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# Some new properties of Lucas-balancing and Lucas-cobalancing number

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**Abstract** In recent year Behera and Panda introduced a new number sequence that is solutions of the Diophantine equation$1+2+3\ldots+(n-1)=(n+1)+(n+2)\ldots+(n+r)$, where $n$ and $r$ are positive integers. If the pairs$(n,r)$ constitutes a solution of above equation then $n$ is called balancing number and $r$ is the corresponding balancer. The concept of balancing number is extended by introducing the notion of cobalancing which solution of the Diophantine equation$1+2+3\ldots+N=(N+1)+(N+2)\cdots+(N+R)$,where $N$ is called cobalancing number and $R$ is called corresponding cobalancer. Further, Panda introduced the concepts of corresponding Lucas-balancing defined as $C_n=\sqrt{8B_n^2+1}$ and Lucas-cobalancing as $c_n=\sqrt{8b_n^2+8b_n+1}$, where $B_n$ is $n^{\text{th}}$ balancing number and $b_n$ is $n^{\text{th}}$cobalancing number. In this paper, we investigate some new properties of Lucas-balancing and Lucas-cobalancing.

**Keywords** Diophantine equation; Balancing numbers; Cobalancing number; Lucas-balancing numbers; Lucas-cobalancing numbers

**AMS mathematics subject classification** 11B83, 11B37

## 1 Introduction

The concept of balancing number is introduced by Behera and Panda [1] in connection with Diophantine equation

$$1+2+3\ldots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1}$$

for some $r\in Z^+$.Here $r$ is called the balancer corresponding to balancing number $n$. For example, 6 is balancing number with balancer 2. Balancing number follow recurrence relation $B_{n+1}=6B_n-B_{n-1}$. Properties of balancing number are very similar to Fibonacci numbers. The study of balancing number and cobalancing number can be seen in [2].

Panda and Ray [3] modified the notion of balancing number to cobalancing number, in which natural $N$ is called cobalancing number if

$$1+2+3\ldots+N=(N+1)+(N+2)+\cdots+(N+R) \tag{2}$$

for some natural number $R,$where $R$ is called cobalancer of cobalancing number $N$. The first three cobalancing numbers are 2, 14 and 84 with cobalancers 1, 6 and 35 respectively.

Panda [4] introduced the Lucas-balancing number as $C_n=\sqrt{8B_n^2+1}$, the first three Lucas-balancing number are 3,17 and 99. Further, he defined Lucas- cobalancing number as $c_n=\sqrt{8b_n^2+8b_n+1}$, first three Lucas-cobalancing are 1, 7 and 40, where $n\in Z^+$.

The Binet formula of balancing number, Lucas-balancing number and Lucas-cobalancing number are $B_n=\frac{\alpha_1^{2n}-\alpha_2^{2n}}{4\sqrt{2}}, C_n=\frac{\alpha_1^{2n}+\alpha_2^{2n}}{2}$ and $c_n=\frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2}$, respectively, where $\alpha_1=1+\sqrt{2}$ and $\alpha_2=1-\sqrt{2}$ (see [4, 5]).

In this paper, we investigate the properties of Lucas-balancing number and Lucas-cobalancing number. In this investigation we find the generating function of both sequences and their relation with balancing number.

Throughout the whole paper the balancing number starts from $B_1 = 1$and$B_2 = 6$, Lucas-balancing number starts from $C_1 = 3$ and $C_2 = 17$and Lucas-cobalancing number starts from $c_1 = 1$ and $c_2 = 7$.

## 2 Generating function of Lucas-balancing and Lucas- cobalancing sequence

In this section, we introduce generating function of Lucas-balancing and Lucas-cobalancing numbers. We use Lockwood [6] identity which expands $x^n + y^n$ and applying it to Binet form of the given sequence,

$$x^n + y^n = (x+y)^n + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \left[ \binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x+y)^{n-2k} \quad (3)$$

where $\lfloor m \rfloor$ denotes the floor of real number $m$, that is, greatest integer less than or equal to $m$.

**Theorem 2.1** *The generating function of Lucas-balancing number is*

$$\sum_{k=0}^{n} \left[ \binom{2n-k}{k} + \binom{2n-k-1}{k-1} \right] 2^{2n-2k-1},$$

*where we define* $\binom{r}{-1} = 0$.

**Proof** Considering the Binet form of Lucas-balancing number and simplifying it using (3), we get the generating function of Lucas-balancing number as

$$C_n = \frac{2^{2n} + \sum_{k=1}^{n} (-1)^k \left[ \binom{2n-k}{k} + \binom{2n-k-1}{k-1} \right] (-1)^k 2^{2n-2k}}{2}$$

$$= \sum_{k=0}^{n} \left[ \binom{2n-k}{k} + \binom{2n-k-1}{k-1} \right] 2^{2n-2k-1}$$

where we define $\binom{r}{-1} = 0$. This proved the theorem. □

**Theorem 2.2** *The generating function of Lucas-cobalancing number is*

$$c_n = \sum_{k=0}^{n-1} \left[ \binom{(2n-1)-k}{k} + \binom{(2n-1)-k-1}{k-1} \right] 2^{2n-2k-2},$$

*where we define* $\binom{r}{-1} = 0$.

