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Fuzzy closure matroids

Talal Al-Hawary
Department of Mathematics, Yarmouk University
Irbid, Jordan
e-mail: talalhawary@yahoo.com

Abstract In 1988, Geotschel and Voxman [1] introduced and explored the concept of fuzzy matroids via the notion of fuzzy independent sets. In 2013, Al-Hawary [2] introduced fuzzy matroids via the notion of fuzzy flats. In this paper, we introduce the notion of fuzzy closure matroids, a new class of fuzzy matroids. We study properties of this class and classify it. In addition, the class of fuzzy modular matroids is defined and characterized.

Keywords Fuzzy matroid; fuzzy flat; fuzzy closure; fuzzy strong map; fuzzy hesitant map.

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1 Introduction

Matroid theory has several interesting applications in system analysis, operations research and economics. Since most of the time the aspects of matroid problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy logic. The notion of fuzzy matroids was first introduced by Geotschel and Voxman in their landmark paper [1] using the notion of fuzzy independent set. The notion of fuzzy independent set was also explored in [3, 4]. Some constructions, fuzzy spanning sets, fuzzy rank and fuzzy closure axioms were also studied in [5–8]. Several other fuzzifications of matroids were also discussed in [9, 10]. Since the notion of flats in traditional matroids is one of the most significant notions that plays a very important rule in characterizing strong maps (see for example [11-13]), in [2], the notions of fuzzy flats and fuzzy C flats were introduced and several examples were provided. Thus in [2], fuzzy matroids are defined via fuzzy flats axioms and it was shown that the levels of the fuzzy matroid introduced are indeed crisp matroids. Moreover, fuzzy strong maps and fuzzy hesitant maps are introduced and explored. We remark that this approach in [2] is different from those mentioned above.

Our goal in this paper is to introduce and study the notion of fuzzy closure matroids. Several properties of this class are discussed. In addition, the class of fuzzy modular matroids is defined and characterized.

Let E be any be any non-empty set. By $\wp(1)$ we denote the set of all fuzzy sets on E. That is $\wp(1) = [0, 1]^E$, which is a completely distributive lattice. Thus let 0^E and 1^E denote its greatest and smallest elements, respectively. That is $0^E(e) = 0$ and $1^E(e) = 1$ for every $e \in E$. A fuzzy set μ_1 is a subset of μ_2 , written $\mu_1 \leq \mu_2$, if $\mu_1(e) \leq \mu_2(e)$ for all $e \in E$. If $\mu_1 \leq \mu_2$ and $\mu_1 \neq \mu_2$, then μ_1 is a proper subset of μ_2, written $\mu_1 < \mu_2$. When F is a subset of fuzzy sets and $\mu_1, \mu_2 \in F$, we write $\mu_1 \prec \mu_2$ if $\mu_1 < \mu_2$ and there does not exist $\mu_3 \in F$ such that $\mu_1 < \mu_3 < \mu_2$. Finally, $\mu_1 \vee \mu_2 = \sup\{\mu_1, \mu_2\}$ and $\mu_1 \wedge \mu_2 = \inf\{\mu_1, \mu_2\}$.

Next we recall some basic definitions and results from [2].

Definition 1 Let E be a finite set and let $\mathfrak{F}$ be a finite family of fuzzy sets satisfying the following three conditions:

(i) $1^E \in \mathfrak{F}$.
(ii) If $\mu_1, \mu_2 \in \mathfrak{F}$, then $\mu_1 \wedge \mu_2 \in \mathfrak{F}$.
(iii) If $\mu \in \mathfrak{F}$ and $\mu_1, \mu_2, ..., \mu_n$ are all minimal members of $\mathfrak{F}$ such that $\mu \prec \mu_i$ for all $i = 1, 2, ..., n$, then $\vee_{i=1}^{n} \mu_i = 1^E$.
Then the system $FM = (E, \mathfrak{F})$ is called *fuzzy matroid* and the elements of $\mathfrak{F}$ are *fuzzy flats of* FM.

