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Finite p-groups in which each absolute central automorphism is elementary abelian

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Abstract In this paper we give necessary and sufficient conditions on a p-group G for each absolute central automorphism of G to be elementary abelian. Also we classify the absolute centre of extraspecial p-groups and show that the absolute central automorphisms of these groups are elementary abelian.

Keywords absolute centre; absolute central automorphisms; elementary abelian; finite p-groups; extraspecial p-groups.

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1 Introduction and basic lemmas

Throughout this paper p denotes a prime number. Let G be a group. We denote C_m, G', $Z(G)$, $L(G)$, $\exp(G)$, $\mathrm{Hom}(G, H)$, $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$, as the cyclic group of order m, the commutator subgroup, the centre, the absolute centre, the exponent, the group of homomorphisms of G into an abelian group H, the full automorphism group and the inner automorphism group of G, respectively. An automorphism α of G is called a *central automorphism* if and only if it induces the identity on $G/Z(G)$. The central automorphisms of G, denoted by $\mathrm{Aut}_c(G)$, fix G' elementwise and form a normal subgroup of the full automorphism group of G. The properties of $\mathrm{Aut}_c(G)$ are studied by many authors, see for instance [1], [2] and [3]. Hegarty in [4] generalized the concept of centre into absolute centre. The *absolute centre* of a group G, $L(G)$, is the subgroup consisting of all those elements that are fixed under all automorphisms of G. Also he introduced the concept of the absolute central automorphisms. An automorphism β of G is called an *absolute central* automorphism if and only if it acts trivially on the factor group $G/L(G)$, or equivalently, $x^{-1}\beta(x) \in L(G)$ for each $x \in G$. We denote the set of all absolute central automorphisms of G by $\mathrm{Aut}_l(G)$. Notice that $\mathrm{Aut}_l(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ contained in $\mathrm{Aut}_c(G)$. In 2006 Jafari [5] gave necessary and sufficient conditions on a finite purely nonabelian p-group G for the group $\mathrm{Aut}_c(G)$ to be elementary abelian. In this paper we give conditions on a p-group G such that $\mathrm{Aut}_l(G)$ to be elementary abelian.

The extraspecial p-groups are of great importance in the investigation of finite p-groups. Finally upon determining the absolute centre of extraspecial p-groups, we prove absolute central automorphisms of these groups are elementary abelian.

Here we give two basic lemmas that will be used in the proof of the results.

Lemma 1 [6, Lemma 2.1] *Let G be a finite nilpotent group of class 2. Then*
(i) $G' \leq Z(G)$,
(ii) $\exp(G') = \exp(G/Z(G))$.

Lemma 2 [6, Lemma 2.2] *Let A, C and U be abelian groups.*

(i) $\mathrm{Hom}(A, C \times U) \cong \mathrm{Hom}(A, C) \times \mathrm{Hom}(A, U)$,

(ii) $\mathrm{Hom}(A \times C, U) \cong \mathrm{Hom}(A, U) \times \mathrm{Hom}(C, U)$,

(iii) $\mathrm{Hom}(C_m, C_n) \cong C_d$, *where* $d = \gcd(m, n)$.

2 Absolute central automorphisms that are elementary abelian

In this section, we firstly state some definitions and elementary results which shall be used in the main results.

Definition 1 [4] Let G be a group. The subgroup $L(G)$, consisting of all elements of G fixed by every automorphism of G is called the *absolute centre* of G. Then

$$L(G) = \{g \in G \mid g^{-1}\alpha(g) = 1,\ \forall \alpha \in \mathrm{Aut}(G)\}.$$

Clearly, $L(G)$ is a central characteristic subgroup of G.

Definition 2 [4] An automorphism α of G is called *absolute central*, if $g^{-1}\alpha(g) \in L(G)$ for each $g \in G$. We denote the set of all absolute central automorphisms of G by $\mathrm{Aut}_l(G)$. $\mathrm{Aut}_l(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ contained in $\mathrm{Aut}_c(G)$.

Lemma 3 [6, Lemma 2.7] *Let G be a finite group. Then $G/L(G)$ is abelian if and only if* $\mathrm{Inn}(G) \leq \mathrm{Aut}_l(G)$.

