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Subbalancing Numbers

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Abstract A natural number n is called balancing number (with balancer r) if it satisfies the Diophantine equation $1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$. However, if for some pair of natural numbers (n, r), $1+2+\cdots+(n-1) < (n+1)+(n+2)+\cdots+(n+r)$ and equality is achieved after adding a natural number D to the left hand side then we call n a D-subbalancing number with D-subbalancer number r. In this paper, such numbers are studied for certain values of D.

Keywords Balancing and Lucas-balancing numbers, cobalancing numbers, Supercobalancing numbers

Mathematics Subject Classification 11B39, 11B83

1 Introduction

Behera and Panda in [1] stated that a natural number n is called a balancing number with balancer r if

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r).$$

Furthermore, they stated that if n is a balancing number then $8n^2 + 1$ is a perfect square. The k^{th} balancing number is denoted by B_k and $C_k = \sqrt{8B_k^2 + 1}$ is called the k^{th} Lucas-balancing number [2]. The balancing and Lucas-balancing numbers satisfy the recurrence relation $x_{n+1} = x_n - x_{n-1}$ with initial terms $B_0 = 0, B_1 = 1$ and $C_0 = 1, C_1 = 3$ respectively. On other hand, n is called a cobalancing number [3] with cobalancer r if

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r).$$

The n^{th} cobalancing number is denoted by b_n and cobalancing numbers satisfy the nonhomogeneous recurrence $b_1 = 0, b_2 = 2, b_{n+1} = 6b_n - b_{n-1} + 2$. The Binet forms are

$$B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}},\ \ C_n = \frac{\alpha^{2n} + \beta^{2n}}{2},\ \ b_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}.$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

Rout and Panda [4] generalized the concept of balancing numbers and introduced gap balancing numbers. If k is odd, they call a natural number n a k-gap balancing number if

$$1 + 2 + \cdots + \left(n - \frac{k+1}{2}\right) = \left(n + \frac{k+1}{2}\right) + \left(n + \frac{k+3}{2}\right) + \cdots + (n + r)$$

for some natural number r, which is a k-gap balancer corresponding to n, while for k even, if

$$1 + 2 + \cdots + \left(n - \frac{k}{2}\right) = \left(n + \frac{k}{2} + 1\right) + \left(n + \frac{k}{2} + 2\right) + \cdots + (n + r)$$

for some natural numbers n and r then they call $2n + 1$ a k-gap balancing number and r is the corresponding k-gap balancer. In [5], Davala and Panda called n, a D-supercobalancing number if for a fixed positive integer D, n satisfies the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r) + D$$

for some natural number r, which they call as D-supercobalancer corresponding to n. Panda and Panda [6] defined almost balancing numbers as the values of n satisfying the Diophantine equation

$$1 + 2 + \cdots + (n - 1) + 1 = (n + 1) + (n + 2) + \cdots + (n + r) \tag{1}$$

respectively for some r, which they called an almost balancer corresponding to n. They observed that there are two classes of almost balancing numbers. The almost balancing numbers admits a generalization. The last term 1 of the left hand side of (1) may be replaced by an arbitrary integer D resulting the definition of subbalancing numbers.

2 Subbalancing Numbers

Definition 1 *For a fixed positive integer D, we call a positive integer n, a D-subbalancing number if*

$$1 + 2 + \cdots + (n - 1) + D = (n + 1) + (n + 2) + \cdots + (n + r) \tag{2}$$

for some natural number r, which we call the D-subbalancer corresponding to D-subbalancing number n. If D is a negative integer, say $D = -R$, we call n a R-superbalancing number and r, a R-superbalancer corresponding to n.

Since, without D, the left hand side of (2) is less than the right hand side, we prefer the name subbalancing number for n. A similar justification applies when D is negative. Observe that when $D = 0$, the above definition coincides with that of balancing numbers; hence, we prefer to exclude the case $D = 0$ from the above definition. Let $D > 0$ and simplifying equation (2), we get

$$n^2 + D = \frac{(n + r)(n + r + 1)}{2}.$$

Thus, n is a D-subbalancing number then $n^2 + D$ is a triangular number or, equivalently, $8n^2 + 8D + 1$ is a perfect square. The D-subbalancer r corresponding to n is given by

$$r = \frac{1}{2}[-(1 + 2n) + \sqrt{8n^2 + 8D + 1}\ ]. \tag{3}$$

Observe that the value of n will generally depend on the choice of D and the existence of n is not ascertained for each value of D, for example, if $D = 7$, then $8n^2 + 8D + 1 = 8n^2 + 57$ is not a perfect square for any natural number n. Hence, the choice of D plays a crucial role.