Definition 2 Let E be any set with n-elements and $\mathfrak{F} = \{\chi_A : A \leq E, |A| = n \text{ or } |A| < m\}$ where m is a positive integer such that $m \leq n$. Then $(E, \mathfrak{F})$ is a fuzzy matroid called the *fuzzy uniform matroid on n-elements and rank m*, denoted by $F_{m,n}$. $F_{m,m}$ is called the *free fuzzy uniform matroid on n-elements.*

We remark that the rank notion in the preceding definition coincides with that in [6].

Definition 3 Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and $\mu \in \wp(1)$. Then the *fuzzy closure of* μ is $\bar{\mu} = \wedge_{\substack{\lambda \in \mathfrak{F} \\ \mu \leq \lambda}} \lambda$.

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Definition 4 A fuzzy matroid FM is called *a closure fuzzy matroid* if $\overline{\mu} \vee \overline{\eta} = \overline{\mu \vee \eta}$ for all fuzzy subsets μ and η of $E(FM)$.

Consider the fuzzy uniform matroid $F_{2,3}$. If μ and η are any two fuzzy elements of its ground set, then each is a fuzzy flat and hence each equals its fuzzy closure. On the other hand, $\overline{\mu \vee \eta} = E(U_{2,3})$. Thus, $F_{2,3}$ is not a fuzzy closure matroid.

Definition 5 A fuzzy matroid FM is fuzzy modular if and only if every fuzzy flat μ in FM is fuzzy modular, that is if for every other fuzzy flat η, $r(\mu)+r(\eta) = r(\mu\vee\eta)+r(\mu\wedge\eta)$.

Equivalently, FM is a fuzzy modular matroid if and only if for all flats μ and η such that $\mu \wedge \eta = \overline{0}$, we have $r(\mu \vee \eta) = r(\mu) + r(\eta)$. Note that the fuzzy uniform matroid $F_{2,3}$ is a fuzzy modular matroid.

Next, we prove the following lemma which will be a basic tool in distinguishing fuzzy closure matroids from fuzzy modular matroids.

Lemma 1 *A fuzzy matroid FM is a fuzzy closure matroid if and only if unions of fuzzy flats in FM are again fuzzy flats in FM.*

Proof Let FM be a fuzzy closure matroid. If μ_1 and μ_2 are fuzzy flats in FM, then $\mu_1 \vee \mu_2 = \overline{\mu_1} \vee \overline{\mu_2} = \overline{\mu_1 \vee \mu_2}$ which is a fuzzy flat in FM.

Conversely, let FM be a fuzzy matroid in which unions of fuzzy flats are fuzzy flats, and let α and β be subsets of $E(FM)$. As $\overline{\alpha}$ and $\overline{\beta}$ are fuzzy flats in FM, $\overline{\alpha} \vee \overline{\beta}$ is a fuzzy flat in FM. Now $\alpha \vee \beta \leq \overline{\alpha} \vee \overline{\beta}$ and as $\overline{\alpha \vee \beta}$ is the smallest fuzzy flat containing $\alpha \vee \beta$, $\overline{\alpha \vee \beta} \leq \overline{\alpha} \vee \overline{\beta}$. On the other hand, $\overline{\alpha} \vee \overline{\beta} \leq \overline{\alpha \vee \beta}$ because $\overline{\alpha}$ and $\overline{\beta}$ are subsets of $\overline{\alpha \vee \beta}$. Therefore, $\overline{\alpha \vee \beta} = \overline{\alpha} \vee \overline{\beta}$. □

Theorem 1 *A fuzzy closure matroid is a fuzzy modular matroid.*

Proof Let FM be a fuzzy closure matroid, and let α and β be fuzzy flats in FM such that $\alpha \wedge \beta = \overline{0}$. By the fuzzy semimodular property, we have

$$r(\alpha \vee \beta) \leq r(\alpha) + r(\beta). \tag{1}$$

On the other hand, assume that $r(\alpha) = n$ and $r(\beta) = m$ where $m \geq n$. Then there exist flats $\alpha_1, \alpha_2, ..., \alpha_{n-1}$ and $\beta_1, \beta_2, ..., \beta_{m-1}$ such that

$$\overline{0} < \alpha_1 < \alpha_2 < ... < \alpha_{n-1} < \alpha \text{ and } \overline{0} < \beta_1 < \beta_2 < ... < \beta_{m-1} < \beta.$$

