(-1)^{p^{3n}-2p^n-1}, (p^{2n}-p^n-1)^{p^n+1}\}.$$

Proof We have $Z(A(n,p)) = \{V(0,b,0) : b \in F\}$ and so $|Z(A(n,p))| = p^n$. The centralizers of non-central elements of $A(n,p)$ are given by

(i) If $b, c \in F$ and $c \neq 0$ then the centralizer of $V(0,b,c)$ in $A(n,p)$ is $\{V(0,b',c') : b', c' \in F\}$ having order $|p^{2n}|$.

(ii) If $a, b \in F$ and $a \neq 0$ then the centralizer of $V(a,b,0)$ in $A(n,p)$ is $\{V(a',b',0) : a', b' \in F\}$ having order $|p^{2n}|$.

(iii) If $a, b, c \in F$ and $a \neq 0, c \neq 0$ then the centralizer of $V(a,b,c)$ in $A(n,p)$ is $\{V(a',b',ca'a^{-1}) : a', b' \in F\}$ having order $|p^{2n}|$.

It can be seen that all the centralizers of non-central elements of $A(n,p)$ are abelian. Hence $A(n,p)$ is an AC-group and so

$$\Gamma_{A(n,p)} = K_{p^{2n}-p^n} \sqcup K_{p^{2n}-p^n} \sqcup (p^n-1)K_{p^{2n}-p^n} = (p^n+1)K_{p^{2n}-p^n}.$$

Hence the result follows from (1). □

We would like to mention here that the groups considered in Proposition 2.4-2.5 are constructed by Hanaki (see [14]). These groups are also considered in [15], in order to compute their numbers of distinct centralizers.

3 Some applications

In this section, we show that the commuting graph of a finite non-abelian group G is integral if G is not isomorphic to S_4 and the commuting graph of G is planar. We also show that the commuting graph of a finite non-abelian group G is integral if the commuting graph of G is toroidal. We shall use the following results.

Theorem 3.1 *Let G be a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then*

$$\mathrm{Spec}(\Gamma_G) = \{(-1)^{3|Z(G)|-3}, (|Z(G)| - 1)^3\}.$$

Proof The result follows from Theorem 2.1 noting that G is an AC-group with 3 distinct centralizers of non-central elements and all of them have order $2|Z(G)|$. □

Proposition 3.1 *Let $D_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1}\rangle$ be the dihedral group of order $2m$, where $m > 2$. Then*

$$\mathrm{Spec}(\Gamma_{D_{2m}}) = \begin{cases} \{(-1)^{m-2}, 0^m, (m-2)^1\} \text{ if } m \text{ is odd} \\ \{(-1)^{(3m/2)-3}, 1^{m/2}, (m-3)^1\} \text{ if } m \text{ is even.} \end{cases}$$

Proof Note that D_{2m} is a non-abelian AC-group. If m is even then $|Z(D_{2m})| = 2$ and D_{2m} has $\frac{m}{2} + 1$ distinct centralizers of non-central elements. Out of these centralizers one has order m and the rests have order 4. Therefore $\Gamma_{D_{2m}} = K_{m-2} \sqcup \frac{m}{2} K_2$. If m is odd then $|Z(D_{2m})| = 1$ and D_{2m} has $m + 1$ distinct centralizers of non-central elements. In this case, one centralizer has order m and the rests have order 2. Therefore $\Gamma_{D_{2m}} = K_{m-1} \sqcup mK_1$. Hence the result follows from (1). □

Proposition 3.2 *The spectrum of the commuting graph of the generalized quaternion group $Q_{4n} = \langle x, y : y^{2n} = 1, x^2 = y^n, xyx^{-1} = y^{-1}\rangle$, where $n \geq 2$, is given by*

$$\mathrm{Spec}(\Gamma_{Q_{4n}}) = \{(-1)^{3n-3}, 1^n, (2n-3)^1\}.$$

Proof Note that Q_{4n} is a non-abelian AC-group with $n + 1$ distinct centralizers of non-central elements. Out of these centralizers one has order $2n$ and the rests have order 4. Also $|Z(Q_{4n})| = 2$. Therefore $\Gamma_{Q_{4n}} = K_{2n-2} \sqcup nK_2$. Hence the result follows from (1). □

As an application of Theorem 3.1 we have the following lemma.