It is well-known that for each positive integer n, $8b_n^2 + 8b_n + 1$ is a perfect square. A variant of this result is given in the following lemma.

Lemma 1 *For* $m \in \mathbb{N}$,

(1) $B_m^2 + b_{2m} = \frac{1}{2}\left[\frac{B_{m-1}+5B_m-1}{2}\right]\left[\frac{B_{m-1}+5B_m+1}{2}\right]$

(2i) $B_m^2 + b_{2m+1} = \frac{1}{2}\left[\frac{B_{m+1}+5B_m-1}{2}\right]\left[\frac{B_{m+1}+5B_m+1}{2}\right]$

Proof Since $8B_m^2 + 1 = C_m^2$ and $B_{m-1} = 3B_m - C_m$, we have

$$\begin{aligned}
(B_{m-1} + 5B_m)^2 - C_m^2 &= (B_{m-1} + 5B_m)^2 - (3B_m - B_{m-1})^2 \\
&= 16(B_{m-1}B_m + B_m^2) \\
&= 16\left[\frac{\alpha^{2m-2} - \beta^{2m-2}}{4\sqrt{2}} \cdot \frac{\alpha^{2m} - \beta^{2m}}{4\sqrt{2}} + \left(\frac{\alpha^{2m} - \beta^{2m}}{4\sqrt{2}}\right)^2\right] \\
&= \frac{1}{2}[\alpha^{4m-2} + \beta^{4m-2} - \alpha^2 - \beta^2 + \alpha^{4m} - 2 + \beta^{4m}] \\
&= \frac{1}{2}[2\sqrt{2}\alpha^{4m-1} - 2\sqrt{2}\beta^{4m-1} - 8] \\
&= 8\ b_{2m},
\end{aligned}$$

hence (1) follows. The proof of (2) follows from

$$\begin{aligned}
(B_{m+1} + 5B_m)^2 - C_m^2 &= (B_{m+1} + 5B_m)^2 - (3B_m - B_{m-1})^2 \\
&= 16(B_{m+1}B_m + B_m^2) \\
&= 16\left[\frac{\alpha^{2m+2} - \beta^{2m+2}}{4\sqrt{2}} \cdot \frac{\alpha^{2m} - \beta^{2m}}{4\sqrt{2}} + \left(\frac{\alpha^{2m} - \beta^{2m}}{4\sqrt{2}}\right)^2\right] \\
&= \frac{1}{2}[\alpha^{4m+2} + \beta^{4m+2} - \alpha^2 - \beta^2 + \alpha^{4m} - 2 + \beta^{4m}] \\
&= \frac{1}{2}[2\sqrt{2}\alpha^{4m+1} - 2\sqrt{2}\beta^{4m+1} - 8] \\
&= 8\ b_{2m+1}.
\end{aligned}$$

□

The above lemma ensures the existence of subbalancing numbers when D is restricted to cobalancing numbers.

If $k = 1$, then $b_k = 0$ and the concept of b_1-subbalancing numbers coincides with that of balancing numbers. The definition of subbalancing numbers excludes this case. If $k = 2$ then $b_k = 2$ and the requirement for a positive integer n to be a b_2-subbalancing number is that $8n^2 + 17$ be a perfect square. But according to Rout and Panda [4], such numbers are 3-gap balancing numbers and are of the form $5B_n \pm C_n$.

2.1 b_3-subbalancing Numbers

By virtue of Definition 1, a natural number n is a b_3-subbalancing number if

$$1 + 2 + \cdots + (n-1) + 14 = (n+1) + (n+2) + \cdots + (n+r) \tag{4}$$

for some natural number r, which is a b_3-subbalancer corresponding to n.

Example 1 Since $0 + 14 = 2 + 3 + 4 + 5$, we accept 1 as a b_3-subbalancing number with b_3-subbalancer 4. Further $1+2+\cdots+7+14 = 9+10+11+12$, 8 is a b_3-subbalancing number with b_3-subbalancer 4.