Moreover, these are maximal chains of flats in α and β, respectively. By Lemma 1, $\alpha \vee \beta$, $\alpha_i \vee \beta_i$ and $\alpha \vee \beta_j$ are fuzzy flats in FM, for all $i = 1, 2, ..., n-1$ and $j = n, n+1, ..., m-1$. Hence,

$$\overline{0} < \alpha_1 \vee \beta_1 < \alpha_2 \vee \beta_2 < ... < \alpha_{n-1} \vee \beta_{n-1} < \alpha \vee \beta_n < \alpha \vee \beta_{n+1} < ... < \alpha \vee \beta_m < \alpha \vee \beta$$

is a chain of fuzzy flats in $\alpha \vee \beta$ of size $n + m$. Therefore, $r(\alpha \vee \beta) \geq n + m = r(\alpha) + r(\beta)$. □

Next, we characterize fuzzy closure matroids in terms of the fuzzy rank function.

Theorem 2 *The following statements are equivalent for a fuzzy matroid FM :*

(i) *FM is a fuzzy closure matroid,*

(ii) *FM has no fuzzy submatroid isomorphic to $F_{2,3}$ and for all fuzzy subsets α and β of $E(FM)$ such that $\overline{\alpha} \wedge \overline{\beta} = \overline{0}$ and $r(\alpha) + r(\beta) = r(\alpha \vee \beta)$,*

(iii) *FM has no fuzzy submatroid isomorphic to $F_{2,3}$ and for all fuzzy subsets α and β of $E(FM)$ such that $\overline{\alpha} \wedge \overline{\beta} = \overline{0}$ and $\overline{\overline{\alpha} \vee \overline{\beta}} = 1$, $r(\alpha) + r(\beta) = r(FM)$.*

Proof Obviously, (ii) implies (iii). We shall show that (i) implies (ii), (ii) implies (i) and (iii) implies (ii).

Assume that (i) holds. Let α and β be fuzzy subsets of $E(FM)$ such that $\overline{\alpha} \wedge \overline{\beta} = 0$. By Theorem 1, FM is fuzzy modular and as $\overline{\alpha}$ and $\overline{\beta}$ are fuzzy flats in FM such that $\overline{\alpha} \wedge \overline{\beta} = \overline{0}$, then $r(\overline{\alpha}) + r(\overline{\beta}) = r(\overline{\alpha} \vee \overline{\beta})$. Thus, $r(\alpha) + r(\beta) = r(\alpha \vee \beta)$. If FM has a fuzzy submatroid FN isomorphic to $F_{2,3}$, then there exist x and y in $E(FN)$ such that $\overline{x} = x$, $\overline{y} = y$ and $\overline{x \vee y} \neq x \vee y$, a contradiction. Thus, (ii) holds.

Assume that (ii) holds, and let α and β be fuzzy flats in FM. By Lemma 1, we need only show that $\alpha \vee \beta$ is a fuzzy flat in FM. That is, the set $\gamma := \overline{\alpha \vee \beta} \backslash \alpha \vee \beta = 0$.