**Proof** Considering the Binet form of Lucas-cobalancing number and again simplifying it using (3), we get the generating function of Lucas-cobalancing number as

$$c_n = \sum_{k=0}^{n-1} \left[ \begin{pmatrix} (2n-1)-k \\ k \end{pmatrix} + \begin{pmatrix} (2n-1)-k-1 \\ k-1 \end{pmatrix} \right] 2^{2n-2k-2}$$

where we define $\begin{pmatrix} r \\ -1 \end{pmatrix} = 0$. This proved the theorem. □

## 3 Connection between balancing, Lucas-balancing and Lucas- cobalancing sequence

In this section, we give an important theorem that links Lucas-balancing and Lucas-cobalancing through a recurrence relation and later, we introduce some of its properties. Throughout this section we have $\alpha_1 = 1+\sqrt{2}$, $\alpha_2 = 1-\sqrt{2}$ and $\alpha_1\alpha_2 = -1$.

Rationalizing $\alpha_1$, we get

$$\alpha_1 = 1+\sqrt{2} = \left(1+\sqrt{2}\right)\left(\frac{1-\sqrt{2}}{1-\sqrt{2}}\right)$$
$$= \frac{-1}{1-\sqrt{2}} = -\alpha_2^{-1}.$$

**Theorem 3.1** *The sequence of Lucas-balancing and Lucas-cobalancing satisfy recurrence relation* $C_n - C_{n-1} = 2c_n$*, where* $n \in N$.

**Proof** Considering Binet form of Lucas-balancing and Lucas-cobalancing number and simplifying $C_n - C_{n-1}$,

$$C_n - C_{n-1} = \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2} - \frac{\alpha_1^{2n-2} + \alpha_2^{2n-2}}{2}.$$

Since $\alpha_1 = -\alpha_2^{-1}$ we simplify above equation as follows,

$$\begin{aligned} \frac{\alpha_1^{2n}\left(1-\alpha_1^{-2}\right) + \alpha_2^{2n}\left(1-\alpha_2^{-2}\right)}{2} &= \frac{\alpha_1^{2n}\left(1-\left(1-\sqrt{2}\right)^2\right) + \alpha_2^{2n}\left(1-\left(1+\sqrt{2}\right)^2\right)}{2} \\ &= \frac{-2\alpha_1^{2n}\left(1-\sqrt{2}\right) - 2\alpha_2^{2n}\left(1+\sqrt{2}\right)}{2} \\ &= \frac{-2\alpha_1^{2n}\left(-\alpha_1^{-1}\right) - 2\alpha_2^{2n}\left(-\alpha_2^{-1}\right)}{2} \\ &= 2\left(\frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2}\right) = 2c_n. \end{aligned}$$ □

**Theorem 3.2** For any positive integer $n$ we have

$$c_n^2 = \frac{C_{2n-1} - 1}{2}.$$

**Proof** Evaluating $c_n^2$ using Binet form of Lucas-cobalancing number we get,

$$\begin{aligned} c_n^2 &= \left(\frac{\alpha_1^{2n-1}+\alpha_2^{2n-1}}{2}\right)^2 \\ &= \frac{\alpha_1^{2(2n-1)}+\alpha_2^{2(2n-1)}+2\alpha_1^{2n-1}\alpha_2^{2n-1}}{4} \\ &= \frac{\alpha_1^{2(2n-1)}+\alpha_2^{2(2n-1)}-2}{4} = \frac{C_{2n-1}-1}{2}. \end{aligned}$$

□

**Corollary 3.1** *For any positive integer $n$ we have*

(1) $\sum_{i=1}^{n} c_i = \dfrac{C_n - 1}{2}$

(2) $\sum_{i=1}^{n} c_i^2 = \dfrac{B_{2n} - 2n}{4}$.

**Proof** For (1), consider relation from Theorem 3.1 that is $C_i - C_{i-1} = 2c_i$ and taking summation from 1 to $n$ we get $C_n = 2\sum_{i=1}^{n} c_i + 1$, by assuming $C_0 = 1$ .

