

Linear program fixed-point representation

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Abstract From a linear program and its asymmetric dual, invariant, primal and dual problems are constructed. Regular mappings are defined between the solution spaces of the original and invariant problems. The notion of centrality is introduced and subsets of regular mappings are shown to be inversely related surjections of central elements, thus representing the original problems as invariant problems. A fixed-point problem involving an idempotent symmetric matrix is constructed from the invariant problems and the notion of centrality carried over to it; the non-negative central fixed-points are shown to map one-to-one to the central solutions to the invariant problems, thus representing the invariant problems as a fixed-point problem and, by transitivity, the original problems as a fixed-point problem.

Keywords optimization; representation; linear programming; fixed-point; non-negative

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1 Introduction

The problem this paper addresses is to maximize $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} \geq \mathbf{b}$, where A is an $m \times n$ matrix, \mathbf{b} is an m -dimensional vector, \mathbf{c} is an n -dimensional vector, and \mathbf{x} is an n -dimensional vector; The problem is written as

$$\max\{\mathbf{c}^T \mathbf{x} : A\mathbf{x} \geq \mathbf{b}\}, \quad (1)$$

and is called the original primal linear program (LP), or simply the original primal. The aim of the paper is to represent Problem 1 as a fixed-point problem to make it more amenable to solution.

The LP and its asymmetric dual are represented as a fixed-point problem of the form $\mathfrak{P}\boldsymbol{\omega} = \boldsymbol{\omega} \geq \mathbf{0}$, where \mathfrak{P} is an idempotent symmetric matrix. Preliminary notions of function pairs, regular pairs, central elements and invariance are introduced. Next the original linear program and its asymmetric dual problem are introduced and their invariant forms are constructed. Then primal and dual function pairs are introduced which map between and establish the equivalence of the original and invariant problems as well as allowing central forms of all the problems to be defined. Finally the notion of centrality is carried over to the fixed-point problem which is then shown to be equivalent to solving the central forms of the invariant problems, thus completing the representation.

The representation is related to that of Pyle [1], [2], Cline [3], Nguyen [4], Bruni *et al.* [5] where successive increases in simplicity and precision have occurred, particularly with the later two authors; the aim of this paper is to continue this trend.

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2 Preliminaries

2.1 The Moore-Penrose pseudo-inverse

The development requires the Moore-Penrose pseudo-inverse: if X is a matrix then there exists a unique pseudo-inverse X^+ satisfying the four non-constructive conditions:

$$\begin{aligned}
 XX^+X &= X, & (i) \\
 X^+XX^+ &= X^+, & (ii) \\
 (XX^+)^T &= XX^+ \text{ and} & (iii) \\
 (X^+X)^T &= X^+X. & (iv)
 \end{aligned} \tag{2}$$

Refer to Albert [6] for a detailed exposition.

2.2 Function pairs

Two functions are said to be a *function pair* if the domain of each is the codomain of the other.

2.2.1 Regular pairs

Motivated by semigroup nomenclature [7], a function pair, say $\{t, u\}$, is called a *regular function pair* if $tut = t$ and $utu = u$.

2.2.2 Central elements

Given a function pair $\{t, u\}$, with $t : V \rightarrow W$ and $u : W \rightarrow V$, the sets $V_c = Wu$ and $W_c = Vt$ are called *central*; elements in central sets are called central, that is

Definition 1 *An element is central in the domain of one function of a function pair if the element is in the range of the other function of the pair.*

Functions of the form $t_c = t|_{V_c}$ with codomain Vt , and $u_c = u|_{W_c}$ with codomain Wu are called *central functions*, and function pairs of the form $\{t_c, u_c\}$ are called *central function pairs*.

Lemma 1 *Given a regular function pair with $t : V \rightarrow W$ and $u : W \rightarrow V$,*

- (a) t_c is bijective,
- (b) u_c is bijective,
- (c) $V_c = W_c u; Wu = Vt u$
- (d) $W_c = V_c t, Vt = Wut$ and
- (e) t_c and u_c are mutual inverses.

Proof

- (a) t_c is 1:1: $(v_1, v_2 \in Wu) \wedge (v_1 t = v_2 t) \Rightarrow (v_1 = w_1 u) \wedge (v_2 = w_2 u) \wedge (v_1 t = v_2 t) \Rightarrow (v_1 = w_1 u) \wedge (v_2 = w_2 u) \wedge (w_1 u t = w_2 u t) \Rightarrow (v_1 = w_1 u) \wedge (v_2 = w_2 u) \wedge (w_1 u t u = w_2 u t u) \Rightarrow (v_1 = w_1 u) \wedge (v_2 = w_2 u) \wedge (w_1 u = w_2 u) \Rightarrow v_1 = v_2$, further t_c is onto W_c : $\text{image}(t_c) = V_c t_c = V_c t \subseteq Vt = W_c = Vt = Vt u t \subseteq Wut = V_c t = V_c t_c = \text{image}(t_c)$, that is $\text{image}(t_c) = W_c$, so t_c is bijective.

- (b) The proof is similar to (a).
- (c) $V_c = Wu = Wutu \subseteq Vtu = W_c u \subseteq Wu = V_c$, so $V_c = W_c u$; $Wu = Wutu \subseteq Vtu \subseteq Wu$, so $Wu = Vtu$.
- (d) the proof is similar to (c).
- (e) For $v \in V_c, v = wu \exists w \in W \Rightarrow vt_c u_c = wut_c u_c = wutu_c = wutu = wu = v$, so $t_c u_c$ is the identity map on V_c , similarly it is found that $u_c t_c$ is the identity map on W_c , so t_c and u_c are mutual inverses. \square

Lemma 2 Given the regular function pair $\{f, \mathfrak{f}\}$ with $f : X \rightarrow \mathfrak{X}$ and $\mathfrak{f} : \mathfrak{X} \rightarrow X$,

- (a) \mathbf{x} is central iff $\mathbf{x} = \mathbf{x}f\mathfrak{f}$.
- (b) \mathfrak{r} is central iff $\mathfrak{r} = \mathfrak{r}\mathfrak{f}f$.

Proof

- (a) \mathbf{x} is central $\Leftrightarrow \mathbf{x} = \mathfrak{r}\mathfrak{f} \exists \mathfrak{r} \in \mathfrak{X} \Rightarrow (\mathbf{x}f\mathfrak{f} = \mathfrak{r}\mathfrak{f}\mathfrak{f}) \wedge (\mathbf{x} = \mathfrak{r}\mathfrak{f}) \exists \mathfrak{r} \in \mathfrak{X} \Rightarrow (\mathbf{x}f\mathfrak{f} = \mathfrak{r}\mathfrak{f}) \wedge (\mathbf{x} = \mathfrak{r}\mathfrak{f}) \exists \mathfrak{r} \in \mathfrak{X} \Rightarrow \mathbf{x} = \mathbf{x}f\mathfrak{f} \Rightarrow \mathbf{x} = \mathfrak{r}\mathfrak{f}$ where $\mathfrak{r} = \mathbf{x}f \Leftrightarrow \mathbf{x}$ is central, that is \mathbf{x} is central iff $\mathbf{x} = \mathbf{x}f\mathfrak{f}$.
- (b) The proof is exactly similar to that of (a). \square

2.3 The invariant framework

Quantities which are invariant under affine transformation of the solution space of Problem 1 are introduced here. These quantities are used to construct the invariant problems of Section 4.

