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Fixed point method and its improvement for the system of Volterra-Fredholm integral equations of the second kind

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Abstract In this paper, we consider the system of Volterra-Fredholm integral equations of the second kind (SVFI-2). We proposed fixed point method (FPM) to solve SVFI-2 and improved fixed point method (IFPM) for solving the problem. In addition, a few theorems and two new algorithms are introduced. They are supported by numerical examples and simulations using Matlab. The results are reasonably good when compared with the exact solutions.

Keywords Fixed point method, improved fixed point method, contraction mapping,and system Volterra-Fredholm integral equations of the second kind.

AMS mathematics subject classification 45D05, 34A12

1 Introduction

Integral equations have been one of the principal instruments in many different fields of science like applied mathematics physics, biology and engineering [1, 2]. Also integral equations are encountered in numerous applications in various areas [3]. In addition, they arise naturally in applications, in many fields of mathematics, science and technology [4]. They have studied both at the theoretical and practicallevels [5].

Solution of integral equations by numerical methods have grown widely grown in the last 25 years. The methods of integral equation are strongly used for treating many problems in mathematical physics [6]. Integral equations have many advantages witnessed by the increasing frequency of the integral equations in the literature and in many areas because some problems have their mathematical representation appear directly [7]. The Volterra-Fredholm integral equations appear from parabolic boundary value problems [6]. Integral equation focused on the numerical method of solution [8].

Jerri [9] was discussed fixed point iteration method to solve linear Volterra integral equation. Sulaiman [10] used fixed point method (FPM) to solve linear Fredholm integral equation of the second kind. Hasan [11] solved a certain system of Fredholm integral equation of the first kind. In addition, Waz waz [12] solved a system of linear Volterra integral equations by FPM.

The proposed method FPM and IFPM are used for obtaining the approximate solution of system Volterra-Fredholm integral equations of the second kind. Illustrative examples will be included to demonstrate the validity and applicability of the presented techniques to highlight the signification of the FPM and improve fixed point method (IFPM).

2 Definitions

Definition 1 [9] *A function $f : R^p \rightarrow R^p$ is said to be Lipschtiz on $B \subseteq R^p$ with Lipschtiz constant $L > 0$ if*

$$\|f(x) - f(y)\| \leq L \|x - y\|,$$

for all $x, y \in B$ for some norm $\|\|$ on R^p.

Definition 2 [13] *A map T is called contraction mapping on the interval $[a, b]$ if it satisfy the following conditions:*
1. *$U \in C[a, b] \rightarrow T(U) \in C[a, b]$.*
2. *$T \in Lip[a, b]$, with Lipchitz constant, $0 < L < 1$.*

Definition 3 [12, 13] *Let V and W be vector spaces over a field F. A function $T : V \rightarrow W$ is called a linear transformation if it satisfies the following conditions:*
1. *$\forall A, B \in V, T(A + B) = T(A) + T(B)$.*
2. *$\forall A \in V$ and $r \in F$, $T(rA) = rT(A)$.*

Definition 4 [6, 14] *The integral equation*

$$y(s) = f(s) + \lambda \int_a^s k(s,t)y(t)dt + \lambda^* \int_a^b g(s,t)y(t)dt,$$

is called a linear Volterra-Fredholm integral equation of the second kind, where the functions $k(s, t)$ and $g(s, t)$ are called kernels of integral equation, $0 < \lambda < 1$ and $0 < \lambda^* < 1$ such that $f(s)$, $k(s, t)$ and $g(s, t)$ are known functions on $R = \{(s, t), a \leq t < s \leq b\}$ and $y(s)$ is unknown function.

Definition 5 [12, 15] *A kernel $k(s, t)$ is said to be a symmetric kernel if $k(s, t) = k(t, s)$ for all $s, t \in R$.*

3 System of linear Volterra-Fredholm integral equations of the second kind

The system of p Volterra-Fredholm integral equations of the second kind (SVFI-2) is given as follows:

$$y_i(s) = f_i(s) + \sum_{k=1}^{m} \lambda_{ik} \int_a^s k_{ik}(s,t)y_k(t)dt + \sum_{k=1}^{m} \lambda_{ik}^* \int_a^b k_{ik}^*(s,t)y_k(t)dt, \qquad (1)$$

for $i = 1, 2, 3, ..., p.f_i(s)$ is a continuous function on $[a, b]$, $k_{ik}(s, t)$ and $k^*_{ik}(s, t)$ denote the given continuous kernel functions on $R = \{(s, t), a \leq t < s \leq b\}$ while $y_i(s)$ is the unknown function to be determined.

4 FPM solution for SVFI-2 with symmetric kernels

The system of linear Volterra- Fredholm integral equation of the second kind given in (1), this system can be written as follows:

$$Y_i = F_i + \sum_{k=1}^{m} \lambda_{ik} K_{ik} Y_k + \sum_{k=1}^{m} \lambda_{ik}^* K_{ik}^* Y_k, \qquad (2)$$

Where $Y_i = y_i(s), F_i = f_i(s), K_{ik} = k_{ik}(s,t)$ and $K^*_{ik} = k^*_{ik}(s,t)$, first, we suppose that the initial solution $Y_i^0 = F_i$, then the first approximation is

$$Y_i^1 = F_i + \sum_{k=1}^{m} \lambda_{ik} K_{ik} Y_k^0 + \sum_{k=1}^{m} \lambda^*_{ik} K^*_{ik} Y_k^0, \tag{3}$$

For second approximation, substitute Y_i^1 into (3) we get

$$Y_i^2 = F_i + \sum_{k=1}^{m} \lambda_{ik} K_{ik} Y_k^1 + \sum_{k=1}^{m} \lambda^*_{ik} K^*_{ik} Y_k^1, \tag{4}$$

In general, the fixed-point method can be written as

$$Y_i^{r+1} = F_i + \sum_{k=1}^{m} \lambda_{ik} K_{ik} Y_k^r + \sum_{k=1}^{m} \lambda^*_{ik} K^*_{ik} Y_k^r, \text{ for } r = 1,2,3,...,\max \tag{5}$$

where max is the maximum number of iterations.

