Finite $p$-groups in which each absolute central automorphism is elementary abelian

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Abstract In this paper we give necessary and sufficient conditions on a $p$-group $G$ for each absolute central automorphism of $G$ to be elementary abelian. Also we classify the absolute centre of extraspecial $p$-groups and show that the absolute central automorphisms of these groups are elementary abelian.

Keywords absolute centre; absolute central automorphisms; elementary abelian; finite $p$-groups; extraspecial $p$-groups.

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1 Introduction and basic lemmas

Throughout this paper $p$ denotes a prime number. Let $G$ be a group. We denote $C_m$, $G'$, $Z(G)$, $L(G)$, $\exp(G)$, $\Hom(G,H)$, $\Aut(G)$ and $\Inn(G)$, as the cyclic group of order $m$, the commutator subgroup, the centre, the absolute centre, the exponent, the group of homomorphisms of $G$ into an abelian group $H$, the full automorphism group and the inner automorphism group of $G$, respectively. An automorphism $\alpha$ of $G$ is called a central automorphism if and only if it induces the identity on $G/Z(G)$. The central automorphisms of $G$, denoted by $\Aut_c(G)$, fix $G'$ elementwise and form a normal subgroup of the full automorphism group of $G$. The properties of $\Aut_c(G)$ are studied by many authors, see for instance [1], [2] and [3]. Hegarty in [4] generalized the concept of centre into absolute centre. The absolute centre of a group $G$, $L(G)$, is the subgroup consisting of all those elements that are fixed under all automorphisms of $G$. Also he introduced the concept of the absolute central automorphisms. An automorphism $\beta$ of $G$ is called an absolute central automorphism if and only if it acts trivially on the factor group $G/L(G)$, or equivalently, $x^{-1}\beta(x) \in L(G)$ for each $x \in G$. We denote the set of all absolute central automorphisms of $G$ by $\Aut_l(G)$. Notice that $\Aut_l(G)$ is a normal subgroup of $\Aut(G)$ contained in $\Aut_c(G)$. In 2006 Jafari [5] gave necessary and sufficient conditions on a finite purely nonabelian $p$-group $G$ for the group $\Aut_l(G)$ to be elementary abelian. In this paper we give conditions on a $p$-group $G$ such that $\Aut_l(G)$ to be elementary abelian.

The extraspecial $p$-groups are of great importance in the investigation of finite $p$-groups. Finally upon determining the absolute centre of extraspecial $p$-groups, we prove absolute central automorphisms of these groups are elementary abelian.

Here we give two basic lemmas that will be used in the proof of the results.

Lemma 1 [6, Lemma 2.1] Let $G$ be a finite nilpotent group of class 2. Then

(i) $G' \leq Z(G)$,
(ii) $\exp(G') = \exp(G/Z(G))$.

Lemma 2 [6, Lemma 2.2] Let $A$, $C$ and $U$ be abelian groups.
(i) \( \text{Hom}(A, C \times U) \cong \text{Hom}(A, C) \times \text{Hom}(A, U) \),
(ii) \( \text{Hom}(A \times C, U) \cong \text{Hom}(A, U) \times \text{Hom}(C, U) \),
(iii) \( \text{Hom}(C_m, C_n) \cong C_d \), where \( d = \gcd(m, n) \).

2 Absolute central automorphisms that are elementary abelian

In this section, we firstly state some definitions and elementary results which shall be used in the main results.

**Definition 1** [4] Let \( G \) be a group. The subgroup \( L(G) \), consisting of all elements of \( G \) fixed by every automorphism of \( G \) is called the absolute centre of \( G \). Then

\[
L(G) = \{ g \in G \mid g^{-1} \alpha(g) = 1, \forall \alpha \in \text{Aut}(G) \}.
\]

Clearly, \( L(G) \) is a central characteristic subgroup of \( G \).

**Definition 2** [4] An automorphism \( \alpha \) of \( G \) is called absolute central, if \( g^{-1} \alpha(g) \in L(G) \) for each \( g \in G \). We denote the set of all absolute central automorphisms of \( G \) by \( \text{Aut}_1(G) \). \( \text{Aut}_1(G) \) is a normal subgroup of \( \text{Aut}(G) \) contained in \( \text{Aut}_c(G) \).

**Lemma 3** [6, Lemma 2.7] Let \( G \) be a finite group. Then \( G/L(G) \) is abelian if and only if \( \text{Inn}(G) \leq \text{Aut}_1(G) \).