Define

$$\begin{aligned}
 \mathfrak{A} &= AA^+, & (a) \\
 \mathfrak{b} &= (I - AA^+)\mathfrak{b}, & (b) \\
 \mathfrak{c} &= A^{T+}\mathfrak{c}, & (c) \\
 \mathfrak{D} &= I - AA^+. & (d)
 \end{aligned} \tag{3}$$

Note that \mathfrak{A} and \mathfrak{D} are symmetric idempotents in view of the latter two non-constructive conditions for the pseudo-inverse. It is a straightforward matter, using the definitions above and the nonconstructive characterization of the Moore-Penrose pseudo-inverse, to show that

$$\begin{array}{llll}
 \mathfrak{A}^T & = & \mathfrak{A} & (a) \\
 \mathfrak{A}^2 & = & \mathfrak{A} & (b) \\
 \mathfrak{D}^T & = & \mathfrak{D} & (c) \\
 \mathfrak{D}^2 & = & \mathfrak{D} & (d) \\
 \mathfrak{A}\mathfrak{D} = \mathfrak{D}\mathfrak{A} & = & 0 & (e)
 \end{array} \left| \begin{array}{ll}
 \mathfrak{A}\mathfrak{b} & = & \mathbf{0} & (f) \\
 \mathfrak{D}\mathfrak{b} & = & \mathfrak{b} & (g) \\
 \mathfrak{A}\mathfrak{c} & = & \mathfrak{c} & (h) \\
 \mathfrak{D}\mathfrak{c} & = & \mathbf{0} & (i) \\
 \mathfrak{b}^T \mathfrak{c} & = & 0 & (j)
 \end{array} \right. \tag{4}$$

The results listed in (4) and a few obvious consequences are used throughout without necessarily referencing them.

As an intermediate step in representing the LP problem as a fixed-point problem, an *invariant primal* and an *invariant dual* are constructed, the term *invariant* being used as these problems can be shown to be invariant under affine transformation of the solution space of Problem 1. For details refer to [8] and [9].

2.4 Quasi-boundedness

If Problem 1 is feasible there exists a solution, say \mathbf{x}_f such that $A\mathbf{x}_f \geq \mathbf{b}$. Now $A\mathbf{x}_f \geq \mathbf{b} \Rightarrow A(x_f + \lambda(I - A^+A)\mathbf{c}) \geq \mathbf{b}$, so $\mathbf{x}_f + \lambda(I - A^+A)\mathbf{c}$ is also feasible and its objective value is $\mathbf{c}^T(x_f + \lambda(I - A^+A)\mathbf{c}) = \mathbf{c}^T x_f + \mathbf{c}^T \lambda(I - A^+A)\mathbf{c} = \mathbf{c}^T x_f + \lambda \mathbf{c}^T (I - A^+A)\mathbf{c} = \mathbf{c}^T x_f + \lambda \|(I - A^+A)\mathbf{c}\|^2$, which is unbounded unless $\|(I - A^+A)\mathbf{c}\| = 0$, that is unless $A^+A\mathbf{c} = \mathbf{c}$, so the following definition is introduced:

Definition 2 Problem 1 is said to be quasi-bounded if $A^+A\mathbf{c} = \mathbf{c}$.

Lemma 3 Problem 1 is feasible bounded \Rightarrow Problem 1 is quasi-bounded.

Note that usually A is of full rank and $m \geq n$, so $A^+A = I$, which implies $A^+A\mathbf{c} = \mathbf{c}$, and so such problem is quasi-bounded.

Lemma 4 Problem 1 is quasi-bounded $\Leftrightarrow A^+A\mathbf{c} = \mathbf{c} \Leftrightarrow A^T\mathbf{c} = \mathbf{c}$.

Proof Problem 1 is quasi-bounded $\Rightarrow A^T\mathbf{c} \stackrel{(3c)}{=} A^T(A^{T+}\mathbf{c}) = (A^T A^{T+})\mathbf{c}$
 $= (A^T A^{T+})\mathbf{c} = (A^+A)^T\mathbf{c} = A^+A\mathbf{c} \stackrel{(\text{Definition 2})}{=} \mathbf{c} \Rightarrow A^T\mathbf{c} = \mathbf{c} \Rightarrow (A^T\mathbf{c} = \mathbf{c}) \wedge (A^+A\mathbf{c} = \mathbf{c}) \Rightarrow (A^T\mathbf{c} = \mathbf{c}) \wedge (A^T\mathbf{c} = A^+A\mathbf{c}) \Rightarrow A^+A\mathbf{c} = \mathbf{c} \stackrel{(\text{Definition 2})}{\Rightarrow}$ Problem 1 is quasi-bounded, so Problem 1 is quasi-bounded $\Leftrightarrow A^T\mathbf{c} = \mathbf{c}$. \square

3 The original problems

3.1 The primal

The form of the LP used is the *original primal* given by (1), that is $\max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \geq \mathbf{b}\}$.

A *feasible vector* is one which satisfies the constraint conditions for the linear program, for example for Problem 1, \mathbf{x} is feasible if $A\mathbf{x} \geq \mathbf{b}$. By feasible bounded is meant the situation where the set of objective values of all feasible solutions is bounded above if the problem is one of maximization, and below if the problem is one of minimization; an “optimal” vector means a feasible vector whose objective value is equal to the maximum for the set of feasible solutions. A feasible linear program is one which has a feasible vector; a *feasible bounded* linear program is one with a solution.

3.2 The asymmetric dual

An asymmetric dual is given by Schrijver [10, p. 95]:

$$\max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\} = \min\{\mathbf{b}^T\mathbf{y} : A^T\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0}\}, \quad \text{if either exists,} \quad (5)$$

From this form

$$\max\{(-\mathbf{c})^T(-\mathbf{x}) : A(-\mathbf{x}) \leq -\mathbf{b}\} = \min\{(-\mathbf{b})^T(-\mathbf{y}) : A^T(-\mathbf{y}) = (-\mathbf{c}), -\mathbf{y} \geq \mathbf{0}\}$$

which implies

$$\max\{\mathbf{c}^T \mathbf{x} : A\mathbf{x} \geq \mathbf{b}\} = \min\{\mathbf{b}^T \mathbf{y} : A^T \mathbf{y} = \mathbf{c}, \mathbf{y} \leq \mathbf{0}\}, \quad (6)$$

We call

$$\min\{\mathbf{b}^T \mathbf{y} : A^T \mathbf{y} = \mathbf{c}, \mathbf{y} \leq \mathbf{0}\} \quad (7)$$

the *original dual*.

4 The invariant problems

In this section the invariant problems are introduced and it is shown how they relate to each other; in the following section these problems are related to the original primal and dual.