If the relationship defined in Equation (5) converges then

$$\lim_{n\to\infty} |Y_i - Y_i^r| = 0, \tag{6}$$

$$Y_i^{r+1} = M(Y_i^r), \tag{7}$$

with

$$M(Y_i^r) = F_i + \sum_{k=0}^{m} \lambda_{ik} K_{ik} Y_k^r + \sum_{k=0}^{m} \lambda^*_{ik} K^*_{ik} Y_k^r, \tag{8}$$

where $Y_k = y_k(s)$, $Y_i^r = y_k^r(s)$.

The condition for the convergence property of the FPM is

$$|\lambda_{ik}| < \frac{1}{M_{ik}},$$

where

$$M_{ik} = \sqrt{\int_a^b \int_a^b [k_{ik}(s,t)]^2 ds\, dt} \text{ and } |\lambda^*_{ik}| < \frac{1}{M^*_{ik}} \text{ where } M^*_{ik} = \sqrt{\int_a^b \int_a^b [k^*_{ik}(s,t)]^2 ds\, dt}.$$

Algorithm for FPM

Step 1 Let $Y_i^0 = F_i$, for $i = 1,2,3,...p$ from equation (1).
Step 2 Find Y_i^{r+1} from Equation (7), $r = 1,2,3,...,\max$.
Step 3 Compute the value of absolute error given by $e_i^r = \left|y_i(s) - y_i^r(s)\right|$.
Step 4 The approximate solutions converges if

$$\lim_{r\to\infty} |y_i(s) - y_i^r(s)| = \varepsilon,$$

where ε is a value close to 0.

Example 1 Consider the following system

$$y_1(s) = 2\sin(s) - \int_0^s y_2(t)dt,$$
$$y_2(s) = \cos(s) - 0.4597 + \int_0^1 y_1(t)dt,$$

with the exact solutions $y_1(s) = \sin(s)$ and $y_2(s) = \cos(s)$.

Solution. By applying FPM and its program, we obtain the approximate solutions of $y_1(s)$ and $y_2(s)$ as follows and shown in Table 1.

$$\begin{aligned}
y_1^0(s) &= 2\sin(s)\\
y_2^0(s) &= \cos(s) - 0.4597\\
y_1^1(s) &= \sin(s) + (0.4597)s\\
y_2^1(s) &= \cos(s) + 0.4596\\
y_1^2(s) &= \sin(s) - (0.4597)s\\
y_2^2(s) &= \cos(s) + 0.2298\\
y_1^3(s) &= \sin(s) - (0.2298)s\\
y_2^3(s) &= \cos(s) - 0.2298
\end{aligned}$$

Table 1: FPM results for example 1 and compared to the exact solution

s	r	Exact solution of $y_1(s) = \sin(s)$	Approximate values of $y_1(s)$	Absolute error $e_1^r = \lvert y_1(s) - {}^*y_1^r(s)\rvert$	Exact solution of $y_2(s) = \cos(s)$	Approximate values of $y_2(s)$	Absolute error $e_2^r = \lvert y_2(s) - {}^*y_2^r(s)\rvert$
	0	0.19866933	0.39733866	1.1986×10^{-1}	0.98006657	0.52036888	4.5969×10^{-1}
0.2	2		0.24463910	4.5969×10^{-2}		1.03752878	5.7462×10^{-2}
	4		0.19292310	5.7462×10^{-3}		0.97288380	7.1827×10^{-3}
	6		0.19938760	7.1827×10^{-4}		0.98096442	8.9784×10^{-4}
	8		0.19866812	2.1212×10^{-5}		0.98084223	2.2346×10^{-4}
	10		0.19866501	4.3245×10^{-6}		0.98009944	3.3134×10^{-5}
	12		0.19866998	6.2156×10^{-7}		0.98006933	3.2355×10^{-6}
	14		0.19866945	1.2678×10^{-7}		0.98006600	5.7762×10^{-7}
	16		0.19866939	6.7693×10^{-8}		0.98006679	2.2759×10^{-7}
	18		0.19866933	8.4678×10^{-9}		0.98006657	4.6793×10^{-8}
	20		0.19866933	2.4468×10^{-9}		0.98006657	5.6784×10^{-9}

Figures 1(a) and 1(b) show a comparison between the exact and the approximate solutions, given in Example 1.

Example 2 Consider the following system

$$y_1(s) = 1 + s + \frac{1}{2}\int_0^s y_2(t)dt - \int_0^1 (st)y_1(t)dt,$$
$$y_2(s) = 2 + s + \int_0^s y_1(t)dt - \int_0^1 (st)y_2(t)dt,$$

with the exact solutions $y_1(s) = e^s$and $y_2(s) = 2e^s$.