**Lemma 4** [7, Proposition 1] Let \( G \) be a group. Then \( \text{Aut}_1(G) \cong \text{Hom}(G/L(G), L(G)) \). So if \( G \) is a \( p \)-group, then \( \text{Aut}_1(G) \) is also a \( p \)-group.

As any homomorphism \( f : G/L(G) \to L(G) \) induces a homomorphism \( \bar{f} : G/G'L(G) \to L(G) \), we see that \( |\text{Hom}(G/L(G), L(G))| = |\text{Hom}(G/G'L(G), L(G))| \).

Now we give the following main results.

**Lemma 5** Let \( G \) be a \( p \)-group such that \( G/L(G) \) is abelian. If \( L(G) \) is elementary abelian of rank \( s \), then \( \text{Aut}_1(G) \) is elementary abelian of order \( p^s \) that \( t = \text{rank}(G/L(G)) \).

**Proof** It follows from lemmas 2 and 4.

**Lemma 6** Let \( G \) be a non-abelian \( p \)-group such that \( G/L(G) \) is abelian. If \( \text{Aut}_1(G) \) is elementary abelian, then \( \exp(G') = p \).

**Proof** Since \( G/L(G) \) is abelian, hence, \( \text{Inn}(G) \leq \text{Aut}_1(G) \). Thus \( \text{Inn}(G) \) is elementary abelian. Hence \( \exp(G/Z(G)) = p \). Also \( G \) is nilpotent of class 2 and so \( \exp(G') = \exp(G/Z(G)) = p \).

In Theorem 1, we give necessary and sufficient conditions on \( p \)-group \( G \) that \( \text{Aut}_1(G) \) is elementary abelian. First, notice that for a finite group \( G \) and central subgroup \( M \) of \( G \), if \( f \) be a homomorphism from \( G \) to \( M \), we can define an endomorphism \( \sigma_f \) of \( G \) such that \( \sigma_f(x) = xf(x) \). Also \( \sigma_f \) is an automorphism of \( G \) if and only if for every non-trivial element \( g \in M \), \( f(g) \neq g^{-1} \).

Recall that when \( \text{Aut}_1(G) \) is trivial, clearly it is elementary abelian. So from now on, we discuss on \( p \)-group \( G \) in which \( \text{Aut}_1(G) \) is non-trivial.

**Theorem 1** Let \( G \) be a finite \( p \)-group. \( \text{Aut}_1(G) \) is elementary abelian if and only if \( \exp(L(G)) = p \) or \( \exp(G/G'L(G)) = p \).
Proof Suppose \( \exp(L(G)) = p \) and \( s = \text{rank}(L(G)) \) and also \( t = \text{rank}(G/G'L(G)) \). By lemmas 2 and 4 we have

\[
\text{Aut}_t(G) \cong \text{Hom}\left(\frac{G}{L(G)}, L(G)\right)
\cong \text{Hom}\left(\frac{G}{G'L(G)}, L(G)\right)
\cong \text{Hom}(C_{p^{2s+1}} \times C_{p^{2s}} \times \cdots \times C_{p^{2s+1}}, C_p \times C_p \times \cdots \times C_p)
\cong C_p \times C_p \times \cdots \times C_p.
\]

So \( \text{Aut}_t(G) \) is elementary abelian. Similarly, if \( \exp(G/G'L(G)) = p \), the theorem is true.

Conversely, suppose \( \text{Aut}_t(G) \) is an elementary abelian \( p \)-group with \( \exp(L(G)) > p \) and \( \exp(G/G'L(G)) > p \). Assume \( (l) \) and \( \langle xG'L(G) \rangle \) are cyclic direct factors of \( L(G) \) and \( G/G'L(G) \) with maximum possible orders, respectively. Define

\[
H = \{ \sigma_f : f \in \text{Hom}(\langle xG'L(G) \rangle, \langle l \rangle) \}.
\]

According to the previously-mentioned points, \( \sigma_f \) is an automorphism of \( G \). Also for each \( x \in G, x^{-1}\sigma_f(x) \in L(G) \) and so \( H \) is a subset of \( \text{Aut}_t(G) \). Now since \( \sigma_f \) acts trivially on \( L(G) \), by some calculations one can see that \( H \) is a cyclic subgroup of \( \text{Aut}_t(G) \) of order at least \( p^2 \). This is a contradiction. \( \square \)

We end this section with some examples of groups that satisfy the conditions of Theorem 1.

