Fuzzy L-closed sets

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Abstract Our goal in this paper is to introduce the relatively new notions of fuzzy L-closed and fuzzy L-generalized closed sets. Several properties and connections to other well-known weak and strong fuzzy closed sets are discussed. Fuzzy L-generalized continuous and fuzzy L-generalized irresolute functions and their basic properties and relations to other fuzzy continuities are explored.

Keywords Fuzzy L-open set; fuzzy L-closed set; fuzzy L-generalized closed set; fuzzy L-generalized continuous function.

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1 Introduction

For a set $X$, a fuzzy set in $X$ is a function $\lambda : X \to [0,1]$. Here $\lambda(x)$ represents the degree of membership of $x$ in the fuzzy subset $A$ of $X$ and by $\chi_A$, we mean the fuzzy set that maps every element in $A$ to 1 and every element outside $A$ to 0. Fuzzy topological spaces (simply, spaces) were first introduced by [1, 2]. A fuzzy topology on a set $X$ is a collection $T$ of subsets of $X$ satisfying: $0, 1 \in T$, $T$ is closed under formation of finite intersections and is closed under formation of arbitrary unions. Fuzzy topological spaces were studied by several authors, see for example [1, 3–8]. Let $(X, \mathcal{T})$ be a fuzzy topological space. If $\lambda$ is a fuzzy set, then the closure of $\lambda$ (the smallest fuzzy closed set containing $\lambda$) and the interior of $\lambda$ (the largest fuzzy open set in $\lambda$) will be denoted by $\text{Cl}_T(\lambda)$ and $\text{Int}_T(\lambda)$, respectively. If no ambiguity appears, we use $\lambda$ and $\bar{\lambda}$ instead, respectively. A fuzzy set $\lambda$ is called fuzzy semiopen [6] if there exists a fuzzy open set $\mu$ such that $\mu \leq \lambda \leq \text{Cl}_T(\mu)$. Clearly $\lambda$ is a fuzzy semiopen set if and only if $\lambda \leq \text{Cl}_T(\text{Int}_T(\lambda))$. A complement of a fuzzy semiopen set is called fuzzy semiclosed. The fuzzy semi-interior of $\lambda$ is the union of all fuzzy semi-open subsets contained in $\lambda$ and is denoted by $s\text{Int}(\lambda)$. $\lambda$ is called fuzzy preopen if $\lambda \leq \text{Int}_T(\text{Cl}_T(\lambda))$. A complement of a fuzzy preopen set is called fuzzy semiclosed. Fuzzy generalization of $\lambda$ is the union of all fuzzy semi-open subsets contained in $\lambda$ and is denoted by $s\text{Int}(\lambda)$. $\lambda$ is called fuzzy preopen if $\lambda \leq \text{Int}_T(\text{Cl}_T(\lambda))$. A complement of a fuzzy preopen set is called fuzzy semiclosed.

We introduce the relatively new notions of fuzzy L-closed sets, which is closely related to the class of fuzzy closed subsets. We show that the collection of all fuzzy L-open
subsets of a space \((X, \mathcal{T})\) forms a fuzzy topology that is finer than \(\mathcal{T}\) and we investigate several characterizations of fuzzy \(L\)-open and fuzzy \(L\)-closed notions via the operations of interior and closure. In section 3, we introduce the notion of fuzzy \(L\)-generalized closed sets and study connections to other weak and strong forms of fuzzy generalized closed sets. In addition several interesting properties and constructions of fuzzy \(L\)-generalized closed sets are discussed. Section 4 is devoted to introducing and studying fuzzy \(L\)-generalized continuous and fuzzy \(L\)-generalized irresolute functions and connections to other similar forms of fuzzy continuity.

2 Fuzzy \(L\)-closed sets

We begin this section by introducing the notions of fuzzy \(L\)-open and fuzzy \(L\)-closed subsets.

**Definition 1** Let \(\mu\) be a fuzzy subset of a space \((X, \tau)\). The fuzzy \(L\)-interior of \(\mu\) is the union of all fuzzy subsets of \(X\) whose closures are contained in \(\text{Int}(\mu)\), and is denoted by \(\text{Int}_L(\mu)\). The fuzzy \(L\)-closure of \(\mu\) is \(\text{Cl}_L(\mu)\) is the smallest fuzzy closed set containing \(\mu\).