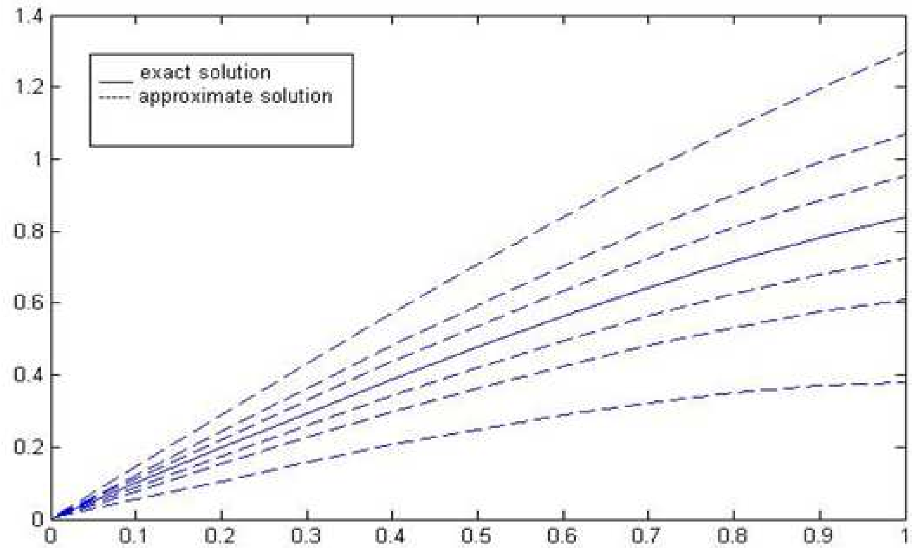


Figure 1(a): Graph $y_1(s) = \sin(s)$

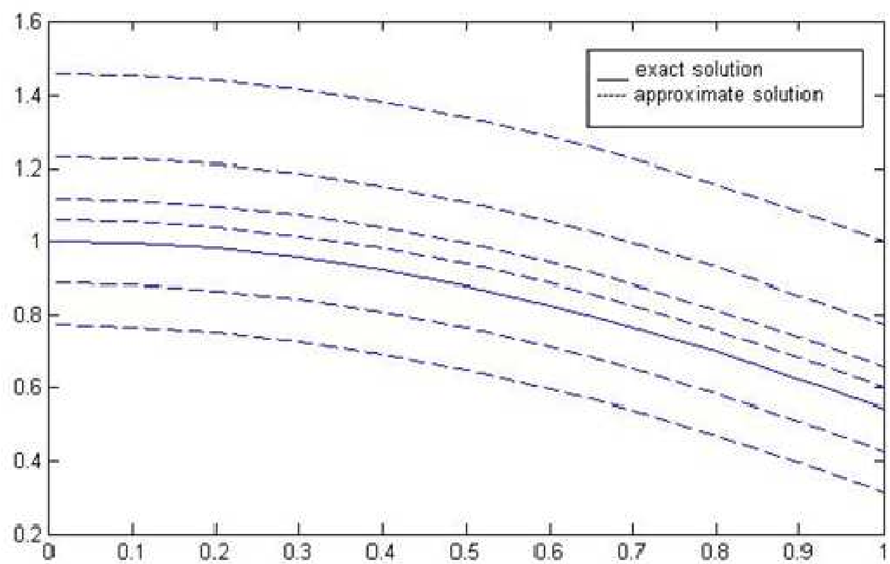


Figure 1(b): Graph $y_2(s) = \cos(s)$

Solution By applying FPM and its program, we obtain the approximate solutions of $y_1(s)$ and $y_2(s)$ as follows and shown in Table 2.

$$
\begin{aligned}
&y_1^0(s) = 1 + s \\
&y_2^0(s) = 2 + s \\
&y_1^1(s) = 1 + 1.1667s + 0.25s^2 \\
&y_2^1(s) = 2 + (2.1867)s + (0.5)s^2 \\
&y_1^2(s) = 1 + (1.2153)s + (0.5417)s^2 + (0.0833)s^3 \\
&y_2^2(s) = 2 + (2.0856)s + (2.1667)s^2 + (0.5)s^3 \\
&y_1^3(s) = 1 + (0.9984)s + (0.0120)s^2 + (0.3611)s^3 + (0.0625)s^4 \\
&y_2^3(s) = 2 + (1.9981)s + (1.0243)s^2 + (0.6667)s^3 + (0.1250)s^4
\end{aligned}
$$

Table 2: FPM results for example 2 and compared to the exact solution

s	r	Exact solution of $y_1(s) = e^s$	Approximate value of $y_1(s)$	Absolute error $e_1^r = \lvert y_1(s) - y_1^r(s)\rvert$	Exact solution of $y_1(s) = 2e^s$	Approximate value of $y_2(s)$	Absolute error $e_2^r = \lvert y_2(s) - y_2^r(s)\rvert$
	0	1.22140275	1.20000000	2.1402×10^{-2}	2.44280551	2.20000000	2.4280×10^{-1}
0.2	2		1.23205555	1.0652×10^{-2}		2.45438888	1.1583×10^{-2}
	4		1.22134014	6.2613×10^{-5}		2.44276323	4.2278×10^{-5}
	6		1.22140196	7.9443×10^{-6}		2.44280090	4.6111×10^{-6}
	8		1.22140331	5.6095×10^{-7}		2.44280607	5.5970×10^{-7}
	10		1.22140276	5.8513×10^{-8}		2.44280552	5.0802×10^{-9}
	12		1.22140275	1.5373×10^{-10}		2.44280551	6.2804×10^{-10}
	14		1.22140275	3.5835×10^{-11}		2.44280551	3.2524×10^{-11}
	16		1.22140275	8.5912×10^{-12}		2.44280551	7.7321×10^{-13}
	18		1.22140275	3.0213×10^{-13}		2.44280551	1.6325×10^{-14}
	20		1.22140275	3.4421×10^{-14}		2.44280551	2.5534×10^{-15}

Figures 2(a) and 2(b) show a comparison between the exact and the approximate solutions, given in Example 2.