Example 1
(1) If \( G \) is the dihedral group of order 8, \( \exp(L(G)) = 2 \) and \( \text{Aut}_t(G) \) is elementary abelian.
(2) \( G = \langle a, b \mid a^{27} = 1, ab = b^{-2}, a^{-1}b^{-4}a = b^2, [b, a] = b^3 \rangle \), a semidirect product of \( C_9 \) and \( C_{27} \), is an example of a group satisfying the conditions of Theorem 1.

In the following, we give an account of \( p \)-groups whose absolute central automorphisms are elementary abelian.

3 The extraspecial \( p \)-groups

A finite \( p \)-group \( G \) is special if \( G \) is elementary abelian or \( Z(G) = G' = \Phi(G) \) and a non-abelian special \( p \)-group \( G \) is extraspecial if \( Z(G) = G' = \Phi(G) \cong C_p \). An extraspecial \( p \)-group \( G \) is the central product of \( n \geq 1 \) non-abelian subgroups of order \( p^3 \) and \( |G| = p^{2n+1} \). For more information you can see [8].

There are two isomorphism classes of extraspecial \( p \)-groups. When \( p = 2 \), \( G \) is of exponent 4 and when \( p \) is odd, one of these isomorphism classes has exponent \( p \) and the other has exponent \( p^2 \). Any extraspecial \( p \)-group \( G \) has generators \( x_1, x_2, \ldots, x_{2n} \) satisfying the following relations where \( z \) is a fixed generator of \( Z(G) \).

1. \( [x_{2i-1}, x_{2i}] = z, 1 \leq i \leq n \)
2. \( [x_i, x_j] = 1, 1 \leq i, j \leq n \mid i - j \mid > 1 \)
3. \( x_i^p \in Z(G), \quad 1 \leq i \leq 2n. \)

In 1972 Winter [9] gave the structure of the automorphism group of extraspecial \( p \)-groups. Here we give those results of Winter [9].

**Theorem 2** [9, Theorem 1] Let \( G \) be an extraspecial \( p \)-group of order \( p^{2n+1} \) and let \( H \) be a normal subgroup of \( \text{Aut}(G) \) which acts trivially on \( Z(G) \). Then \( \text{Aut}(G) = \langle \theta \rangle H \) where \( \theta \) has order \( p - 1 \), \( H \cap \langle \theta \rangle = 1 \) and \( H/\text{Inn}(G) \) is isomorphic to a subgroup of symplectic group \( \text{Sp}(2n, p) \).

Here we define the automorphism \( \theta \) in Theorem 2. Let \( m \) be a primitive root mod \( p \) with \( 0 < m < p \). The automorphism \( \theta \) is defined by \( \theta(x_{2i-1}) = x_{2i-1}^m \) and \( \theta(x_{2i}) = x_{2i} \), for \( 1 \leq i \leq n \) and \( \theta(z) = z^m \).

Recall for \( p = 2 \), \( m = 1 \) and if \( p \) is odd, then \( |m| \geq 2 \). Now in the theorem below we determine the absolute centre of extraspecial \( p \)-groups.

**Theorem 3** Let \( G \) be an extraspecial \( p \)-group.

(I) if \( p = 2 \) then \( L(G) \cong C_2 \).

(II) if \( p \) is odd then \( L(G) \) is trivial.

**Proof** Let \( G \) be an extraspecial \( p \)-group. Since \( L(G) \leq Z(G) \cong C_p \), then either \( L(G) \cong C_p \) or \( L(G) = \langle 1 \rangle \).

(I) let \( p = 2 \). By Theorem 2, \( \text{Aut}(G) = \langle \theta \rangle H \) where \( H \) acts trivially on \( Z(G) \). Also \( m = 1 \). Thus \( \theta(z) = z \) and so each automorphism of \( G \) fixes the centre elementwise.

Hence \( L(G) = Z(G) \cong C_2 \).

(II) let \( p \) be odd. We know \( m \neq 1 \) and \( 0 < m < p \). Hence \( \theta(z) = z^m \neq z \). Therefore \( \text{Aut}(G) \) does not fix the centre of \( G \) elementwise and so \( L(G) \leq Z(G) \). Thus \( L(G) \) is trivial.

Now by determining the absolute centre of extraspecial \( p \)-groups, we can see for \( p \) odd, \( \text{Aut}(G) \) is trivial. Also for \( p = 2 \), \( \exp(L(G)) = 2 \) and by Theorem 1, \( \text{Aut}(G) \) is elementary abelian. Therefore in both cases, \( \text{Aut}(G) \) is elementary abelian.

**References**


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