Clearly \(\text{Int}_L(\mu) \subseteq \text{Int}(\mu) \subseteq \mu\) and hence every fuzzy \(L\)-open set is fuzzy open and thus every fuzzy \(L\)-closed set is fuzzy closed, but the converses needs not be true.

**Example 1** Let \(X = \{a, b, c\}\) and \(\tau = \{0, 1, \chi(a), \chi(b), \chi(a,b)\}\). Then \(\chi(a,c)\) is a fuzzy open set that is not a fuzzy \(L\)-open but \(\text{Int}_L(\chi(a,c)) = 0\).

Next, we show that the collection of all fuzzy \(L\)-open subsets of a space \((X, \tau)\) forms a fuzzy topology \(\tau_{FL}\) that is finer than \(\tau\).

**Theorem 1** If \((X, \tau)\) is a fuzzy space, then \((X, \tau_{FL})\) is a fuzzy space such that \(\tau \supseteq \tau_{FL}\).

**Proof** We only need to show \((X, \tau_{FL})\) is a fuzzy space. Clearly 0 and 1 are fuzzy \(L\)-open sets. If \(\mu, \gamma \in \tau_{FL}\), then \(\mu = \text{Int}_L(\mu)\) and \(\gamma = \text{Int}_L(\gamma)\). Now \(\text{Int}_L(\mu \cap \gamma) = \text{Int}_L(\mu) \cap \text{Int}_L(\gamma) = \text{Int}(\mu \cap \gamma)\). Thus \(\text{Int}_L(\mu \cap \gamma) \supseteq \text{Int}_L(\mu) \land \text{Int}_L(\gamma)\), and so \(\mu \land \gamma \in \tau_{FL}\).

If \(\{\mu_\alpha : \alpha \in \Delta\}\) is a collection of fuzzy \(L\)-open subsets of \(X\), then for every \(\alpha \in \Delta\), \(\text{Int}_L(\mu_\alpha) = \mu_\alpha\). Hence

\[
\text{Int}_L(\bigvee_{\alpha \in \Delta} \mu_\alpha) = \bigvee\{\theta \in \tau : \text{Cl}(\theta) \subseteq \text{Int}_L(\bigvee_{\alpha \in \Delta} \mu_\alpha)\} \\
\supseteq \bigvee\{\theta \in \tau : \text{Cl}(\theta) \subseteq \bigvee_{\alpha \in \Delta} \text{Int}_L(\mu_\alpha)\} \\
\supseteq \bigvee\{\theta \in \tau : \text{Cl}(\theta) \subseteq \mu_\alpha\} \quad \text{for every } \alpha \in \Delta \\
= \text{Int}_L(\mu_\alpha) \quad \text{for every } \alpha \in \Delta \\
= \mu_\alpha \quad \text{for every } \alpha \in \Delta.
\]

Hence \(\bigvee_{\alpha \in \Delta} \mu_\alpha \subseteq \text{Int}_L(\bigvee_{\alpha \in \Delta} \mu_\alpha)\) and thus \(\bigvee_{\alpha \in \Delta} \mu_\alpha\) is fuzzy \(L\)-open.

In classical topology, a set is always contained in its closure, but this is not the case in \(\tau_{FL}\). Next we show that \(\mu \subseteq \text{Cl}(\mu)\) needs not be true.

**Example 2** Let \(X = \{a, b, c, d\}\) and \(\tau = \{0, 1, \chi(a), \chi(a,b), \chi(c,d), \chi(a,c,d)\}\). Then \(\chi(c) \subseteq \chi(a,b,c)\), but \(\chi(c) \not\subseteq \text{Cl}_{FL}(\chi(a,b,c))\).
One might think that a fuzzy subset $\mu$ of a fuzzy space $X$ is fuzzy L-closed if and only if $\mu = Cl_L(\mu)$, but this is not true as shown in the next example.

**Example 3** Consider the space in Example 2 and consider $\mu = \chi_{\{b,c\}}$. Since $Cl(\chi_{\{a\}}) = \chi_{\{a,c\}}, \chi_{\{a\}} \notin Cl_L(\mu)$. Since $Cl_L(\mu) = \mu$, but $\mu$ is not a fuzzy L-open set.

**Lemma 1** For any fuzzy set $\mu$ of a fuzzy space $X$,

(i) $Int(\mu) \leq Cl_L(\mu)$.

(ii) $Int(\mu) = 0$ if and only if $Cl_L(\mu) = 0$.