Substituting \mathfrak{A} for A , \mathbf{b} for \mathbf{b} and \mathbf{c} for \mathbf{c} in (6),

$$\max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}\mathbf{x} \geq \mathbf{b}\} = \min\{\mathbf{b}^T \mathbf{y} : \mathfrak{A}\mathbf{y} = \mathbf{c}, \mathbf{y} \leq \mathbf{0}\}, \quad \text{if either exists.} \quad (8)$$

Now $\min\{\mathbf{b}^T \mathbf{y} : \mathfrak{A}\mathbf{y} = \mathbf{c}, \mathbf{y} \leq \mathbf{0}\}$

$$\begin{aligned} &= \min\{\mathbf{b}^T \mathbf{y} : (I - \mathfrak{D})\mathbf{y} = \mathbf{c}, \mathbf{y} \leq \mathbf{0}\} \\ &= \min\{\mathbf{b}^T \mathbf{y} : \mathbf{y} - \mathfrak{D}\mathbf{y} = \mathbf{c}, \mathbf{y} \leq \mathbf{0}\} \\ &= \min\{\mathbf{b}^T \mathbf{y} : \mathfrak{D}(-\mathbf{y}) - \mathbf{c} = -\mathbf{y}, -\mathbf{y} \geq \mathbf{0}\} \\ &= \min\{\mathbf{b}^T (-\mathbf{y}) : \mathfrak{D}\mathbf{y} - \mathbf{c} = \mathbf{y}, \mathbf{y} \geq \mathbf{0}\} \\ &= \min\{-\mathbf{b}^T \mathbf{y} : \mathfrak{D}\mathbf{y} - \mathbf{c} = \mathbf{y}, \mathbf{y} \geq \mathbf{0}\} \\ &= -\max\{\mathbf{b}^T \mathbf{y} : \mathfrak{D}\mathbf{y} - \mathbf{c} = \mathbf{y} \geq \mathbf{0}\}, \end{aligned}$$

that is

$$\min\{\mathbf{b}^T \mathbf{y} : \mathfrak{A}\mathbf{y} = \mathbf{c}, \mathbf{y} \leq \mathbf{0}\} = -\max\{\mathbf{b}^T \mathbf{y} : \mathfrak{D}\mathbf{y} - \mathbf{c} = \mathbf{y} \geq \mathbf{0}\} \quad (9)$$

and from (8) and (9)

$$\max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}\mathbf{x} \geq \mathbf{b}\} = -\max\{\mathbf{b}^T \mathbf{y} : \mathfrak{D}\mathbf{y} - \mathbf{c} = \mathbf{y} \geq \mathbf{0}\} \quad (10)$$

Lemma 5

- (a) $\max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}\mathbf{x} - \mathbf{b} = \mathbf{x} \geq \mathbf{0}\} = \max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}\mathbf{x} \geq \mathbf{b}\}$
- (b) $\max\{\mathbf{b}^T \mathbf{y} : \mathfrak{D}\mathbf{y} - \mathbf{c} = \mathbf{y} \geq \mathbf{0}\} = \max\{\mathbf{b}^T \mathbf{y} : \mathfrak{D}\mathbf{y} \geq \mathbf{c}\}$

Proof

- (a) $\max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}\mathbf{x} - \mathbf{b} = \mathbf{x} \geq \mathbf{0}\} \leq \max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}\mathbf{x} - \mathbf{b} \geq \mathbf{0}\} = \max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}(\mathfrak{A}\mathbf{x} - \mathbf{b}) - \mathbf{b} = \mathfrak{A}\mathbf{x} - \mathbf{b} \geq \mathbf{0}\} = \max\{\mathbf{c}^T (\mathfrak{A}\mathbf{x} - \mathbf{b}) : \mathfrak{A}(\mathfrak{A}\mathbf{x} - \mathbf{b}) - \mathbf{b} = \mathfrak{A}\mathbf{x} - \mathbf{b} \geq \mathbf{0}\} \leq \max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}\mathbf{x} - \mathbf{b} = \mathbf{x} \geq \mathbf{0}\}.$
- (b) The proof is analogous to proof of (a). \square

Using (10) and Lemma 5 (a) the central invariant problems are related as follows:

$$\max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}\mathbf{x} - \mathbf{b} = \mathbf{x} \geq \mathbf{0}\} + \max\{\mathbf{b}^T \mathbf{y} : \mathfrak{D}\mathbf{y} - \mathbf{c} = \mathbf{y} \geq \mathbf{0}\} = 0, \quad (11)$$

and from (10) and Lemma 5 (b) the peripheral invariant problems are related thus:

$$\max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}\mathbf{x} \geq \mathbf{b}\} + \max\{\mathbf{b}^T \mathbf{y} : \mathfrak{D}\mathbf{y} \geq \mathbf{c}\} = 0. \quad (12)$$

The four problems in Lemma 5 comprise the invariant problems; the form in (10), on the LHS, without an equals, is called “peripheral” and the form, on the RHS, with an equals is called “central”; the problems are classified in Table 1. Later, regular function pairs which map between the original and invariant forms are introduced which justify the “central” appellation and permit the definition of peripheral and central forms of the original problems. Note (a) referring forward, Theorem 1 is the motivation for calling the invariant problems of Table 1 “central”. (b) in view of Lemma 5 the central invariant problems are very similar to the peripheral invariant problems, however precise analysis of the LP problem requires focus on the central invariant problems.

Table 1: Invariant problem classification

	primal	dual
peripheral	$\max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}\mathbf{x} \geq \mathbf{b}\}$	$\max\{\mathbf{b}^T \mathbf{y} : \mathfrak{D}\mathbf{y} \geq \mathbf{c}\}$
central	$\max\{\mathbf{c}^T \mathbf{x} : \mathfrak{A}\mathbf{x} - \mathbf{b} = \mathbf{x} \geq \mathbf{0}\}$	$\max\{\mathbf{b}^T \mathbf{y} : \mathfrak{D}\mathbf{y} - \mathbf{c} = \mathbf{y} \geq \mathbf{0}\}$

5 Original and invariant problem relationship

5.1 The primal and dual function pairs

With solution sets labeled as:

X for the original primal (a copy of \mathfrak{R}^n)

Y for the original dual (a copy of \mathfrak{R}^m)

\mathfrak{X} for the invariant primal (a copy of \mathfrak{R}^m), and

\mathfrak{Y} for the invariant dual (a copy of \mathfrak{R}^m),

the original and invariant problems are related using two function pairs as follows:

the primal function pair $\{f_p, \mathfrak{f}_p\}$ where

$$f_p : X \rightarrow \mathfrak{X}, \mathbf{x} \mapsto A\mathbf{x} - \mathbf{b} \text{ and} \quad (13)$$

$$\mathfrak{f}_p : \mathfrak{X} \rightarrow X, \mathbf{x} \mapsto A^+(\mathbf{x} + \mathbf{b}) \quad (14)$$

and the dual function pair $\{f_d, \mathfrak{f}_d\}$ where

$$f_d : Y \rightarrow \mathfrak{Y}, \mathbf{y} \mapsto -\mathfrak{D}\mathbf{y} - \mathbf{c} \text{ and} \quad (15)$$

$$\mathfrak{f}_d : \mathfrak{Y} \rightarrow Y, \mathbf{y} \mapsto \mathbf{c} - \mathfrak{D}\mathbf{y} \quad (16)$$

For the primal mappings the compositions $\mathbf{x}f_p\mathfrak{f}_p \stackrel{(13)}{=} (A\mathbf{x} - \mathbf{b})\mathfrak{f}_p$