Alternatively,

$$\begin{aligned} \sum_{i=1}^{n} c_i &= \left(\frac{\alpha_1^1+\alpha_1^3+\alpha_1^5+...+\alpha_1^{2n-1}}{2}\right)+\left(\frac{\alpha_2^1+\alpha_2^3+\alpha_2^5+...+\alpha_2^{2n-1}}{2}\right) \\ &= \frac{\alpha_1}{2}\left(\frac{\alpha_1^{2n}-1}{\alpha_1^2-1}\right)+\frac{\alpha_2}{2}\left(\frac{1-\alpha_2^{2n}}{1-\alpha_2^2}\right). \end{aligned}$$

Now simplifying above equation using $\alpha_1 = -\alpha_2^{-1}$, we get

$$\frac{2\left(\alpha_1^{2n}+\alpha_2^{2n}\right)-4}{8}.$$

Therefore, using Binet form of Lucas-cobalancing number, above equation reduces to $\frac{1}{2}(C_n - 1)$.

For (2) taking summation from 1 to $n$ for $2c_n^2 = C_{2n-1} - 1$ in Theorem 3.2 we get,

$$2\sum_{i=1}^{n} c_i^2 = \sum_{i=1}^{n} C_{2n-1} - n. \tag{4}$$

Now we evaluate $\sum_{i=1}^{n} C_{2n-1}$,

$$\begin{aligned} \sum_{i=1}^{n} C_{2n-1} &= C_1 + C_3 + ... + C_{2n-1} \\ &= \left(\frac{\alpha_1^2+\alpha_1^6+\alpha_1^{10}+...+\alpha_1^{2n-1}}{2}\right)+\left(\frac{\alpha_2^2+\alpha_2^6+\alpha_2^{10}+...+\alpha_2^{2n-1}}{2}\right). \end{aligned}$$

Since, the series are in geometric progressions where $\alpha_1 = 1+\sqrt{2} > 1$ and $\alpha_2 = 1-\sqrt{2} < 1$. So, we get summation of $\sum_{i=1}^{n} C_{2n-1}$as

$$\alpha_1^2\left(\frac{\alpha_1^{4n}-1}{\alpha_1^4-1}\right)+\alpha_2^2\left(\frac{1-\alpha_2^{4n}}{1-\alpha_2^4}\right).$$

Using $\alpha_1 = -\alpha_2^{-1}$ and

$$B_n = \frac{\alpha_1^{2n}-\alpha_2^{2n}}{4\sqrt{2}},$$

simplifying the above equation we get $\frac{B_{2n}}{2}$. So, putting this in equation (4) reduces it to

$$\frac{B_{2n}-2n}{4}.$$

This proved our corollary. □

**Theorem 3.3** For any positive integer $n$ we have

(1) $C_{n-1}C_{n+1} = \dfrac{C_{2n}+17}{2}$

(2) $c_{n-1}c_{n+1} = \dfrac{C_{2n-1}-17}{2}$

(3) $C_{n-1}C_{n+1} + c_{n-1}c_{n+1} = \dfrac{C_{2n}+C_{2n-1}}{2}$

**Proof**

For (1), considering Binet form of Lucas-balancing number and evaluating $C_{n-1}C_{n+1}$,

$$\left(\frac{\alpha_1^{2(n-1)}+\alpha_2^{2(n-1)}}{2}\right)\left(\frac{\alpha_1^{2(n+1)}+\alpha_2^{2(n+1)}}{2}\right).$$

Now simplifying above equation using $\alpha_1 = -\alpha_2^{-1}$ we get $\frac{1}{2}(C_{2n}+17)$.

For (2), considering Binet form of Lucas-cobalancing number and evaluating $c_{n-1}c_{n+1}$,

$$\left(\frac{\alpha_1^{2n-3}+\alpha_2^{2n-3}}{2}\right)\left(\frac{\alpha_1^{2n+1}+\alpha_2^{2n+1}}{2}\right).$$

Now simplifying the above equation using $\alpha_1 = -\alpha_2^{-1}$, we get $\frac{1}{2}(C_{2n-1}-17)$.

For (3), just add (1) and (2). □

**Theorem 3.4** *For any positive integer $n$ we have*

(1) $\lim\limits_{n\to\infty}\dfrac{C_{n+1}}{C_n} = 3+2\sqrt{2}$

(2) $\lim\limits_{n\to\infty}\dfrac{c_{n+1}}{c_n} = 3+2\sqrt{2}$

**Proof**

For (1), we consider Binet form of Lucas-balancing number,

$$\lim_{n\to\infty}\frac{C_{n+1}}{C_n}=\frac{\alpha_1^{2(n+1)}+\alpha_2^{2(n+1)}}{\alpha_1^{2n}+\alpha_2^{2n}}.$$

Since$\alpha_1>\alpha_2$, so dividing above equation by $\alpha_1^{2n}$ we get$\alpha_1^2=3+2\sqrt{2}$.