Lemma 4 [7, Proposition 1] *Let G be a group. Then* $\mathrm{Aut}_l(G) \cong \mathrm{Hom}(G/L(G), L(G))$. *So if G is a p-group, then* $\mathrm{Aut}_l(G)$ *is also a p-group.*

As any homomorphism $f : G/L(G) \to L(G)$ induces a homorphism $\bar{f} : G/G'L(G) \to L(G)$, we see that $|\mathrm{Hom}(G/L(G), L(G))| = |\mathrm{Hom}(G/G'L(G), L(G))|$.

Now we give the following main results.

Lemma 5 *Let G be a p-group such that $G/L(G)$ is abelian. If $L(G)$ is elementary abelian of rank s, then* $\mathrm{Aut}_l(G)$ *is elementary abelian of order p^{ts} that* $t = \mathrm{rank}(G/L(G))$.

Proof It follows from lemmas 2 and 4. □

Lemma 6 *Let G be a non-abelian p-group such that $G/L(G)$ is abelian. If* $\mathrm{Aut}_l(G)$ *is elementary abelian, then* $\exp(G') = p$.

Proof Since $G/L(G)$ is abelian, hence, $\mathrm{Inn}(G) \leq \mathrm{Aut}_l(G)$. Thus $\mathrm{Inn}(G)$ is elementary abelian. Hence $\exp(G/Z(G)) = p$. Also G is nilpotent of class 2 and so $\exp(G') = \exp(G/Z(G)) = p$. □

In Theorem 1, we give necessary and sufficient conditions on p-group G that $\mathrm{Aut}_l(G)$ is elementary abelian. First, notice that for a finite group G and central subgroup M of G, if f be a homomorphism from G to M, we can define an endomorphism σ_f of G such that $\sigma_f(x) = xf(x)$. Also σ_f is an automorphism of G if and only if for every non-trivial element $g \in M$, $f(g) \neq g^{-1}$.

Recall that when $\mathrm{Aut}_l(G)$ is trivial, clearly it is elementary abelian. So from now on, we discuss on p-group G in which $\mathrm{Aut}_l(G)$ is non-trivial.

Theorem 1 *Let G be a finite p-group.* $\mathrm{Aut}_l(G)$ *is elementary abelian if and only if* $\exp(L(G)) = p$ *or* $\exp(G/G'L(G)) = p$.

Proof Suppose $\exp(L(G)) = p$ and $s = \text{rank}(L(G))$ and also $t = \text{rank}(G/G'L(G))$. By lemmas 2 and 4 we have

$$\begin{aligned}
\text{Aut}_l(G) &\cong \text{Hom}(\frac{G}{L(G)}, L(G)) \\
&\cong \text{Hom}(\frac{G}{G'L(G)}, L(G)) \\
&\cong \text{Hom}(C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_t}}, \underbrace{C_p \times C_p \times \cdots \times C_p}_{s-times}) \\
&\cong \underbrace{C_p \times C_p \times \cdots \times C_p.}_{st-times}
\end{aligned}$$

So $\text{Aut}_l(G)$ is elementary abelian. Similary, if $\exp(G/G'L(G)) = p$, the theorem is true.

Conversely, suppose $\text{Aut}_l(G)$ is an elementary abelian p-group with $\exp(L(G)) > p$ and $\exp(G/G'L(G)) > p$. Assume $\langle l \rangle$ and $\langle xG'L(G) \rangle$ are cyclic direct factors of $L(G)$ and $G/G'L(G)$ with maximum possible orders, respectively. Define

$$H = \{\sigma_f : f \in \text{Hom}(\langle xG'L(G) \rangle, \langle l \rangle)\}.$$

According to the previously-mentioned points, σ_f is an automorphism of G. Also for each $x \in G$, $x^{-1}\sigma_f(x) \in L(G)$ and so H is a subset of $\text{Aut}_l(G)$. Now since σ_f acts trivially on $L(G)$, by some calculations one can see that H is a cyclic subgroup of $\text{Aut}_l(G)$ of order at least p^2. This is a contradiction. □

We end this section with some examples of groups that satisfy the conditions of Theorem 1.