$\stackrel{(14)}{=} A^+((A\mathbf{x} - \mathbf{b}) + \mathbf{b}) = A^+A\mathbf{x}$, and $\mathbf{x}\mathfrak{f}_pf_p = (A^+(\mathbf{x} + \mathbf{b}))f_p = A(A^+(\mathbf{x} + \mathbf{b})) - \mathbf{b} = AA^+(\mathbf{x} + \mathbf{b}) - \mathbf{b} = \mathfrak{A}\mathbf{x} + \mathfrak{A}\mathbf{b} - \mathbf{b} = \mathfrak{A}\mathbf{x} - \mathbf{b}$ are computed, that is

$$\mathbf{x}f_p\mathfrak{f}_p = A^+A\mathbf{x} \quad (17)$$

$$\mathbf{x}\mathfrak{f}_pf_p = \mathfrak{A}\mathbf{x} - \mathbf{b} \quad (18)$$

and for the dual mappings the compositions $\mathbf{y}f_d\mathfrak{f}_d = (-\mathfrak{D}\mathbf{y} - \mathbf{c})\mathfrak{f}_d = \mathbf{c} - \mathfrak{D}(-\mathfrak{D}\mathbf{y} - \mathbf{c}) = \mathbf{c} - (-\mathfrak{D}\mathbf{y}) = \mathbf{c} + \mathfrak{D}\mathbf{y}$, and $\mathfrak{h}\mathfrak{f}_d\mathfrak{f}_d = (\mathbf{c} - \mathfrak{D}\mathfrak{h})\mathfrak{f}_d = -\mathfrak{D}(\mathbf{c} - \mathfrak{D}\mathfrak{h}) - \mathbf{c} = \mathfrak{D}\mathfrak{h} - \mathbf{c}$ are computed, that is

$$\mathbf{y}f_d\mathfrak{f}_d = \mathfrak{D}\mathbf{y} + \mathbf{c} \quad (19)$$

$$\mathfrak{h}\mathfrak{f}_d\mathfrak{f}_d = \mathfrak{D}\mathfrak{h} - \mathbf{c} \quad (20)$$

The central sets and optimal central subsets w.r.t. these function pairs for the original and invariant problems are indicated in Figure 1 by the black and green boxes respectively.

For the primal mappings the triple compositions

$$\mathbf{x}f_p\mathfrak{f}_p\mathfrak{f}_p \stackrel{(17)}{=} (A^+A\mathbf{x})f_p \stackrel{(13)}{=} A(A^+A\mathbf{x}) - \mathbf{b} = A\mathbf{x} - \mathbf{b} \stackrel{(13)}{=} \mathbf{x}f_p,$$

and

$$\mathfrak{r}\mathfrak{f}_p\mathfrak{f}_p\mathfrak{f}_p \stackrel{(18)}{=} (\mathfrak{A}\mathfrak{r} - \mathbf{b})\mathfrak{f}_p \stackrel{(14)}{=} A^+(\mathfrak{A}\mathfrak{r} - \mathbf{b}) + \mathbf{b} = A^+(\mathfrak{A}\mathfrak{r} + \mathbf{b}) = A^+(\mathfrak{r} + \mathbf{b}) \stackrel{(14)}{=} \mathfrak{r}\mathfrak{f}_p$$

are computed, that is

$$f_p\mathfrak{f}_p\mathfrak{f}_p = f_p \quad (21)$$

$$\mathfrak{f}_p\mathfrak{f}_p\mathfrak{f}_p = \mathfrak{f}_p \quad (22)$$

and for the dual mappings the triple compositions

$$\mathbf{y}f_d\mathfrak{f}_d\mathfrak{f}_d \stackrel{(19)}{=} (\mathfrak{D}\mathbf{y} + \mathbf{c})f_d \stackrel{(15)}{=} -\mathfrak{D}(\mathfrak{D}\mathbf{y} + \mathbf{c}) - \mathbf{c} = -\mathfrak{D}\mathbf{y} - \mathbf{c} \stackrel{(15)}{=} \mathbf{y}f_d, \text{ and}$$

$$\mathfrak{h}\mathfrak{f}_d\mathfrak{f}_d\mathfrak{f}_d \stackrel{(20)}{=} (\mathfrak{D}\mathfrak{h} - \mathbf{c})\mathfrak{f}_d \stackrel{(16)}{=} \mathbf{c} - \mathfrak{D}(\mathfrak{D}\mathfrak{h} - \mathbf{c}) = \mathbf{c} - \mathfrak{D}\mathfrak{h} \stackrel{(16)}{=} \mathfrak{h}\mathfrak{f}_d \text{ are computed, that is}$$

$$f_d\mathfrak{f}_d\mathfrak{f}_d = f_d \quad (23)$$

$$\mathfrak{f}_d\mathfrak{f}_d\mathfrak{f}_d = \mathfrak{f}_d \quad (24)$$

From (21), (22), (23) and (24), the function pairs $\{f_p, \mathfrak{f}_p\}$ and $\{f_d, \mathfrak{f}_d\}$ are regular so Lemma 1 applies, giving in the context of these functions:

Theorem 1

- (a) *The four central mappings $f_{pc}, \mathfrak{f}_{pc}, f_{dc}$ and \mathfrak{f}_{dc} are bijective.*
- (b) *The primal and dual central function pairs $\{f_{pc}, \mathfrak{f}_{pc}\}$ and $\{f_{dc}, \mathfrak{f}_{dc}\}$ comprise mutually inverse functions.*
- (c) *Conditions $\mathbf{x} = A^+A\mathbf{x}$, $\mathbf{y} = \mathfrak{D}\mathbf{y} + \mathbf{c}$, $\mathfrak{r} = \mathfrak{A}\mathfrak{r} - \mathbf{b}$, and $\mathfrak{h} = \mathfrak{D}\mathfrak{h} - \mathbf{c}$ define precisely the central elements of X, Y, \mathfrak{X} , and \mathfrak{Y} respectively.*

Proof (a) and (b) follow since the function pairs $\{f_p, \mathfrak{f}_p\}$ and $\{f_d, \mathfrak{f}_d\}$ satisfy the conditions for Lemma 1. (c) Follows from Lemma 2, and from (17), (18), (19) and (20) respectively. \square

Theorem 2

- (a) $\mathbf{x} = A^+A\mathbf{x}$,
- (b) $\mathbf{y} = \mathfrak{D}\mathbf{y} + \mathbf{c}$,
- (c) $\mathfrak{r} = \mathfrak{A}\mathfrak{r} - \mathbf{b}$, and
- (d) $\mathfrak{h} = \mathfrak{D}\mathfrak{h} - \mathbf{c}$ define precisely the central elements of X, Y, \mathfrak{X} , and \mathfrak{Y} respectively.

Proof The theorem follows from Lemma 2, and from (17), (18), (19) and (20) respectively. \square

Four linear programs are now considered: the original primal (1), original dual (7), invariant primal (mentioned in Lemma 5 a) and invariant dual (mentioned in Lemma 5 b); the original and invariant programs are related using the primal and dual function pairs defined in Section 5.1.