If $\alpha \leq \beta$ or $\beta \leq \alpha$, then clearly $\gamma = 0$. If $r(\alpha) = 0$ or $r(\beta) = 0$, then $\alpha = \overline{0}$ or $\beta = \overline{0}$, respectively. Hence, $\gamma = 0$. Suppose α and β are fuzzy flats of nonzero ranks such that none is a subset of the other. Then there exist $a \in \alpha - \beta$ and $b \in \beta - \alpha$ such that $r(a) = r(b) = 1$. As $\overline{a} \wedge \overline{b} = \overline{0}$, by (ii), $r(a \vee b) = r(\overline{a} \vee \overline{b}) = 2$. If there exists $c \in \gamma$, then $r(c) = 1$ and $\overline{a} \wedge \overline{c} = \overline{c} \wedge \overline{b} = \overline{0}$ and by (ii), $r(a \vee c) = r(\overline{a} \vee \overline{c}) = 2 = r(\overline{c} \vee \overline{b}) = r(b \vee c)$. As $r(\alpha \vee \beta \vee c) = r(\alpha \vee \beta)$, $r(a \vee b \vee c) = 2$. Thus, $FM|_{\{a,b,c\}} \cong F_{2,3}$, a contradiction. Hence, $\gamma = 0$. Therefore, $\alpha \vee \beta$ is a flat and then (i) holds.

Assume that (iii) holds, but (ii) does not. That is, suppose there exist fuzzy subsets α and β of $E(FM)$ such that $\overline{\alpha} \wedge \overline{\beta} = \overline{0}$ but $r(\alpha) + r(\beta) \neq r(\alpha \vee \beta)$. If $\overline{\overline{\alpha} \vee \overline{\beta}} = 1$ then by (iii), $r(\alpha \vee \beta) \geq r(\alpha) + r(\beta) = r(FM)$ and hence equality holds throughout. So, we may assume

$\overline{\overline{\alpha}\vee\overline{\beta}}$ is a proper subset of $E(FM)$. Thus, $r(\alpha\vee\beta)=r(\overline{\overline{\alpha}\vee\overline{\beta}})<r(FM)$. Let γ be a fuzzy basis for the contraction $FM/(\overline{\alpha}\vee\overline{\beta})$. We show $\overline{\gamma\vee\overline{\beta}}\wedge\overline{\alpha}=\overline{0}$ and $\overline{\overline{\gamma\vee\overline{\beta}}\vee\overline{\alpha}}=1$. Clearly $\overline{\overline{\gamma\vee\overline{\beta}}\vee\overline{\alpha}}=1$. Suppose a is a non-loop of $\overline{\gamma\vee\overline{\beta}}\wedge\overline{\alpha}$. Then $a\notin\overline{\beta}$ since $\overline{\alpha}\wedge\overline{\beta}=\overline{0}$. Thus if δ is a fuzzy basis of $\overline{\beta}$, then $\delta\vee a$ is fuzzy independent in the restriction $FM|(\overline{\alpha}\vee\overline{\beta})$. But γ is fuzzy independent in $FM/(\overline{\alpha}\vee\overline{\beta})$. Therefore, $\gamma\vee\delta\vee a$ is fuzzy independent in FM, a contradiction to the fact that $a\in\overline{\gamma\vee\overline{\beta}}$. Hence $\overline{\gamma\vee\overline{\beta}}\wedge\overline{\alpha}=\overline{0}$. Now by (iii),

$$\begin{aligned}
r(FM) &= r(\alpha)+r(\overline{\gamma\vee\overline{\beta}})\\
&= r(\alpha)+r(\gamma\vee\overline{\beta})\\
&= r(\alpha)+r(\beta)+|\gamma|\\
&= r(\alpha)+r(\beta)+r(FM)-r(\alpha\vee\beta).
\end{aligned}$$

Therefore $r(\alpha)+r(\beta)=r(\alpha\vee\beta)$. This contradiction completes the proof of the theorem.□

The following is an immediate result of Theorem 2.

Corollary 1 *A fuzzy matroid FM is a fuzzy closure matroid if and only if FM is a fuzzy modular matroid that has no fuzzy submatroid isomorphic to $F_{2,3}$.*

Theorem 3 *A fuzzy matroid FM is a fuzzy closure matroid if and only if $\widetilde{FM}$ is free.*

Proof Let FM be a fuzzy matroid such that $\widetilde{FM}$ is free, and let α and β be fuzzy flats in FM. By Lemma 1, we need only show $\alpha\vee\beta$ is a fuzzy flat. Let T be the set of all fuzzy loops and parallel elements in $\alpha\vee\beta$. Then as $(\alpha\vee\beta)-T$ is a subset of $E(\widetilde{FM})$, $(\alpha\vee\beta)-T$ is a fuzzy flat in $\widetilde{FM}$ and hence, $\alpha\vee\beta=T\vee(\alpha\vee\beta)-T$ is a fuzzy flat in FM. Therefore, FM is a fuzzy closure matroid.