It follows from equations (3) and (4) that, if n is a b_3-subbalancing number then the corresponding b_3-subbalancer is

$$r = \frac{-(2n+1) + \sqrt{8n^2+113}}{2}.$$

From the above discussion, we notice that if x is a b_3-subbalancing number then $8x^2 + 113$ is a perfect square. Since 113 is prime, the congruence

$$121x^2 \equiv 8x^2 + 113 (\text{mod } 113)$$

gives

$$11x \equiv \pm\sqrt{8x^2+113}\,(\text{mod } 113).$$

Thus, $\frac{11x+\sqrt{8x^2+113}}{113}$ or $\frac{11x-\sqrt{8x^2+113}}{113}$ is a natural number. Since

$$8\left[\frac{11x \pm \sqrt{8x^2+113}}{113}\right]^2 + 1 = \left[\frac{11\sqrt{8x^2+113} \pm 8x}{113}\right]^2,$$

it follows that either

$$\frac{11x + \sqrt{8x^2+113}}{113} \quad \text{or} \quad \frac{11x - \sqrt{8x^2+113}}{113}$$

is a balancing number [1, p.98] Letting

$$B = \frac{11x \pm \sqrt{8x^2+113}}{113},$$

we obtain

$$(113B - 11x)^2 = 8x^2 + 113,$$

which leads to the quadratic equation

$$x^2 - 22Bx + 113B^2 - 1 = 0.$$

whose solutions are $x = 11B \pm C$, (C is the Lucas-balancing, $C = \sqrt{8B^2+1}$). We further observe that

$$8(11B \pm C)^2 + 113 = (8B \pm 11C)^2.$$

Thus, the b_3-subbalancing numbers are of the form $11B \pm C$ and hence the set

$$\{11B_l + C_l, 11B_{l+1} - C_{l+1} : l = 0, 1, \cdots\} \tag{5}$$

lists all the b_3-subbalancing numbers.

The above discussion confirms that the set in (5) is the exhaustive list of b_3-subbalancing numbers and hence we have the following theorem:

Theorem 1 *The b_3-subbalancing numbers partition in two classes of the form* $11B_l+C_l, 11B_{l+1}-C_{l+1}, l \geq 0$.

2.2 Computation of b_5-subbalancing Numbers

In view of the definition 1, a natural number n is a b_5-subbalancing number if

$$1+2+\cdots+(n-1)+492=(n+1)+(n+2)+\cdots+(n+r) \tag{6}$$

for some natural number r, which is the b_5-subbalancer corresponding to n.

Example 2 From the following examples $2, 6, 47$ and 57 are b_5-subbalancing numbers with $29, 26, 26$ and 29 as corresponding b_5-subbalancers.

1. $1+492=3+4+\cdots+31$
2. $1+2+3+4+5+492=7+8+\cdots+32$
3. $1+2+\cdots+46+492=48+49+\cdots+73$
4. $1+2+\cdots+56+492=58+59+\cdots+86$

It is easy to see that if n is a b_5-subbalancing number then the corresponding b_5-subbalancer is $r=\frac{1}{2}[-(2n+1)+\sqrt{8n^2+3937}]$. We infer from the above discussion that, if x is a b_5-subbalancing number then $8x^2+3937$ is a perfect square. Thus, computation of b_5-subbalancing numbers reduces to solving the Diophantine equation

$$8x^2+3937=y^2. \tag{7}$$

To find all the b_5-subbalancing numbers one needs to solve the generalized Pell's equation $y^2-8x^2=3937$. The bounds for its fundamental solutions are $|y| \leq \sqrt{7874}<89$ and $0 \leq x \leq \sqrt{3937/8}<23$ (see [7]). Thus, we need to find those integers x in the interval $[0, 23)$ such that $8x^2+3937$ is a perfect square. This happens for $(x, y)=(2, \pm 63)$ and $(6, \pm 65)$ from which it is easy to see that there are four fundamental solutions $-63+2\sqrt{8}$, $-65+6\sqrt{8}$, $63+2\sqrt{8}$ and $65+6\sqrt{8}$ respectively. Using techniques similar to that used in [6], we get the following four classes of solutions for b_5-subbalancing numbers in terms of balancing and Lucas-balancing numbers:

$$\{63B_l+2C_l,\ 63B_{l+1}-2C_{l+1},\ 65B_l+6C_l,\ 65B_{l+1}-6C_{l+1}: l=0,1,\cdots\}$$

Thus, we have the following theorem:

Theorem 2 *The b_5-subbalancing numbers can be classified in four classes of the form* $63B_l+2C_l,\ 63B_{l+1}-2C_{l+1},\ 65B_l+6C_l,\ 65B_{l+1}-6C_{l+1}: l \geq 0$.