For (2), we consider Binet form of Lucas-cobalancing number,

$$\lim_{n\to\infty}\frac{c_{n+1}}{c_n}=\frac{\alpha_1^{2n+1}+\alpha_2^{2n+1}}{\alpha_1^{2n-1}+\alpha_2^{2n-1}}.$$

Since$\alpha_1>\alpha_2$, so dividing above equation by $\alpha_1^{2n}$ we get $\alpha_1^2=3+2\sqrt{2}$. This proved our theorem. □

**Theorem 3.5** *For any positive integer n we have*

(1) $\sum_{i=1}^{n}C_i=\dfrac{c_{n+1}-1}{2}$

(2) $\sum_{i=1}^{n}C_i^2=\dfrac{c_{2(n+1)}+c_{2n+1}-8+16n}{32}$

**Proof** For (1), consider the Binet form of Lucas-balancing number and evaluating$\sum_{i=1}^{n}C_i$,

$$\sum_{i=1}^{n}C_i=\left(\frac{\alpha_1^2+\alpha_1^4+\alpha_1^6+...+\alpha_1^{2n}}{2}\right)+\left(\frac{\alpha_2^2+\alpha_2^4+\alpha_2^6+...+\alpha_2^{2n}}{2}\right)$$
$$=\frac{\alpha_1^2}{2}\left(\frac{\alpha_1^{2n}-1}{\alpha_1^2-1}\right)+\frac{\alpha_2^2}{2}\left(\frac{1-\alpha_2^{2n}}{1-\alpha_2^2}\right).$$

Simplifying the above equation using $\alpha_1=-\alpha_2^{-1}$ we get$\frac{1}{2}(c_{n+1}-1)$.

For (2), we have $C_i^2=\frac{1}{4}(\alpha_1^{4n}+\alpha_4^{4n}+2)$, so evaluating $\sum_{i=1}^{n}C_i^2$

$$\sum_{i=1}^{n}C_i^2=\frac{\left(\alpha_1^4+\alpha_1^8+\alpha_1^{12}+...+\alpha_1^{4n}\right)+\left(\alpha_2^4+\alpha_2^8+\alpha_2^{12}+...+\alpha_2^{4n}\right)+2n}{4}.$$

Simplifying the above equation using Binet form of Lucas-balancing number and $\alpha_1=-\alpha_2^{-1}$ we get

$$\frac{C_{2(n+1)}-C_{2n}-16}{64}+\frac{n}{2}. \quad (5)$$

Now, by Theorem 3.1 we get $C_{2(n+1)}-C_{2n+1}=2c_{2(n+1)}$ and $C_{2n+1}-C_{2n}=2c_{2n+1}$ which reduces the equation (5) to

$$\frac{c_{2(n+1)}+c_{2n+1}-8+16n}{32}.$$

This completes the proof. □

**Lemma 3.1** $2c_nC_{n-1} = c_{2n-1} + 1$, $n \in N$.

**Proof** Considering $c_nC_{n-1}$ and evaluating it using Binet form of Lucas-balancing and Lucas-cobalancing number

$$\frac{\left(\alpha_1^{2n-1} + \alpha_2^{2n-1}\right)\left(\alpha_1^{2(n-1)} + \alpha_2^{2(n-1)}\right)}{4} = \frac{\alpha_1^{4n-3} + \alpha_2^{4n-3} + \alpha_1^{2n-1}\alpha_2^{2n-2} + \alpha_1^{2n-1}\alpha_2^{2n-2}}{4}$$
$$= \frac{\alpha_1^{4n-3} + \alpha_2^{4n-3} + 2}{4} = \frac{c_{2n-1} + 1}{2}$$

This proved the lemma. □

**Theorem 3.6** *For any positive integer $n$, balancing, Lucas-balancing and Lucas-cobalancing sequence satisfy recurrence relation $C_n + c_n = 4B_n$.*

**Proof** By the definition of Lucas-balancing we get recurrence relation $C_n^2 = 8B_n^2 + 1$, now simplifying it using Theorem 3.1 and $C_{n-1}^2 = 8B_{n-1}^2 + 1$ we get

$$C_{n-1}^2 + 4c_n^2 + 4C_{n-1}c_n = 8B_n^2 + 1.$$

Then by using Theorem 3.2, Lemma 3.1 and $B_{2n-1} = B_n^2 - B_{n-1}^2$ (see [7]) the above equation is further simplified to $C_{2n-1} + c_{2n-1} = 4B_{2n-1}$. Replacing $n$ by $\left(\frac{2n+1}{2}\right) - 1$ in previous equation we get $C_{2n} + c_{2n} = 4B_{2n}$. So we can generalize the relation between balancing, Lucas-balancing and Lucas-cobalancing sequence through a recurrence relation $C_n + c_n = 4B_n$. This completes the proof. □

## 4 Conclusion

In this paper, we investigate some new properties of Lucas-balancing and Lucas-cobalancing. We connect balancing number, Lucas-balancing and Lucas-coblancing through a recurrence relation.

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