**Proof**

(i) $\lambda \notin Cl_L(\mu)$ implies that there exists a fuzzy open set $\theta$ containing $\lambda$ such that $Cl(\theta) \land Int(\mu) = 0$. Hence $\lambda \notin Int(\mu)$.

(ii) If $\lambda \leq Cl_L(\mu)$, then for every fuzzy open subset $\theta$ containing $\lambda$, $Cl(\theta) \land Int(\mu) \neq 0$.

Hence there exists $\gamma \leq Cl(\theta) \land Int(\mu)$ and as $Int(\mu)$ is fuzzy open, $\theta \land Int(\mu) \neq 0$.

Therefore $Int(\mu) \neq 0$.

Conversely if $Cl_L(\mu) = 0$, then by (i) as $Int(\mu) \leq Cl_L(\mu)$, $Int(\mu) = 0$.

**Lemma 2** The union of a fuzzy open set with a fuzzy L-open-set is fuzzy open.

**Proof** Let $\mu$ be an open set and $\eta$ be a fuzzy L-open set. For all $\gamma \leq \mu \lor \eta$, $\gamma \leq \mu$ or $\gamma \leq \eta$ and so $\gamma \leq Int(\mu) \land Int(\eta)$ or $\gamma \leq Int_L(\mu) \land Int(\eta) \leq Int(\mu \lor \eta)$.

**Corollary 1** The intersection of a fuzzy closed set with a fuzzy L-closed set is fuzzy closed.

**Lemma 3** If $\lambda$ is a fuzzy semiopen set of a fuzzy space $X$, $Cl_L(\mu) = Cl(\mu)$.

**Proof** If $\theta$ is a fuzzy open set containing $\lambda$ such that $Cl(\theta) \land Int(\mu) \neq 0$, then there exists $\gamma \leq Cl(\theta) \land Int(\mu)$. Thus $\theta \land Int(\mu) \neq 0$ and so $\theta \land Int(\mu) \neq 0$. Therefore $Cl_L(\mu) \leq Cl(\mu)$.

Conversely if for every fuzzy open set $\theta$ containing $\mu$ we have $\theta \land \mu \neq 0$, $\theta \land Int(Cl(\mu)) \neq 0$, since $\mu$ is fuzzy semiopen. Thus there exists $\gamma \leq \theta \land Int(\mu)$ and so $\theta \land Int(\mu) \neq 0$ which implies that $Cl(\theta) \land Int(\mu) \neq 0$. Hence $Cl(\mu) \leq Cl_L(\mu)$.

**Corollary 2**

(i) For any fuzzy subset $\mu$ of $X$, $Cl_L(\mu) \leq Cl(\mu)$.

(ii) If $\mu$ is a fuzzy semiopen subset of a space $X$, then $\mu \leq Cl_L(\mu)$.

**Lemma 4** If $\mu$ is a fuzzy L-closed set in a fuzzy space $X$, then $Cl_L(\mu) \leq \mu$.

**Proof** If $\mu$ is a fuzzy L-closed subset, then $\mu$ is fuzzy closed and thus by Corollary 2 (i), $Cl_L(\mu) \leq \mu$.

Next, we show that a fuzzy preclosed set that is also fuzzy semiopen equals its fuzzy L-closure.

**Theorem 2** If $\mu$ is a fuzzy regular closed subset of a fuzzy space $X$, then $Cl_L(\mu) \leq \mu$.

**Proof** $Cl_L(\mu) \leq Cl(\mu) \leq Cl(Cl(Int(\mu))) = Cl(Int(\mu)) \leq \mu$. This together with Corollary 2 implies that $\mu = Cl_L(\mu)$. 

□
3 Fuzzy L-generalized closed sets

In this section, we introduce the notion of fuzzy L-generalized closed set. Moreover, several interesting properties and constructions of these subsets are discussed.

**Definition 2** A fuzzy subset $\mu$ of a fuzzy space $X$ is called fuzzy L-generalized closed set if whenever $\theta$ is a fuzzy open subset containing $\mu$, we have $\text{Cl}_L(\mu) \leq \theta$. $\mu$ is fuzzy L-generalized open if $1 - \mu$ is fuzzy L-generalized closed set.

**Theorem 3** A subset $\mu$ of $(X, \tau)$ is fuzzy L-generalized open if and only if $\eta \leq \text{Int}_L(\mu)$, whenever $\eta \leq \mu$ and $\eta$ is fuzzy closed in $(X, \tau)$.