5 IFPM to solve SVFI-2 with symmetric kernels

Improved fixed-point method is obtained by adding $\alpha_i Y_i$ in both sides of Equation (7) where $\alpha_i \neq -1$:

$$(1 + \alpha_i)Y_i = \alpha_i Y_i + M(Y_i) \tag{9}$$

we get

$$Y_i = \frac{1}{1 + \alpha_i} M(Y_i) + \frac{\alpha_i}{1 + \alpha_i} Y_i = M_{\alpha_i}(^*Y_i). \tag{10}$$

If $^*Y_i$ verifies Equation (7) then it also verifies Equation (10). This means $^*Y_i$ is a solution of Equation (5) expressed in the iterative form, as follows:

$$^*Y_i^{r+1} = M_{\alpha_i}(^*Y_i^r), \tag{11}$$

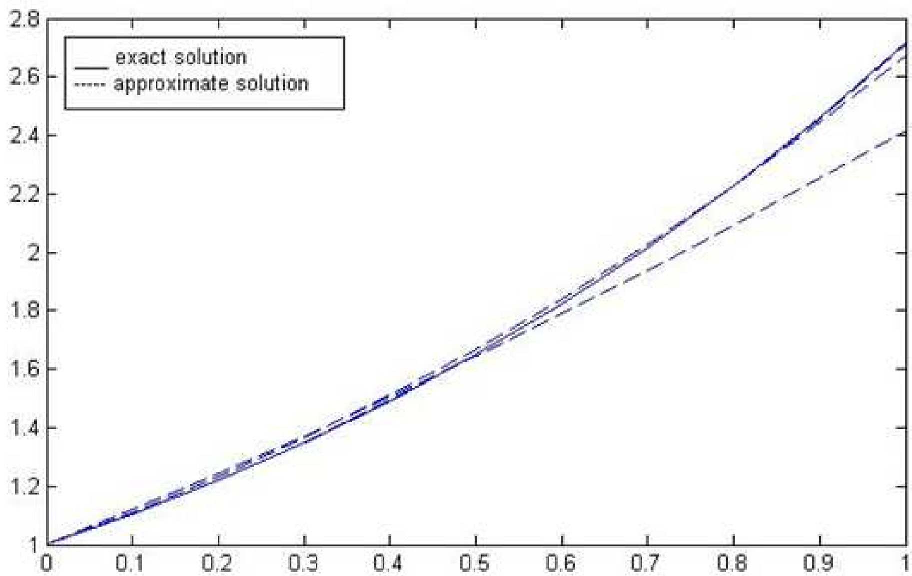


Figure 2(a): Graph $y_1(s) = e^s$

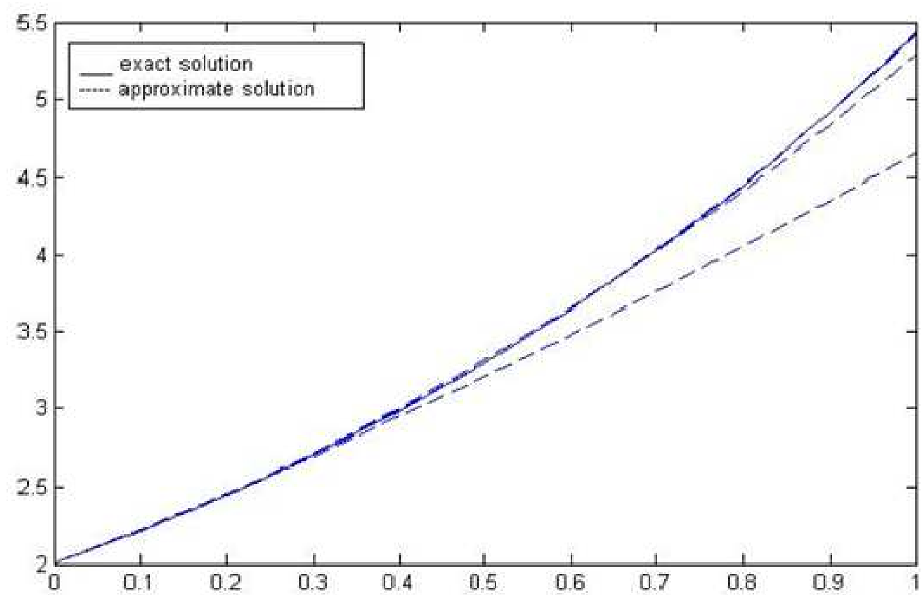


Figure 2(b): Graph $y_2(s) = 2e^s$

The approximate solutions in Equation (11) should converge faster than the iterative form defined in Equation (7) to the exact solution of Equation (1).