From the above discussion, it is clear that for some values of k, there can be more than two classes of b_k-subbalancing numbers. Obtaining each class is a difficult task; however, we employ the same techniques used by the author in [5] for obtaining two classes of solutions.

In the following theorems we explore two classes of b_k-subbalancing numbers corresponding to even and odd positive integer k.

Theorem 3 *For $m > 1$, the values of x satisfying the Diophantine equation*

$$1 + 2 + \cdots + (x-1) + b_{2m} = (x+1) + (x+2) + \cdots + (x+r)$$

for some r, may result in multiple classes. Two such classes of solutions are

$$(B_{m-1} + 5B_m)B_l + B_m C_l \quad \textit{and} \quad (B_{m-1} + 5B_m)B_{l+1} - B_m C_{l+1}$$

for $l \geqslant 0$.

Proof In view of (3), $8x^2 + 8b_{2m} + 1$ is perfect square. Since, $8B_m^2 + 8b_{2m} + 1$ is a perfect square (Lemma 1), the congruence

$$(8B_m^2 + 8b_{2m} + 1)x^2 \equiv B_m^2(8x^2 + 8b_{2m} + 1) \pmod{8b_{2m} + 1} \tag{8}$$

holds. Hence any x satisfying the congruences

$$\sqrt{8B_m^2 + 8b_{2m} + 1}\, x \equiv \pm B_m \sqrt{8x^2 + 8b_{2m} + 1} \pmod{8b_{2m} + 1} \tag{9}$$

is also a solution of the congruence (8). Thus, to obtain two classes of b_{2m}-subbalancing numbers, we need to solve the congruences (9).

Equation (9) implies that

$$\frac{x\sqrt{8B_m^2 + 8b_{2m} + 1} + B_m\sqrt{8x^2 + 8b_{2m} + 1}}{8b_{2m} + 1}$$

or

$$\frac{x\sqrt{8B_m^2 + 8b_{2m} + 1} - B_m\sqrt{8x^2 + 8b_{2m} + 1}}{8b_{2m} + 1}$$

is a natural number. Since

$$8\left[\frac{x\sqrt{8B_m^2 + 8b_{2m} + 1} \pm B_m\sqrt{8x^2 + 8b_{2m} + 1}}{8b_{2m} + 1}\right]^2 + 1$$
$$= \left[\frac{8xB_m \pm \sqrt{8x^2 + 8b_{2m} + 1}\sqrt{8B_m^2 + 8b_{2m} + 1}}{8b_{2m} + 1}\right]^2,$$

it follows that either

$$\frac{x\sqrt{8B_m^2 + 8b_{2m} + 1} + B_m\sqrt{8x^2 + 8b_{2m} + 1}}{8b_{2m} + 1}$$

or

$$\frac{x\sqrt{8B_m^2+8b_{2m}+1}-B_m\sqrt{8x^2+8b_{2m}+1}}{8b_{2m}+1}$$

is a balancing number [1]. Letting

$$B=\frac{(B_{m-1}+5B_m)x\pm B_m\sqrt{8x^2+8b_{2m}+1}}{8b_{2m}+1}$$

we get

$$[(B_{m-1}+5B_m)x-B(8b_{2m}+1)]^2=B_m^2(8x^2+8b_{2m}+1)$$

which can be transformed to the quadratic equation

$$x^2-2(B_{m-1}+5B_m)B\ x+B^2(8b_{2m}+1)-B_m^2=0$$

whose solutions are $x=(B_{m-1}+5B_m)B\pm B_mC$. We further observe that

$$8[(B_{m-1}+5B_m)B\pm B_mC]^2+8b_{2m}+1=[(B_{m-1}+5B_m)C\pm 8B_mB]^2.$$

Thus, two classes of b_{2m}-subbalancing numbers are $(B_{m-1}+5B_m)B_l+B_mC_l$ and $(B_{m-1}+5B_m)B_{l+1}-B_mC_{l+1}$ for $l\geq 0$. □

Theorem 4 *For $m\geq 1$, the values of x satisfying the Diophantine equation*

$$1+2+\cdots+(x-1)+b_{2m+1}=(x+1)+(x+2)+\cdots+(x+r)$$

may result in multiple classes. Two such classes of solutions are given by

$$(B_{m+1}+5B_m)B_l+B_mC_l \quad \textit{and} \quad (B_{m+1}+5B_m)B_{l+1}-B_mC_{l+1}$$

for $l\geqslant 0$.