Lemma 3.1 *Let G be a group isomorphic to any of the following groups*

(i) $\mathbb{Z}_2 \times D_8$

(ii) $\mathbb{Z}_2 \times Q_8$

(iii) $M_{16} = \langle a, b : a^8 = b^2 = 1, bab = a^5\rangle$

(iv) $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 = \langle a, b : a^4 = b^4 = 1, bab^{-1} = a^{-1}\rangle$

(v) $D_8 * \mathbb{Z}_4 = \langle a, b, c : a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = a^2cb\rangle$

(vi) $SG(16, 3) = \langle a, b : a^4 = b^4 = 1, ab = b^{-1}a^{-1}, ab^{-1} = ba^{-1}\rangle$.

Then $\mathrm{Spec}(\Gamma_G) = \{(-1)^9, 3^3\}$.

Proof If G is isomorphic to any of the above listed groups, then $|G| = 16$ and $|Z(G)| = 4$. Therefore, $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus the result follows from Theorem 3.1. $\square$

The next lemma is also useful in this section.

Lemma 3.2 *Let G be a non-abelian group of order pq, where p and q are primes with $p \mid (q-1)$. Then*

$$\operatorname{Spec}(\Gamma_G) = \{(-1)^{pq-q-1}, (p-2)^q, (q-2)^1\}.$$

Proof It is easy to see that $|Z(G)| = 1$ and G is an AC-group. Also the centralizers of non-central elements of G are precisely the Sylow subgroups of G. The number of Sylow q-subgroups and Sylow p-subgroups of G are one and q respectively. Therefore, by Lemma 2.1 we have $\Gamma_G = K_{q-1} \sqcup qK_{p-1}$. Hence, the result follows from (1). $\square$

Now we state and proof the main results of this section.

Theorem 3.2 *Let Γ_G be the commuting graph of a finite non-abelian group G. If G is not isomorphic to S_4 and Γ_G is planar then Γ_G is integral.*

Proof By Theorem 2.2 of [10] we have that Γ_G is planar if and only if G is isomorphic to either $D_6, D_8, D_{10}, D_{12}, Q_8, Q_{12}, \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, M_{16}, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8 * \mathbb{Z}_4, SG(16,3), A_4, A_5, S_4, SL(2,3)$ or $Sz(2) = \langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^3\rangle$.

If $G \cong D_6, D_8, D_{10}$ or D_{12} then by Proposition 3.1, one may conclude that Γ_G is integral. If $G \cong Q_8$ or Q_{12} then by Proposition 3.2, Γ_G becomes integral. If $G \cong \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, M_{16}, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8 * \mathbb{Z}_4$ or $SG(16,3)$ then by Lemma 3.1, Γ_G becomes integral.

If $G \cong A_4 = \langle a, b : a^2 = b^3 = (ab)^3 = 1\rangle$ then the distinct centralizers of non-central elements of G are $C_G(a) = \{1, a, bab^2, b^2ab\}, C_G(b) = \{1, b, b^2\}, C_G(ab) = \{1, ab, b^2a\}, C_G(ba) = \{1, ba, ab^2\}$ and $C_G(aba) = \{1, aba, bab\}$. Note that these centralizers are abelian subgroups of G. Therefore, $\Gamma_G = K_3 \sqcup 4K_2$ and

$$\operatorname{Spec}(\Gamma_G) = \{(-1)^6, 2^1, 1^4\}.$$

Thus Γ_G is integral.

If $G \cong Sz(2)$ then by Theorem 2.2, we have

$$\operatorname{Spec}(\Gamma_G) = \{(-1)^{13}, (3)^1, (2)^5\}.$$

Hence, Γ_G is integral.