5.2 Feasibility

The quantities $A\mathbf{x} - \mathbf{b}$, \mathfrak{y} , $\mathfrak{A}\mathfrak{x} - \mathbf{b}$ and $\mathfrak{D}\mathfrak{y} - \mathbf{c}$ are related by the defined function pairs as follows:

Lemma 6

$$\begin{aligned} \text{(a)} \quad \mathfrak{A}(\mathbf{x}f_p) - \mathbf{b} &= A\mathbf{x} - \mathbf{b} \\ \text{(b)} \quad A(\mathfrak{x}f_p) - \mathbf{b} &= \mathfrak{A}\mathfrak{x} - \mathbf{b} \\ \text{(c)} \quad \mathfrak{D}(\mathbf{y}f_d) - \mathbf{c} &= -\mathbf{y} \quad \text{if } A^T\mathbf{y} = \mathbf{c} \\ \text{(d)} \quad A^T(\mathfrak{y}f_d) = \mathbf{c} \wedge \mathfrak{y}f_d &= -(\mathfrak{D}\mathfrak{y} - \mathbf{c}) \quad \text{if Problem 1 is quasi-bounded} \end{aligned}$$

Proof

$$\begin{aligned} \text{(a)} \quad \mathfrak{A}(\mathbf{x}f_p) - \mathbf{b} &\stackrel{(18)}{=} (\mathbf{x}f_p)f_p f_p = \mathbf{x}(f_p f_p f_p) \stackrel{(21)}{=} \mathbf{x}f_p \stackrel{(13)}{=} A\mathbf{x} - \mathbf{b}, \\ \text{(b)} \quad A(\mathfrak{x}f_p) - \mathbf{b} &\stackrel{(13)}{=} (\mathfrak{x}f_p)f_p \stackrel{(18)}{=} \mathfrak{A}\mathfrak{x} - \mathbf{b}, \\ \text{(c)} \quad \mathfrak{D}(\mathbf{y}f_d) - \mathbf{c} &\stackrel{(20)}{=} \mathbf{y}f_d(f_d f_d) = \mathbf{y}(f_d f_d f_d) \stackrel{(23)}{=} \mathbf{y}f_d \stackrel{(15)}{=} -\mathfrak{D}\mathbf{y} - \mathbf{c} = -(I - \mathfrak{A})\mathbf{y} - \mathbf{c} = \\ &-\mathbf{y} + AA^+\mathbf{y} - \mathbf{c} = -\mathbf{y} + A^{+T}A^T\mathbf{y} - \mathbf{c} = -\mathbf{y} + A^{+T}\mathbf{c} - \mathbf{c} = -\mathbf{y} + \mathbf{c} - \mathbf{c} = -\mathbf{y}, \\ \text{(d)} \quad \mathfrak{y}f_d = \mathbf{c} - \mathfrak{D}\mathfrak{y} &= -(\mathfrak{D}\mathfrak{y} - \mathbf{c}), \text{ so } A^T(\mathfrak{y}f_d) = A^T(\mathbf{c} - \mathfrak{D}\mathfrak{y}) = A^T(\mathbf{c} - (I - AA^+)\mathfrak{y}) = \\ A^T\mathbf{c} &\stackrel{(\text{Lemma 4c})}{=} \mathbf{c}. \quad \square \end{aligned}$$

From Lemma 6 it is seen that feasible solutions are mapped to feasible solutions:

Corollary 1

$$\begin{aligned} \text{(a)} \quad A\mathbf{x} \geq \mathbf{b} &\Leftrightarrow \mathfrak{A}(\mathbf{x}f_p) \geq \mathbf{b}, \\ \text{(b)} \quad \mathfrak{A}\mathfrak{x} \geq \mathbf{b} &\Leftrightarrow A(\mathfrak{x}f_p) \geq \mathbf{b}, \\ \text{(c)} \quad \mathbf{y} \leq 0 &\Leftrightarrow \mathfrak{D}(\mathbf{y}f_d) \geq \mathbf{c}, \quad \text{if } A^T\mathbf{y} = \mathbf{c}, \\ \text{(d)} \quad \mathfrak{D}\mathfrak{y} \geq \mathbf{c} &\Leftrightarrow A^T(\mathfrak{y}f_d) = \mathbf{c} \wedge \mathfrak{y}f_d \leq 0, \quad \text{if Problem 1 is quasi-bounded.} \end{aligned}$$

Corollary 1 shows that the two functions of the primal function pair map between feasible original primal solutions and feasible invariant primal solutions, while the two functions of the dual function pair map between feasible original dual solutions and feasible invariant dual solutions.

5.3 Optimality

It is shown that the objective values of corresponding primal vectors differ by a fixed amount, and that the objective values of corresponding dual vectors move in lockstep - that is they differ in sign and by a fixed amount, and thus solutions are mapped to solutions:

Lemma 7

- (a) $\mathbf{c}^T(\mathbf{x}f_p) = \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{c}$
- (b) $\mathbf{c}^T(\mathbf{x}f_p) = \mathbf{c}^T\mathbf{x} + \mathbf{b}^T\mathbf{c}$
- (c) $\mathbf{b}^T(\mathbf{y}f_d) = \mathbf{b}^T\mathbf{c} - \mathbf{b}^T\mathbf{y}$ if $A^T\mathbf{y} = \mathbf{c}$
- (d) $\mathbf{b}^T(\mathbf{y}f_d) = \mathbf{b}^T\mathbf{c} - \mathbf{b}^T\mathbf{y}$

Proof

- (a) $\mathbf{c}^T(\mathbf{x}f_p) = \mathbf{c}^T(A\mathbf{x} - \mathbf{b}) = \mathbf{c}^TA\mathbf{x} - \mathbf{c}^T\mathbf{b} = (A^{T+}\mathbf{c})^TA\mathbf{x} - \mathbf{c}^T\mathbf{b} = \mathbf{c}^TA^+A\mathbf{x} - \mathbf{c}^T\mathbf{b} = (A^+A\mathbf{c})^T\mathbf{x} - \mathbf{c}^T\mathbf{b} \stackrel{\text{(Definition 2)}}{=} \mathbf{c}^T\mathbf{x} - \mathbf{c}^T\mathbf{b} = \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{c};$
- (b) $\mathbf{c}^T(\mathbf{x}f_p) = \mathbf{c}^T(A^+\mathbf{x} + \mathbf{b}) = (A^{+T}\mathbf{c})^T(\mathbf{x} + \mathbf{b}) = (A^{T+}\mathbf{c})^T(\mathbf{x} + \mathbf{b}) \stackrel{(3c)}{=} \mathbf{c}^T(\mathbf{x} + \mathbf{b}) = \mathbf{c}^T\mathbf{x} + \mathbf{c}^T\mathbf{b} = \mathbf{c}^T\mathbf{x} + \mathbf{b}^T\mathbf{c};$
- (c) $\mathbf{b}^T(\mathbf{y}f_d) = \mathbf{b}^T(-\mathcal{D}\mathbf{y} - \mathbf{c}) = -\mathbf{b}^T\mathcal{D}\mathbf{y} = \mathbf{b}^T(AA^+ - I)\mathbf{y} = \mathbf{b}^T(AA^+)\mathbf{y} - \mathbf{b}^T\mathbf{y} = \mathbf{b}^TA^{+T}A^T\mathbf{y} - \mathbf{b}^T\mathbf{y} = \mathbf{b}^TA^{+T}\mathbf{c} - \mathbf{b}^T\mathbf{y} = \mathbf{b}^T\mathbf{c} - \mathbf{b}^T\mathbf{y};$
- (d) $\mathbf{b}^T(\mathbf{y}f_d) = \mathbf{b}^T(\mathbf{c} - \mathcal{D}\mathbf{y}) = \mathbf{b}^T\mathbf{c} - \mathbf{b}^T\mathcal{D}\mathbf{y} = \mathbf{b}^T\mathbf{c} - \mathbf{b}^T\mathbf{y}. \quad \square$

Corollary 2

- (a) $\mathbf{c}^T(\mathbf{x}f_p) + \mathbf{b}^T(\mathbf{y}f_d) = \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y}$ if $A^T\mathbf{y} = \mathbf{c}$
- (b) $\mathbf{c}^T(\mathbf{x}f_p) - \mathbf{b}^T(\mathbf{y}f_d) = \mathbf{c}^T\mathbf{x} + \mathbf{b}^T\mathbf{y}.$