Proof By virtue of equation (3), $8x^2+8b_{2m+1}+1$ is perfect square. Since, $8B_m^2+8b_{2m+1}+1$ is a perfect square (Lemma 1), the congruence

$$(8B_m^2+8b_{2m+1}+1)x^2\equiv B_m^2(8x^2+8b_{2m+1}+1)\ (\mathrm{mod}\ 8b_{2m+1}+1)$$

holds and is implied by

$$x\sqrt{8B_m^2+8b_{2m+1}+1}\equiv\pm B_m\sqrt{8x^2+8b_{2m+1}+1}\ (\mathrm{mod}\ 8b_{2m+1}+1)$$

and any solution of the latter congruence is a solution of the former and is a b_{2m+1}-subbalancing number. In view of the latter congruence

$$\frac{x\sqrt{8B_m^2+8b_{2m+1}+1}+B_m\sqrt{8x^2+8b_{2m+1}+1}}{8b_{2m+1}+1}$$

or

$$\frac{x\sqrt{8B_m^2+8b_{2m+1}+1}-B_m\sqrt{8x^2+8b_{2m+1}+1}}{8b_{2m+1}+1}$$

is a natural number. Since

$$8\left[\frac{x\sqrt{8B_m^2+8b_{2m+1}+1}\pm B_m\sqrt{8x^2+8b_{2m+1}+1}}{8b_{2m+1}+1}\right]^2+1$$
$$=\left[\frac{8xB_m\pm\sqrt{8x^2+8b_{2m+1}+1}\sqrt{8B_m^2+8b_{2m+1}+1}}{8b_{2m+1}+1}\right]^2$$

it follows that either

$$\frac{x\sqrt{8B_m^2+8b_{2m+1}+1}+B_m\sqrt{8x^2+8b_{2m+1}+1}}{8b_{2m+1}+1}$$

or

$$\frac{x\sqrt{8B_m^2+8b_{2m+1}+1}-B_m\sqrt{8x^2+8b_{2m+1}+1}}{8b_{2m+1}+1}$$

is a balancing number [1]. Letting

$$B=\frac{(B_{m+1}+5B_m)x\pm B_m\sqrt{8x^2+8b_{2m+1}+1}}{8b_{2m+1}+1}$$

we get

$$[(B_{m+1}+5B_m)x-B(8b_{2m+1}+1)]^2=B_m^2(8x^2+8b_{2m+1}+1),$$

which, on rearrangement, results in the quadratic equation

$$x^2-2(B_{m+1}+5B_m)B\ x+B^2(8b_{2m+1}+1)-B_m^2=0,$$

whose solutions are $x=(B_{m+1}+5B_m)B\pm B_mC$. We further observe that

$$8[(B_{m+1}+5B_m)B\pm B_mC]^2+8b_{2m+1}+1=[(B_{m+1}+5B_m)C\pm 8B_mB]^2.$$

Thus, two classes of b_{2m+1}-subbalancing numbers are $(B_{m+1}+5B_m)B_l+B_mC_l$ and $(B_{m+1}+5B_m)B_{l+1}-B_mC_{l+1}$ for $l\geq 0$. □

In view of the above theorems, for a fixed positive integer m, the b_{2m}-subbalancing numbers are given by $x_{2l}=(B_{m-1}+5B_m)B_l+B_mC_l$, $x_{2l+1}=(B_{m-1}+5B_m)B_{l+1}-B_mC_{l+1}$ and the b_{2m+1}-subbalancing numbers are $x_{2l}=(B_{m+1}+5B_m)B_l+B_mC_l$, $x_{2l+1}=(B_{m+1}+5B_m)B_{l+1}-B_mC_{l+1}$ for $l\geq 0$. Since balancing and Lucas-balancing numbers satisfy the recurrence relation $y_{n+1}=6y_n-y_{n-1}$, it follows that b_m-subbalancing numbers satisfy the recurrence relation $x_{n+2}=6x_n-x_{n-2}, n\geq 3$.

In the following theorem we give functions that transforms from balancing to subbalancing numbers and subbalancing numbers to balancing numbers.