Example 1

(1) If G is the dihedral group of order 8, $\exp(L(G)) = 2$ and $\text{Aut}_l(G)$ is elementary abelian.
(2) $G = \langle a, b \mid a^{27} = 1, aba^{-1} = b^{-2}, a^{-1}b^{-4}a = b^2, [b, a] = b^3 \rangle$, a semidirect product of C_9 and C_{27}, is an example of a group satisfying the conditions of Theorem 1.

In the following, we give an account of p-groups whose absolute central automorphisms are elementary abelian.

3 The extraspecial p-groups

A finite p-group G is special if G is elementary abelian or $Z(G) = G' = \Phi(G)$ and a non-abelian special p-group G is extraspecial if $Z(G) = G' = \Phi(G) \cong C_p$. An extraspecial p-group G is the central product of $n \geq 1$ non-abelian subgroups of order p^3 and $|G| = p^{2n+1}$. For more information you can see [8].

There are two isomorphism classes of extraspecial p-groups. When $p = 2$, G is of exponent 4 and when p is odd, one of these isomorphism classes has exponent p and the other has exponent p^2. Any extraspecial p-group G has generators $x_1, x_2, \ldots, x_{2n}$ satisfying the following relations where z is a fixed generator of $Z(G)$.

1. $[x_{2i-1}, x_{2i}] = z, \qquad 1 \leq i \leq n$
2. $[x_i, x_j] = 1, \qquad 1 \leq i, j \leq n \quad |i - j| > 1$

3. $x_i^p \in Z(G), \qquad 1 \le i \le 2n.$

In 1972 Winter [9] gave the structure of the automorphism group of extraspecial p-groups. Here we give those results of Winter [9].

Theorem 2 [9, Theorem 1] *Let G be an extraspecial p-group of order p^{2n+1} and let H be a normal subgroup of $\mathrm{Aut}(G)$ which acts trivially on $Z(G)$. Then $\mathrm{Aut}(G) = \langle\theta\rangle H$ where θ has order $p-1$, $H \cap \langle\theta\rangle = 1$ and $H/\mathrm{Inn}(G)$ is isomorphic to a subgroup of symplectic group $Sp(2n, p)$.*

Here we define the automorphim θ in Theorem 2. Let m be a primitive root mod p with $0 < m < p$. The automorphim θ is defined by $\theta(x_{2i-1}) = x_{2i-1}^m$ and $\theta(x_{2i}) = x_{2i}$, for $1 \le i \le n$ and $\theta(z) = z^m$.

Recall for $p = 2$, $m = 1$ and if p is odd, then $|m| \ge 2$. Now in the theorem below we determine the absolute centre of extraspecial p-groups.

Theorem 3 *Let G be an extraspecial p-group.*
(I) *if $p = 2$ then $L(G) \cong C_2$.*
(II) *if p is odd then $L(G)$ is trivial.*

Proof Let G be an extraspecial p-group. Since $L(G) \le Z(G) \cong C_p$, then either $L(G) \cong C_p$ or $L(G) = \langle 1\rangle$.

(I) let $p = 2$. By Theorem 2, $\mathrm{Aut}(G) = \langle\theta\rangle H$ where H acts trivially on $Z(G)$. Also $m = 1$. Thus $\theta(z) = z$ and so each automorphism of G fixes the centre elementwise. Hence $L(G) = Z(G) \cong C_2$.

(II) let p be odd. We know $m \ne 1$ and $0 < m < p$. Hence $\theta(z) = z^m \ne z$. Therefore $\mathrm{Aut}(G)$ does not fix the centre of G elementwise and so $L(G) \lneq Z(G)$. Thus $L(G)$ is trivial. □

Now by determining the absolute centre of extraspecial p-groups, we can see for p odd, $\mathrm{Aut}_l(G)$ is trivial. Also for $p = 2$, $\exp(L(G)) = 2$ and by Theorem 1, $\mathrm{Aut}_l(G)$ is elementary abelian. Therefore in both cases, $\mathrm{Aut}_l(G)$ is elementary abelian.

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