Proof

- (a) Adding the results in Lemma 7 a and 7 c, $\mathbf{c}^T(\mathbf{x}f_p) + \mathbf{b}^T(\mathbf{y}f_d) = \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y}$ is obtained.
- (b) Subtracting Lemma 7 d from 7 b yields $\mathbf{c}^T(\mathbf{x}f_p) - \mathbf{b}^T(\mathbf{y}f_d) = \mathbf{c}^T\mathbf{x} + \mathbf{b}^T\mathbf{y}.$ \square

From Corollaries 1 and 2 it is seen that solutions to the original primal are mapped to solutions to the invariant primal and *vice versa*, and solutions to the original dual are mapped to solutions to the invariant dual and *vice versa*, making the representation of the primal problems as invariant problems explicit:

Theorem 3

Provided the original primal is quasi-bounded,

- (a) *if \mathbf{x}_s and \mathbf{y}_s are \langle feasible|solutions \rangle for the original primal and dual respectively then \mathbf{x}_sf_p and \mathbf{y}_sf_d are central \langle feasible|solutions \rangle for the invariant primal and dual respectively.*
- (b) *if \mathbf{x}_s and \mathbf{y}_s are \langle feasible|solutions \rangle for the invariant primal and dual respectively then \mathbf{x}_sf_p and \mathbf{y}_sf_d are central \langle feasible|solutions \rangle for the original primal and dual respectively,*
- (c) *The four mappings, restricted to central [feasible|solution] vectors are bijections.*

Proof

- (a) If \mathbf{x}_s and \mathbf{y}_s are feasible then, from Corollary 1 (a) and (c), \mathbf{x}_sf_p and \mathbf{y}_sf_d are feasible, while centrality follows from Definition 1; further, given solutions \mathbf{x}_s and \mathbf{y}_s ,

$$\mathbf{c}^T\mathbf{x}_s - \mathbf{b}^T\mathbf{y}_s = 0 \stackrel{\text{(Corollary 2a)}}{\Rightarrow} \mathbf{c}^T(\mathbf{x}_sf_p) + \mathbf{b}^T(\mathbf{y}_sf_d) = 0 \stackrel{(12)}{\Rightarrow} \mathbf{x}_sf_p$$

and $\mathbf{y}_s f_d$ are optimal, and central since they are in the ranges of f_p and f_d respectively, and are therefor respective solutions for the invariant primal and dual.

- (b) If \mathbf{x}_s and \mathbf{y}_s are feasible then, from Corollary 1 (b) and (d), $\mathbf{x}_s f_p$ and $\mathbf{y}_s f_d$ are feasible while centrality follows from Definition 1; further, if \mathbf{x}_s and \mathbf{y}_s are solutions then

$$\mathbf{c}^T \mathbf{x}_s + \mathbf{b}^T \mathbf{y}_s = 0 \stackrel{\text{(Corollary 2b)}}{\Rightarrow} \mathbf{c}^T (\mathbf{x}_s f_p) - \mathbf{b}^T (\mathbf{y}_s f_d) = 0 \stackrel{(6)}{\Rightarrow} \mathbf{x}_s f_p$$

and $\mathbf{y}_s f_d$ are respective solutions for the original primal and dual.

- (c) For central vectors the result follows from Lemma 1; for central feasible vectors the result follows from Corollary 1 (b) and (d); for central solutions the result follows from Corollary 1 (b) and (d), and Corollary 2. \square

Remark 1 From Lemmas 1 and 3, the central function pairs establish a one-to-one correspondence between the central solutions to the original problems and the central solutions to the invariant problems, and a one-to-one correspondence between the central solutions to the original problems and the central solutions to the invariant problems.

The representation of the original problems as invariant problems is summarized by the lower two-thirds of Figure 1. Feasible conditions and solutions are in thin-lined boxes, central feasible conditions and solutions are in thick-lined boxes, and optimal central feasible conditions and solutions are in green boxes; the two function pairs and their central forms are shown linking the original and invariant solutions; the two functions $\mathcal{F}_p = f_p f_p$ and $\mathcal{F}_d = f_d f_d$ which map arbitrary invariant solutions to central invariant solutions are shown.

5.4 Summary

Invariant problems have been constructed and mappings have been defined which establish a 1:1 relationship between the solutions to the central original and central invariant problems.

6 Invariant and fixed-point problem relationship

A fixed-point problem is constructed which involves finding a non-negative vector

$$\omega = \begin{bmatrix} \xi \\ \zeta \end{bmatrix}$$

which is transformed to itself by an idempotent symmetric matrix \mathfrak{B} . The notion of centrality is introduced to elements in the domain of \mathfrak{N} via the backmap of the natural mapping

$$\mathfrak{N} : \omega = \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \in \mathbb{R}^{2m} \mapsto (\xi, \zeta), \xi \in \mathbb{R}^m, \zeta \in \mathbb{R}^m$$

(where ξ and ζ are interpreted as being vectors in the solution spaces of the invariant primal and dual respectively). It is then shown that \mathfrak{N} with domain restricted to central solutions is a surjection onto the set of central invariant solutions, thus completing the representation of the LP problem as a fixed-point problem.

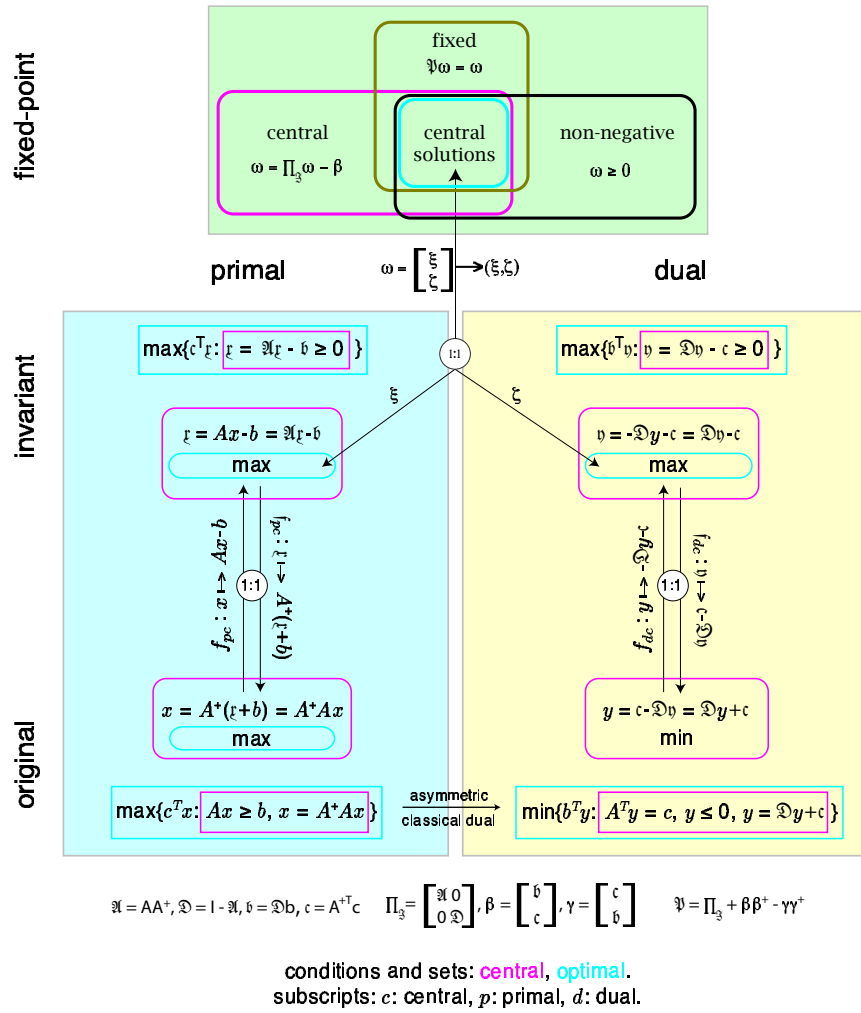


Figure 1: Original, invariant and fixed-point problem relationship

6.1 The fixed-point matrix

The following notation is introduced:

$$\begin{aligned}
 \Pi_3 &= \begin{bmatrix} \mathfrak{A} & 0 \\ 0 & \mathfrak{D} \end{bmatrix} & (a) \\
 \beta &= \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} & (b) \\
 \gamma &= \begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix} & (c) \\
 \Pi_\beta &= \beta\beta^+ & (d) \\
 \Pi_\gamma &= \gamma\gamma^+ & (e)
 \end{aligned} \tag{25}$$

then

$$\begin{array}{lcl}
 \Pi_3^2 = \Pi_3^T & = & \Pi_3 \text{ (a)} \\
 \Pi_3\Pi_\beta & = & 0 \text{ (b)} \\
 \Pi_3\Pi_\gamma & = & \Pi_\gamma \text{ (c)} \\
 \Pi_3\beta & = & 0 \text{ (d)} \\
 \Pi_3\gamma & = & \gamma \text{ (e)} \\
 \Pi_\beta^2 = \Pi_\beta^T & = & \Pi_\beta \text{ (f)}
 \end{array} \left| \begin{array}{lcl}
 \Pi_\beta\Pi_\gamma = \Pi_\gamma\Pi_\beta & = & 0 \text{ (g)} \\
 \Pi_\beta\beta & = & \beta \text{ (h)} \\
 \Pi_\beta\gamma & = & 0 \text{ (i)} \\
 \Pi_\gamma^2 = \Pi_\gamma^T & = & \Pi_\gamma \text{ (j)} \\
 \Pi_\gamma\beta & = & 0 \text{ (k)} \\
 \Pi_\gamma\gamma & = & \gamma \text{ (l)}
 \end{array} \right. \tag{26}$$

Definition 3 The fixed-point matrix is defined as $\mathfrak{P} = \Pi_3 + \Pi_\beta - \Pi_\gamma$, that is

$$\mathfrak{P} = \begin{bmatrix} \mathfrak{A} & 0 \\ 0 & \mathfrak{D} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}^+ - \begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix}^+ .$$

Symmetry of \mathfrak{P} follows from the symmetry of its components Π_3 , Π_β and Π_γ , while $\mathfrak{P}^2 \stackrel{(3)}{=} \mathfrak{P}$

$$\begin{aligned}
 (\Pi_3 + \Pi_\beta - \Pi_\gamma)^2 &= \Pi_3^2 + \Pi_3\Pi_\beta - \Pi_3\Pi_\gamma + \Pi_\beta\Pi_3 + \Pi_\beta^2 - \Pi_\beta\Pi_\gamma - \Pi_\gamma\Pi_3 - \Pi_\gamma\Pi_\beta + \Pi_\gamma^2 \stackrel{(26)}{=} \Pi_3 + \\
 0 - \Pi_\gamma + 0 + \Pi_\beta - 0 - \Pi_\gamma - 0 + \Pi_\gamma &= \Pi_3 + \Pi_\beta - \Pi_\gamma \stackrel{(3)}{=} \mathfrak{P}, \text{ that is}
 \end{aligned}$$

$$\mathfrak{P}^T = \mathfrak{P} \text{ and } \mathfrak{P}^2 = \mathfrak{P}, \tag{27}$$

in other words \mathfrak{P} is a projection matrix.

$$\text{Further, } \mathfrak{P}\beta \stackrel{(3)}{=} (\Pi_3 + \Pi_\beta - \Pi_\gamma)\beta = \Pi_3\beta + \Pi_\beta\beta - \Pi_\gamma\beta \stackrel{(26)}{=} 0 + \beta - 0 = \beta, \text{ and}$$

$$\mathfrak{P}\gamma \stackrel{(3)}{=} (\Pi_3 + \Pi_\beta - \Pi_\gamma)\gamma = \Pi_3\gamma + \Pi_\beta\gamma - \Pi_\gamma\gamma \stackrel{(26)}{=} \gamma + 0 - \gamma = 0, \text{ that is}$$

$$\begin{aligned}
 \mathfrak{P}\beta &= \beta, & (a) \\
 \mathfrak{P}\gamma &= \mathbf{0}. & (b)
 \end{aligned} \tag{28}$$

and, obviously,

$$\begin{aligned}
 \beta^T\mathfrak{P} &= \beta^T, & (a) \\
 \gamma^T\mathfrak{P} &= \mathbf{0}^T. & (b)
 \end{aligned} \tag{29}$$

Definition 4 A point ω is said to be a fixed-point, if $\mathfrak{P}\omega = \omega$.

Remark 2 It follows that for fixed-point ω ,

$$\gamma^T \omega = 0, \Pi_\gamma \omega = \mathbf{0}, \text{ and } \mathfrak{P}\omega = (\Pi_3 + \Pi_\beta)\omega .$$

6.2 The surjection

6.2.1 Point classification

Attributes are ascribed to pairs as follows: (ξ, ζ) inherits a particular characteristic iff ξ and ζ both have this characteristic. Attributes are ascribed to points as follows: $\begin{bmatrix} \xi \\ \zeta \end{bmatrix}$ inherits a particular characteristic iff (ξ, ζ) has this characteristic so, for example, given

Definition 5 (ξ, ζ) is quasi-optimal iff $\mathbf{c}^T \xi + \mathbf{b}^T \zeta = 0$.

If (ξ_q, ζ_q) is quasi-optimal then, by ascription, $\begin{bmatrix} \xi_q \\ \zeta_q \end{bmatrix}$ is quasi-optimal

Thus feasibility, centrality and solution characteristics can be ascribed to pairs and points in a natural manner.

Lemma 8 $\omega = \begin{bmatrix} \xi \\ \zeta \end{bmatrix}$ is a fixed-point \Rightarrow (a) $\gamma^T \omega = 0$; (b) $\Pi_\gamma \omega = \mathbf{0}$.

Proof

$$(a) \omega = \mathfrak{P}\omega \Rightarrow \gamma^T \omega = \gamma^T \mathfrak{P}\omega \stackrel{(29b)}{\Rightarrow} \gamma^T \omega = \mathbf{0}^T \omega = 0.$$

$$(b) \gamma^T \omega = 0 \Rightarrow \gamma^+ \omega = 0 \Rightarrow \gamma \gamma^+ \omega = \mathbf{0} \Rightarrow \Pi_\gamma \omega = \mathbf{0}. \quad \square$$

6.2.2 The discriminant and centrality

Definition 6 $d(\omega) = -\beta^+ \omega$ is called the discriminant of ω .

Lemma 9 ω is a central-point $\Rightarrow d(\omega) = \beta^+ \beta \in \{0, 1\}$.