The selection for the optimal values of the scalar α_i is based on the following role

$$\alpha_i = \frac{(\nu_i + \rho_i)}{2} \ , \tag{12}$$

where

$$\nu_i = \sup_{[a,b]} \{M(Y_i) - M(Y_{i-1})\} \ \text{and} \ \rho_i = \inf_{[a,b]} \{M(Y_i) - M(Y_{i-1})\} \, . \tag{13}$$

Algorithm for IFPM

Step 1 Find the optimal values of the scalar α_i in Equations (12) and (13).
Step 2 Let $Y_i^0 = F_i$, for $i = 1, 2, 3, ..., p$, from Equation (1) which is the initial solution.
Step 3 Adding $\alpha_i Y_i$ to both sides of Equation (7).
Step 4 Compute ${}^*Y_i^{r+1}$ From Equation (11).
Step 5 Calculate $e_i^r = \left|y_i(s) - {}^*y_i^r(s)\right|$.
Step 6 The approximate solutions converges faster if $\lim\limits_{r\to\infty} |y_i(s) - y_i^r(s)| = \varepsilon$.

6 Numerical examples about SVFI-2 and results by using IFPM

The method of section 5 is very useful for finding the numerical solutions of SVFI-2. The computations associated with the examples were performed using MATLAB version 12.

Example 3 Find approximate solution ofa SVFI-2, in example 1 by using IFPM.

Solution Applying the numerical technique which is IFPM, we obtained the results for approximate solutions of $y_1(s)$ and $y_2(s)$, where the optimal values are $\alpha_1 = -0.190$ and $\alpha_2 = 1.4597$, as shown in Table 3.

$$\begin{aligned}
y_1^0(s) &= 2\sin(s) \\
y_2^0(s) &= \cos(s) - 0.4597 \\
y_1^1(s) &= (0.7654)\sin(s) + (0.5675)s \\
y_2^1(s) &= \cos(s) - 0.0859 \\
y_1^2(s) &= (1.0550)\sin(s) - (0.0271)s \\
y_2^2(s) &= \cos(s) + 0.0205 \\
y_1^3(s) &= (0.9871)\sin(s) - (0.0190)s \\
y_2^3(s) &= \cos(s) + 0.0170
\end{aligned}$$

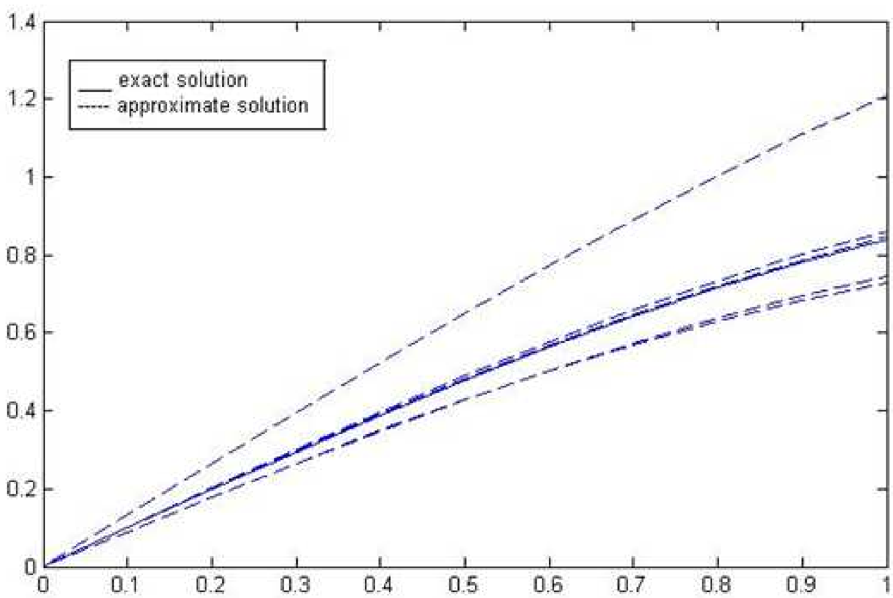
Figures 3(a) and 3(b) show a comparison between the exact and the approximate solutions, given in Example 3.

Example 4 Find approximate solution ofa SVFI-2 in example 2 by using IFPM.

Solution Applying the numerical technique which is IFPM we obtained the results for approximate solutions of $y_1(s)$ and $y_2(s)$, where the optimal values are $\alpha_1 = 0.128$and

Table 3: IFPM results for example 3 and compared to the exact solution

s	r	Exact solution of $y_1(s)=\sin(s)$	Approximate Values of $y_1(s)$	Absolute error $e_1^r = \lvert y_1(s) - {}^*y_1^r(s)\rvert$	Exact solution of $y_2(s)=\cos(s)$	Approximate Values of $y_2(s)$	Absolute error $e_2^r = \lvert y_2(s) - {}^*y_2^r(s)\rvert$
	0	0.19866933	0.39733866	1.9866×10^{-1}	0.98006657	0.52036888	4.5969×10^{-1}
0.2	2		0.20418913	5.5198×10^{-3}		1.00060762	2.0541×10^{-2}
	4		0.19597184	2.6974×10^{-3}		0.98386228	3.7957×10^{-3}
	6		0.19887275	2.0342×10^{-4}		0.97945051	6.1606×10^{-4}
	8		0.19868480	1.3549×10^{-5}		0.98007752	1.0946×10^{-5}
	10		0.19866538	3.9434×10^{-6}		0.98007323	6.6574×10^{-6}
	12		0.19866951	1.8462×10^{-7}		0.98006581	7.6728×10^{-7}
	14		0.19866936	1.0472×10^{-8}		0.98006656	1.0471×10^{-8}
	16		0.19866932	5.3205×10^{-9}		0.98006658	1.0574×10^{-9}
	18		0.19866933	1.0570×10^{-10}		0.98006657	8.7082×10^{-10}
	20		0.19866933	5.6451×10^{-11}		0.98006657	4.9588×10^{-11}