If G is isomorphic to

$$SL(2,3) = \langle a, b, c : a^3 = b^4 = 1, b^2 = c^2, \quad c^{-1}bc = b^{-1}, a^{-1}ba = b^{-1}c^{-1}, a^{-1}ca = b^{-1}\rangle$$

then $Z(G) = \{1, b^2\}$. It can be seen that

$$\begin{aligned}
C_G(b) &= \{1, b, b^2, b^3\} = \langle b\rangle,\\
C_G(c) &= \{1, c, c^2, c^3\} = \langle c\rangle,\\
C_G(bc) &= \{1, b^2, bc, cb\} = \langle bc\rangle,\\
C_G(a^2b^2) &= \{1, b^2, a, a^2, a^2b^2, ab^2\} = \langle a^2b^2\rangle,\\
C_G(ac) &= \{1, b^2, ac, ca^2, a^2bc, ab^2c\} = \langle ac\rangle,\\
C_G(ca) &= \{1, b^2, ca, a^2c, ba^2, ab\} = \langle ca\rangle \quad \text{and}\\
C_G(a^2b) &= \{1, b^2, a^2b, ba, b^3a, (ba)^2\} = \langle a^2b\rangle
\end{aligned}$$

are the only distinct centralizers of non-central elements of G. Note that these centralizers are abelian subgroups of G. Therefore, $\Gamma_G = 3K_2 \sqcup 4K_4$ and

$$\mathrm{Spec}(\Gamma_G) = \{(-1)^{15}, 1^3, 3^4\}.$$

Thus Γ_G is integral.

If $G \cong A_5$ then by Proposition 2.2, we have

$$\mathrm{Spec}(\Gamma_G) = \{(-1)^{38}, 1^{10}, 2^5, 3^6\}$$

noting that $PSL(2,4) \cong A_5$. Thus Γ_G is integral.

Finally, if $G \cong S_4$ then it can be seen that the characteristic polynomial of Γ_G is $(x-1)^7(x+1)^{10}(x^2-5)^2(x^2-3x-2)$ and so

$$\mathrm{Spec}(\Gamma_G) = \left\{1^7, (-1)^{10}, (\sqrt{5})^2, (-\sqrt{5})^2, \left(\frac{3+\sqrt{17}}{2}\right)^1, \left(\frac{3-\sqrt{17}}{2}\right)^1\right\}.$$

Hence, Γ_G is not integral. This completes the proof. □

In [10, Theorem 2.3], Afkhami *et al.* have classified all finite non-abelian groups whose commuting graphs are toroidal. Unfortunately, the statement of Theorem 2.3 in [10] is printed incorrectly. We list the correct version of [10, Theorem 2.3] below, since we are going to use this.

Theorem 3.3 *Let G be a finite non-abelian group. Then Γ_G is toroidal if and only if Γ_G is projective if and only if G is isomorphic to either D_{14}, D_{16}, Q_{16}, $QD_{16}, D_6 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$.*

Theorem 3.4 *Let Γ_G be the commuting graph of a finite non-abelian group G. Then Γ_G is integral if Γ_G is toroidal.*

Proof By Theorem 3.3 we have that Γ_G is toroidal if and only if G is isomorphic to either $D_{14}, D_{16}, Q_{16}, QD_{16}, D_6 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

If $G \cong D_{14}$ or D_{16} then by Proposition 3.1, one may conclude that Γ_G is integral. If $G \cong Q_{16}$ then by Proposition 3.2, Γ_G becomes integral. If $G \cong QD_{16}$ then by Proposition 2.1, Γ_G becomes integral. If $G \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ then Γ_G is integral, by Lemma 3.2. If G is isomorphic to $D_6 \times \mathbb{Z}_3$ or $A_4 \times \mathbb{Z}_2$ then Γ_G becomes integral by Corollary 2.1, since D_6 and A_4 are AC-groups. This completes the proof. □

We shall conclude the paper with the following result.

Proposition 3.3 *Let Γ_G be the commuting graph of a finite non-abelian group G. Then Γ_G is integral if the complement of Γ_G is planar.*

Proof If the complement of Γ_G is planar then by Proposition 2.3 of [12] we have that G is isomorphic to either D_6, D_8 or Q_8. If $G \cong D_6$ or D_8 then by Proposition 3.1, Γ_G is integral. If $G \cong Q_8$ then by Proposition 3.2, Γ_G becomes integral. This completes the proof. □

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