Conversely, let FM be a fuzzy closure matroid such that $\left|E(\widetilde{FM})\right|=n$. Then every element in $E(\widetilde{FM})$ is a fuzzy flat in $\widetilde{FM}$. If $\alpha\leq E(\widetilde{FM})$, then $\alpha=\underset{x\in\alpha}{\vee}x$. By Lemma 1 and since FM is a fuzzy closure matroid, α is a fuzzy flat of $\widetilde{FM}$. Hence, $\widetilde{FM}\cong F_{n,n}$. □

Next, we show that direct sums of fuzzy closure matroids are fuzzy closure matroids. Moreover, we show that if the direct sum of loopless fuzzy matroids FM_1 and FM_2 is a fuzzy closure matroid, then FM_1 and FM_2 are fuzzy closure matroids.

Theorem 4 *Let FM_1 and FM_2 be loopless fuzzy matroids on disjoint ground sets. Then $FM_1\oplus FM_2$ is a fuzzy closure matroid if and only if FM_1 and FM_2 are fuzzy closure matroids.*

Proof Assume $FM_1\oplus FM_2$ is a fuzzy closure matroid, and let α and β be subsets of $E(FM_1)$. Then $\overline{\alpha}^{FM_1}$ and $\overline{\beta}^{FM_1}$ are both fuzzy flats in FM_1 and as FM_1 is a loopless fuzzy matroid, $\overline{\alpha}^{FM_1}$ and $\overline{\beta}^{FM_1}$ are fuzzy flats in $FM_1\oplus FM_2$. By Lemma 1 and since $FM_1\oplus FM_2$ is a fuzzy closure matroid, $\overline{\alpha}^{FM_1}\vee\overline{\beta}^{FM_1}$ is a fuzzy flat in $FM_1\oplus FM_2$. Thus, $\overline{\alpha}^{FM_1}\vee\overline{\beta}^{FM_1}=(\overline{\alpha}^{FM_1}\vee\overline{\beta}^{FM_1})\wedge E(FM_1)$ is a fuzzy flat in FM_1. Therefore, FM_1 is a fuzzy closure matroid. Similarly, FM_2 is a fuzzy closure matroid.

Conversely, assume FM_i is a fuzzy closure matroid for $i = 1, 2$, and let α and β be fuzzy subsets of $E(FM_1) \vee E(FM_2)$. Then $\overline{\alpha}^{FM_1 \oplus FM_2} \wedge E(FM_i)$ and $\overline{\beta}^{FM_1 \oplus FM_2} \wedge E(FM_i)$ are fuzzy flats in FM_i. By Lemma 1 and since FM_i is a fuzzy closure matroid,

$$(\overline{\alpha}^{FM_1 \oplus FM_2} \vee \overline{\beta}^{FM_1 \oplus FM_2}) \wedge E(FM_i) = (\overline{\alpha}^{FM_1 \oplus FM_2} \wedge E(FM_i)) \vee (\overline{\beta}^{FM_1 \oplus FM_2} \wedge E(FM_i))$$

is a fuzzy flat in FM_i. Thus, $\overline{\alpha}^{FM_1 \oplus FM_2} \vee \overline{\beta}^{FM_1 \oplus FM_2}$ is a fuzzy flat in $FM_1 \oplus FM_2$. Therefore by Lemma 1, $FM_1 \oplus FM_2$ is a fuzzy closure matroid. □

The following result, in which we classify all fuzzy closure matroids, follows immediately from Theorem 3 combined with Theorem 4.

Corollary 2 *A fuzzy matroid FM is a fuzzy closure matroid if and only if FM is the direct sum of a parallel extension of a free fuzzy matroid and $F_{0,m}$ for some positive integer m.*

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