Figigure 3(a): Graph $y_1(s) = \sin(s)$

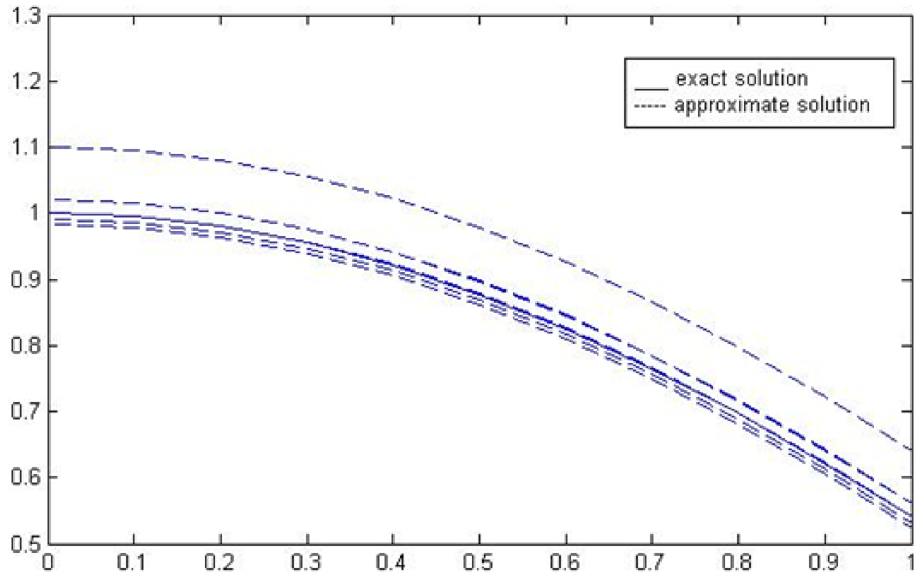


Figure 3(b): Graph $y_2(s) = \cos(s)$

$\alpha_2 = 0.315$, as shown in Table 4.

$$
\begin{aligned}
y_1^0(s) &= 1 + s \\
y_2^0(s) &= 2 + s \\
y_1^1(s) &= 1 + (1.1478)s + (0.2216)s^2 \\
y_2^1(s) &= 12 + (1.8872)s + (0.3802)s^2 \\
y_1^2(s) &= 1 + (1.0717)s + (0.4434)s^2 + (0.0562)s^3 \\
y_2^2(s) &= 2 + (2.0201)s + (0.8086)s^2 + (0.0964)s^3 \\
y_1^3(s) &= 1 + (1.0265)s + (0.4980)s^2 + (0.1259)s^3 + (0.0107)s^4 \\
y_2^3(s) &= 2 + (2.0204)s + (0.9618)s^2 + (0.2281)s^3 + (0.0183)s^4
\end{aligned}
$$

Figures 4(a) and 4(b) show a comparison between the exact and the approximate solutions, given in Example 4.

7 Fixed point and contractive mapping

The original iterative method is proposed as follows:

$$Y_i^{r+1}(s) = F_i(s) + \sum_{k=0}^{m} \lambda K_{ik} Y_i^r + \sum_{k=0}^{m} \lambda^* K_{ik}^* Y_i^r, \text{ for } r = 0, 1, 2, ..., \max \tag{14}$$

Table 4: IFPM results for example 4 and results compared to the exact solution

s	r	Exact solution of $y_1(s) = e^s$	Approximate Value of $y_1(s)$	Absolute error $e_1^r = \lvert y_1(s) - {}^*y_1^r(s)\rvert$	Exact solution of $y_1(s) = 2e^s$	Approximate Value of $y_2(s)$	Absolute error $e_2^r = \lvert y_2(s) - {}^*y_2^r(s)\rvert$
	0	1.22140275	1.20000000	2.1402×10^{-2}	2.44280551	2.20000000	2.4280×10^{-2}
0.2	2		1.22163981	2.3705×10^{-4}		2.44360370	7.9818×10^{-3}
	4		1.22132818	7.4575×10^{-5}		2.44272708	7.8007×10^{-5}
	6		1.22140268	7.7778×10^{-7}		2.44280553	1.7745×10^{-7}
	8		1.22140275	4.8578×10^{-9}		2.44280551	2.3335×10^{-8}
	10		1.22140275	4.3890×10^{-10}		2.44280551	4.1740×10^{-9}
	12		1.22140275	7.2264×10^{-11}		2.44280551	6.8422×10^{-11}
	14		1.22140272	2.3652×10^{-12}		2.44280551	6.2924×10^{-13}
	16		1.22140275	3.0622×10^{-13}		2.44280551	3.0642×10^{-14}
	18		1.22140225	9.2433×10^{-15}		2.44280551	8.2420×10^{-15}
	20		1.22140275	1.6952×10^{-17}		2.44280551	1.8732×10^{-17}