Proof $\omega = \Pi_3 \omega - \beta \Rightarrow \beta^+ \omega = \beta^+ \Pi_3 \omega - \beta^+ \beta \Rightarrow -\beta^+ \omega = \beta^+ \beta \Rightarrow d(\omega) = \beta^+ \beta$. \square

Note that the functions \mathfrak{C}_1 and \mathfrak{P} are idempotents which commute.

The following lemma shows that the discriminant determines precisely whether or not a fixed-point is central, and guarantees that applying \mathfrak{C}_2 to a fixed-point yields a central fixed-point:

Lemma 10 ω is a central fixed-point iff ω is a fixed-point and $d(\omega) \in \{0, 1\}$.

Proof The forward implication follows from Lemma 9; conversely,

$$\begin{aligned}
& \mathfrak{P}\omega = \omega \wedge d(\omega) \in \{0, 1\} \\
& \Rightarrow (\Pi_3 + \beta\beta^+ - \gamma\gamma^+)\omega = \omega \wedge d(\omega) = \beta^+\beta \\
& \Rightarrow \mathfrak{P}\omega = \omega \wedge (\Pi_3 + \beta\beta^+)\omega = \omega \wedge d(\omega) = \beta^+\beta \\
& \Rightarrow \mathfrak{P}\omega = \omega \wedge \Pi_3\omega - \beta(-\beta^+\omega) = \omega \wedge d(\omega) = \beta^+\beta \\
& \Rightarrow \mathfrak{P}\omega = \omega \wedge \Pi_3\omega - \beta(\beta^+\beta) = \omega \\
& \Rightarrow \mathfrak{P}\omega = \omega \wedge \Pi_3\omega - \beta = \omega \\
& \Rightarrow \omega \text{ is a central fixed-point.} \quad \square
\end{aligned}$$

Consistent with Theorem 2 (c) and (d) in case (a), and with Lemma 10 in case (b), we define centralising functions:

Definition 7

$$\begin{aligned}
(a) \quad \mathfrak{C}_1 : \begin{bmatrix} \xi \\ \zeta \end{bmatrix} &\mapsto \begin{bmatrix} \mathfrak{A}\xi \\ \mathfrak{D}\zeta \end{bmatrix} - \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} \text{ or,} \\
& \text{alternatively, } \mathfrak{C} : \omega \mapsto \Pi_3\omega - \beta, \text{ and} \\
(b) \quad \mathfrak{C}_2 : \omega &\mapsto \omega/d(\omega).
\end{aligned}$$

Lemma 11 ω is a central fixed-point iff ω is a quasi-optimal central point.

Proof If ω is a central fixed-point then by Lemma 8, ω is quasi-optimal central. Conversely if ω is quasi-optimal central, $\mathfrak{P}\omega = (\Pi_3 + \Pi_\beta - \Pi_\gamma)\omega \stackrel{\text{(Lemma 8)}}{=} (\Pi_3 + \Pi_\beta)\omega = (\Pi_3 + \beta\beta^+)\omega = \Pi_3\omega + \beta(\beta^+\omega) \stackrel{\text{(Lemma 9)}}{=} \Pi_3\omega + \beta(-\beta^+\beta) = \Pi_3\omega - \beta \stackrel{\text{(Definition 7)}}{=} \mathfrak{C}_1\omega = \omega$, that is $\mathfrak{P}\omega = \omega$, so ω is a central fixed-point. \square

So, ω_a is an arbitrary non-negative central fixed-point $\stackrel{\text{Lemma 10}}{\Rightarrow} \omega_c = \omega_a/d(\omega_a)$ is a non-negative central fixed-point $\stackrel{\text{Lemma 11}}{\Leftrightarrow} \omega_c$ is a non-negative quasi-optimal central point $\stackrel{(11)}{\Leftrightarrow} \omega_c$ is a central solution point $\Leftrightarrow \mathfrak{N}\omega_c$ is an invariant central solution pair. Thus we have the following theorem.

Theorem 4 The map $\mathfrak{N} : \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \mapsto (\xi, \zeta)$ where $\xi, \zeta \in \mathfrak{R}^m$, with domain equal to the set of non-negative central fixed-points, and codomain equal to the set of invariant central solution pairs is a surjection.

6.2.3 The unusual case

There remains the unusual case of a zero central fixed-point. From Lemma 9, for central fixed-point ω_{cf} , $-\beta^+\omega_{cf} = \beta^+\beta \in \{0, 1\}$ and usually $-\beta^+\omega_{cf} = 1$, however consider the unusual case where $-\beta^+\omega_c = 0$, that is $[\omega_{cf} = \Pi_3\omega_{cf} - \beta] \wedge [\omega_{cf} = \mathfrak{P}\omega_{cf}] \wedge [-\beta^+\omega_{cf} = 0]$

$$\begin{aligned}
& \stackrel{\text{(Lemma 8)}}{\Rightarrow} [\omega_{cf} = \Pi_3\omega_{cf} - \beta] \wedge [\omega_{cf} = \mathfrak{P}\omega_{cf}] \wedge [-\beta^+\omega_{cf} = 0] \wedge [\gamma^+\omega_{cf} = 0] \\
& \Rightarrow [\omega_{cf} = \Pi_3\omega_{cf} - \beta] \wedge [\omega_{cf} = (\Pi_3 + \beta\beta^+)\omega_{cf}] \wedge [-\beta^+\omega_{cf} = 0] \\
& \Rightarrow [\omega_{cf} = \Pi_3\omega_{cf} - \beta] \wedge [\omega_{cf} = \Pi_3\omega_{cf}] \Rightarrow \beta = \mathbf{0}. \text{ If } \beta = \mathbf{0} \text{ then } \mathbf{b} = \mathbf{0} \text{ and } \mathbf{c} = \mathbf{0}, \text{ and}
\end{aligned}$$

the invariant problems are $\max\{\mathbf{0}^T \mathbf{x} : \mathfrak{A}\mathbf{x} \geq \mathbf{0}\}$ and $\max\{\mathbf{0}^T \mathbf{y} : \mathfrak{D}\mathbf{y} \geq \mathbf{0}\}$, that is, find the solutions for the invariant problems $\{\mathbf{x} : \mathfrak{A}\mathbf{x} \geq \mathbf{0}\}$ and $\{\mathbf{y} : \mathfrak{D}\mathbf{y} \geq \mathbf{0}\}$ and the solutions to the original problems $\{A^+(\mathbf{x} + \mathbf{b}) : \mathfrak{A}\mathbf{x} \geq \mathbf{0}\}$ and $\{\mathbf{c} - \mathfrak{D}\mathbf{y} : \mathfrak{D}\mathbf{y} \geq \mathbf{0}\}$. So, under these circumstances, every non-negative central solution fixed-point solution $\mathfrak{z}_s = \begin{bmatrix} \mathbf{x}_s \\ \mathbf{y}_s \end{bmatrix}$ yields solutions $A^+(\mathbf{x}_s + \mathbf{b})$ and $\mathbf{c} - \mathfrak{D}\mathbf{y}_s$ to the original primal and dual respectively.

7 Conclusion

Summing up, the one-to-one relationship between the original and invariant problems is shown in the lower two-thirds of Figure 1, while the one-to-one relationship between the invariant and fixed-point problems is shown in the upper two-thirds of the figure. Thus the central original problems have been represented as the central invariant problems, and the central invariant problems have been represented as the central fixed-point problem; since the central problems encapsulate all solutions, this completes the representation of the LP problem as a fixed-point problem.

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