Theorem 5 *For a balancing number x, $f(x)=(B_{m-1}+5B_m)x+B_m\sqrt{8x^2+1}$ and $g(x)=(B_{m-1}+5B_m)x-B_m\sqrt{8x^2+1}$ are b_{2m}-subbalancing numbers.*

We next find functions that transform b_{2m}-subbalancing numbers to balancing numbers. It is easy to check that the functions are strictly increasing in the domain $[0, \infty)$. Hence their inverses exist and are equal to

$$f^{-1}(y) = \frac{(B_{m-1} + 5B_m)y - B_m\sqrt{8x^2 + 8b_{2m} + 1}}{8b_{2m} + 1}$$

and

$$g^{-1}(y) = \frac{(B_{m-1} + 5B_m)y + B_m\sqrt{8x^2 + 8b_{2m} + 1}}{8b_{2m} + 1}.$$

The above discussion leads to the following theorems.

Theorem 6 *If $y = (B_{m-1} + 5B_m)B_n + B_mC_n$ is b_{2m}-subbalancing number then*

$$B_n = \frac{(B_{m-1} + 5B_m)y - B_m\sqrt{8y^2 + 8b_{2m} + 1}}{8b_{2m} + 1}$$

and

$$C_n = \frac{(B_{m-1} + 5B_m)\sqrt{8y^2 + 8b_{2m} + 1} - 8yB_m}{8b_{2m} + 1}.$$

Theorem 7 *If $y = (B_{m-1} + 5B_m)B_n - B_mC_n$ is b_{2m}-subbalancing number then*

$$B_n = \frac{(B_{m-1} + 5B_m)y + B_m\sqrt{8y^2 + 8b_{2m} + 1}}{8b_{2m} + 1}$$

and

$$C_n = \frac{(B_{m-1} + 5B_m)\sqrt{8y^2 + 8b_{2m} + 1} + 8yB_m}{8b_{2m} + 1}.$$

Theorem 8 *If x is a balancing number then $\alpha(x) = (B_{m+1} + 5B_m)x + B_m\sqrt{8x^2 + 1}$ and $\beta(x) = (B_{m+1} + 5B_m)x - B_m\sqrt{8x^2 + 1}$ are b_{2m+1}-subbalancing numbers.*

We next find functions that transform b_{2m+1}-subbalancing numbers to balancing numbers. It is easy to check that the functions are strictly increasing in the domain $[0, \infty)$. Hence their inverses exist and are equal to

$$\alpha^{-1}(y) = \frac{(B_{m+1} + 5B_m)y - B_m\sqrt{8x^2 + 8b_{2m} + 1}}{8b_{2m+1} + 1}$$

and

$$\beta^{-1}(y) = \frac{(B_{m+1} + 5B_m)y + B_m\sqrt{8x^2 + 8b_{2m} + 1}}{8b_{2m+1} + 1}.$$

The above discussion leads to the following theorems.

Theorem 9 *If $y = (B_{m+1} + 5B_m)B_n + B_mC_n$ is b_{2m+1}-subbalancing number then*

$$B_n = \frac{(B_{m+1} + 5B_m)y - B_m\sqrt{8y^2 + 8b_{2m} + 1}}{8b_{2m+1} + 1}$$

and

$$C_n = \frac{(B_{m+1} + 5B_m)\sqrt{8y^2 + 8b_{2m} + 1} - 8yB_m}{8b_{2m+1} + 1}.$$

Theorem 10 *If* $y = (B_{m+1} + 5B_m)B_n - B_m C_n$ *is* b_{2m+1}*-subbalancing number then*

$$B_n = \frac{(B_{m+1} + 5B_m)y + B_m\sqrt{8y^2 + 8b_{2m} + 1}}{8b_{2m+1} + 1}$$

and

$$C_n = \frac{(B_{m+1} + 5B_m)\sqrt{8y^2 + 8b_{2m} + 1} + 8yB_m}{8b_{2m+1} + 1}.$$

3 Conclusion

In this paper, we define D-subbalancing numbers by restricting D to cobalancing numbers. However, for many values of D, D-subbalancer numbers exist. For example, one can verify that a natural number x is a $6-$subbalancing number if and only if $8x^2 + 49$ is a perfect square and the values of x satisfying $8x^2 + 49 = y^2$ are $5-$gap balancing numbers [4]. Indeed, finding all feasible values of D is an interesting problem.

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