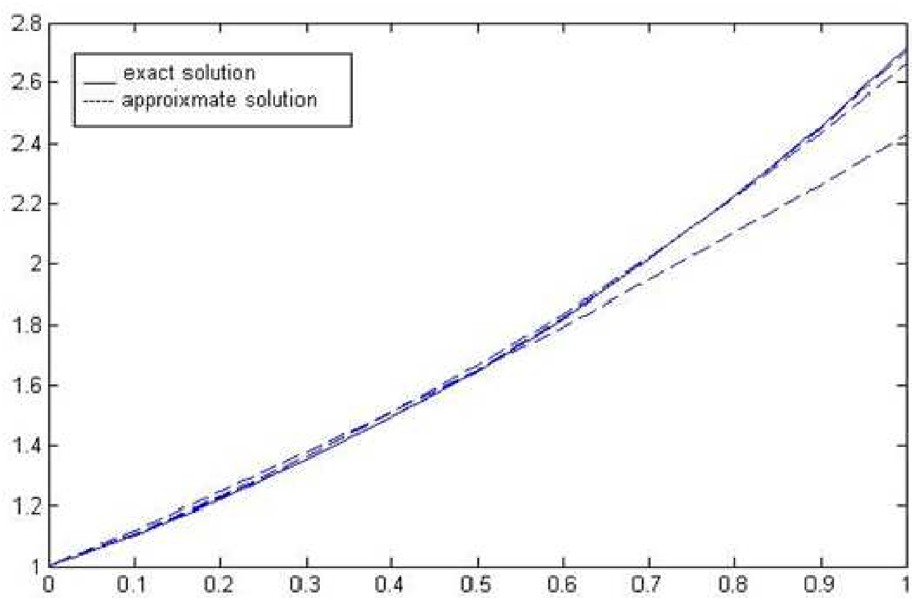


Figure 4(a): Graph $y_1(s) = e^s$

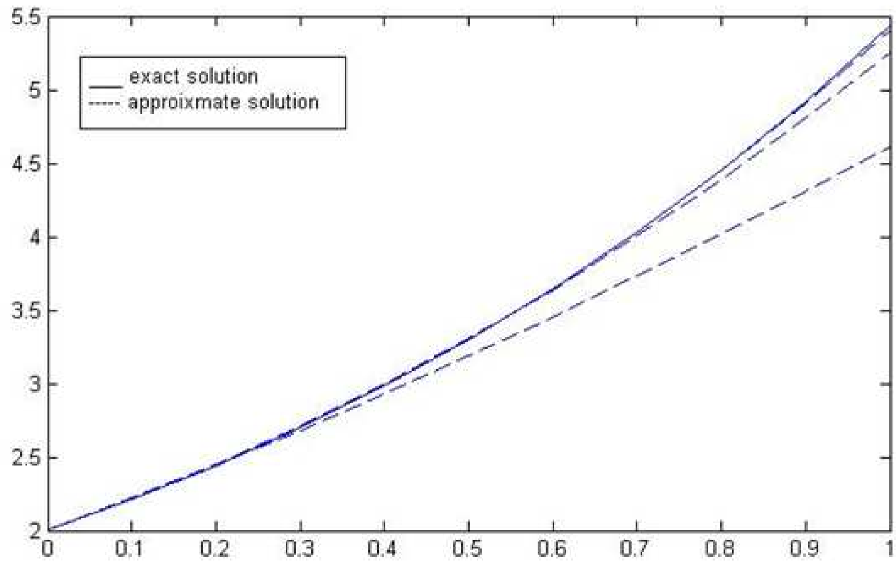


Figure 4(b): Graph $y_2(s) = 2e^s$

The approach of the sequences $Y_i^{r+1}(s)$ to $Y_i(s)$ has been studied for solving Equation (1). Let

$$T(Y_i^r) = F_i(s) + \sum_{k=0}^{m} \lambda K_{ik} Y_i^r + \sum_{k=0}^{m} \lambda^* K_{ik}^* Y_i^r. \tag{15}$$

Our solution in this work has been directed toward solving Equation (1). The integral equation (15) is searched in the following manner, the right hand side is supposed as a transformation T on Y_i^rexpressed by $T(Y_i^r)$, when the left hand side denotes that the transformation had left this element Y_i^{r+1} unchanged. That is

$$Y_i^{r+1} = T(Y_i^r), \tag{16}$$

which mean that the solution Y_i^r which we search for the integral equation (15) expresses an especial values in the domain of the transformation T, called that which remains unaltered of stable under the transformation T. Each element Y_i^r as defined in equation (16) is known a fixed point of the transformation T.

Lemma 1 [12] *If a map T satisfies the Lipchitz condition on the interval $[a, b]$ then there exists a positive constant L, such that $|T(y) - T(w)| \leq |y - w| \, L$, for all values $y, w \in R$. The constant L is called Lipchitz constant.*

Theorem 2 [3] *Every contraction mapping is a Lipschitz function on $[a, b]$.*

Theorem 3 [3] *Every Lipschitz function is a continuous function on $[a, b]$.*

Theorem 4 *Let T be a contraction mapping on R then:*

1. *The relation $Y^{r+1} = T(Y^r)$ has a unique exact solution $Y^* \in R$.*

2. *For any initial solution $Y^0 \in R$ the sequence defined in Equation (16)converges to Y^*.*

Proof

1. Suppose that T is a contraction mapping on R. Then T is Lipchitz function by Theorem 2 and T is continuous function by Theorem 3. Since Y is exact solution, then
$$Y^* = T(Y^*) \tag{17}$$
is continuous on the region R. Now we want to show that, the equation (17) has a unique solution.
By contradiction, suppose that there exists another solution Z^*,such that $Z^* \neq Y^*$ which satisfy
$$Y^* = T(Y^*) \text{ and } Z^* = T(Z^*)$$
Now
$$|Y^* - Z^*| = |T(Y^*) - T(Z^*)| \leq |Y^* - Z^*| \, L,$$
because T is Lipchitz function. Then we get
$$|Y^* - Z^*| \leq |Y^* - Z^*| \, L,$$
Hence $L \geq 1$ which is a contradiction to the definition of the Lipchitz constant, because $L < 1$. Since T is contraction mapping on the region R, then $Z^* = Y^*$. Therefore, Equation (16) has a unique solution.
2. By part one of this theorem we give the closure condition. That's mean If $Y^0 \in R$ then $Y^d \in R$. Now$\left|Y^d - Y^*\right| \leq \left|T(Y^d) - T(Y^*)\right| \leq \left|Y^{d-1} - Y^*\right| L$, also $\left|Y^{d-1} - Y^*\right| \leq \left|T(Y^{d-1}) - T(Y^*)\right| \leq \left|Y^{d-2} - Y^*\right| L^2$, and so on, after d-times we get
$$\left|Y^d - Y^*\right| \leq \left|Y^{d-(d-1)} - Y^*\right| L^{d-1} \leq \left|Y^{d-(d-1)-1} - Y^*\right| L^d \leq \left|Y^0 - Y^*\right| L^d,$$
Take the limit of both sides we get
$$\lim_{d\to\infty} \left|Y^d - Y^*\right| \leq \lim_{d\to\infty} \left|Y^0 - Y^*\right| L^d .$$
Since $0 < L < 1$, if $d \to \infty$ then $L^d \to 0$. That means $Y^d \to Y^*$ for $d \to \infty$. Hence, the sequence defined in equation (16) converges to Y^*. □

8 Conclusion

In this work, we propose two methods called fixed point method and improve fixed point method to solve SVFI-2. Several numerical examples were tested using algorithms for FPM and IFPM for solving a system SVFI-2. The results given in Tables 1, 2, 3 and 4, indicate clearly that both methods achieve good convergence as r increases when the error decreases. The mentioned examples demonstrated the validity and applicability of the techniques. Finally, we concluded that IFPM converges faster than FPM as shown in Tables 5 and 6.

Table 5: Comparison between the absolute error of the present methods, FPM and IFPM methods in Example 1

s	r	Absolute errors of $y_1(s)$ by FPM	Absolute errors of $y_1(s)$ by IFPM	Absolute errors of $y_2(s)$ by FPM	Absolute errors of $y_2(s)$ by IFPM	Running Time second
	0	1.1986×10^{-1}	1.9866×10^{-1}	4.5969×10^{-1}	4.5969×10^{-1}	0.019985
0.2	2	4.5969×10^{-2}	5.5198×10^{-3}	5.7462×10^{-2}	2.0541×10^{-2}	0.083763
	4	5.7462×10^{-3}	2.6974×10^{-3}	7.1827×10^{-3}	3.7957×10^{-3}	0.165676
	6	7.1827×10^{-4}	2.0342×10^{-4}	8.9784×10^{-4}	6.1606×10^{-4}	0.331369
	8	2.1212×10^{-5}	1.3549×10^{-5}	2.2346×10^{-4}	1.0946×10^{-5}	0.388837
	10	4.3245×10^{-6}	3.9434×10^{-6}	3.3134×10^{-5}	6.6574×10^{-6}	0.477969
	12	6.2156×10^{-7}	1.8462×10^{-7}	3.2355×10^{-6}	7.6728×10^{-7}	0.585971
	14	1.2678×10^{-7}	1.0472×10^{-8}	5.7762×10^{-7}	1.0471×10^{-8}	0.655434
	16	6.7693×10^{-8}	5.3205×10^{-9}	2.2759×10^{-7}	1.0574×10^{-9}	0.754577
	18	8.4678×10^{-9}	1.0570×10^{-10}	4.6793×10^{-8}	8.7082×10^{-10}	0.848303
	20	2.4468×10^{-9}	5.6451×10^{-11}	5.6784×10^{-9}	4.9588×10^{-11}	0.961526

Table 6: Comparison between the absolute error of the present methods, FPM and IFPM methods in Example 2

s	r	Absolute errors of $y_1(s)$ by FPM	Absolute errors of $y_1(s)$ by IFPM	Absolute errors of $y_2(s)$ by FPM	Absolute errors of $y_2(s)$ by IFPM	Running Time by second
	0	2.1402×10^{-2}	2.1402×10^{-2}	2.4280×10^{-1}	2.4280×10^{-2}	0.017213
0.2	2	1.0652×10^{-2}	2.3705×10^{-4}	1.1583×10^{-2}	7.9818×10^{-3}	0.097263
	4	6.2613×10^{-5}	7.4575×10^{-5}	4.2278×10^{-5}	7.8007×10^{-5}	0.203957
	6	7.9443×10^{-6}	7.7778×10^{-7}	4.6111×10^{-6}	1.7745×10^{-7}	0.347614
	8	5.6095×10^{-7}	4.8578×10^{-9}	5.5970×10^{-7}	2.3335×10^{-8}	0.513815
	10	5.8513×10^{-8}	4.3890×10^{-10}	5.0802×10^{-9}	4.1740×10^{-9}	0.722167
	12	1.5373×10^{-10}	7.2264×10^{-11}	6.2804×10^{-10}	6.8422×10^{-11}	0.964843
	14	3.5835×10^{-11}	2.3652×10^{-12}	3.2524×10^{-11}	6.2924×10^{-13}	1.230646
	16	8.5912×10^{-12}	3.0622×10^{-13}	7.7321×10^{-13}	3.0642×10^{-14}	1.547392
	18	3.0213×10^{-14}	9.2433×10^{-15}	1.6325×10^{-14}	8.2420×10^{-15}	1.877719
	20	3.4421×10^{-15}	1.6952×10^{-17}	2.5534×10^{-15}	1.8732×10^{-17}	2.237260

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