**Proof** Let $\mu$ be a fuzzy L-generalized open set and $\eta$ be a fuzzy closed subset such that $\eta \leq \mu$. Then $1 - \mu \leq 1 - \eta$. As $1 - \mu$ is fuzzy L-generalized closed set and as $1 - \eta$ is fuzzy open, $\text{Cl}_L(1 - \mu) \leq 1 - \eta$. So $\eta \leq 1 - \text{Cl}_L(1 - \mu) = \text{Int}_L(\mu)$.

Conversely if $1 - \mu \leq \theta$ where $\theta$ is fuzzy open, then the fuzzy closed set $1 - \theta \leq \mu$. Thus $1 - \theta \leq \text{Int}_L(\mu) = 1 - \text{Cl}_L(1 - \mu)$ and so $\text{Cl}_L(1 - \mu) \leq \theta$. $\square$

Next we show that every fuzzy L-closed set is fuzzy L-generalized closed. Moreover, the class of fuzzy generalized closed sets is properly placed between the classes of fuzzy semiopen sets that are fuzzy closed and fuzzy L-generalized closed sets. Clearly every fuzzy closed set that is fuzzy semiopen, by Lemma 1, is a fuzzy L-closed set. A fuzzy closed set is trivially fuzzy generalized closed and every fuzzy generalized closed set is fuzzy L-generalized closed by Corollary 2 (i).

The following result follows from Corollary 2 (i) and the fact that every fuzzy L-closed set is fuzzy L-closed:

**Lemma 5** Every fuzzy L-closed set is fuzzy L-generalized closed.

The converse of the preceding result needs not be true.

**Example 4** Let $X = \{a, b, c, d\}$ and $\tau = \{0, 1, \chi_{\{b\}}, \chi_{\{c\}}, \chi_{\{b,c\}}, \chi_{\{a,b\}}, \chi_{\{a,b,c\}}\}$. Then as $\text{Cl}_L(\chi_{\{a\}}) = 0, \chi_{\{a\}}$ is fuzzy L-generalized closed, but it is not fuzzy L-closed and not fuzzy generalized closed and hence not fuzzy closed. Also $\chi_{\{b,d\}}$ is an an fuzzy generalized closed set that is not fuzzy closed.

The following is an immediate result from Lemma 1:

**Theorem 4** If $\mu$ is a fuzzy semiopen subset of a space $X$, the following are equivalent:

1. $\mu$ is fuzzy L-generalized closed;
2. $\mu$ is fuzzy generalized closed.

Its clear that if $\mu \leq \gamma$, then $\text{Int}_L(\mu) \leq \text{Int}_L(\gamma)$ and $\text{Cl}_L(\mu) \leq \text{Cl}_L(\gamma)$.

**Lemma 6** If $\mu$ and $\gamma$ are fuzzy sets in a fuzzy space $X$, then $\text{Cl}_L(\mu) \vee \text{Cl}_L(\gamma) = \text{Cl}_L(\mu \vee \gamma)$ and $\text{Cl}_L(\mu \wedge \gamma) \leq \text{Cl}_L(\mu) \wedge \text{Cl}_L(\gamma)$.

**Proof** Since $\mu$ and $\gamma$ are subsets of $\mu \vee \gamma$, $\text{Cl}_L(\mu) \vee \text{Cl}_L(\gamma) \leq \text{Cl}_L(\mu \vee \gamma)$. On the other hand, if $\eta \leq \text{Cl}_L(\mu \vee \gamma)$ and $\theta$ is a fuzzy open set containing $\eta$, then $\text{Cl}(\theta) \wedge \text{Int}(\mu \vee \gamma) \neq 0$. Hence either $\text{Cl}(\theta) \wedge \text{Int}(\mu) \neq 0$ or $\text{Cl}(\theta) \wedge \text{Int}(\gamma) \neq 0$. Thus $\eta \leq \text{Cl}_L(\mu \vee \gamma)$. Finally since $\mu \wedge \gamma$ is a fuzzy subset of $\mu$ and $\gamma$, $\text{Cl}_L(\mu \wedge \gamma) \leq \text{Cl}_L(\mu) \wedge \text{Cl}_L(\gamma)$. $\square$

**Corollary 3** Finite union of fuzzy L-generalized closed sets is fuzzy L-generalized closed.

While the finite intersection of fuzzy L-generalized closed sets needs not be fuzzy L-generalized closed.
Example 5 Let \( X = \{a, b, c, d, e\} \) and \( \tau = \{0, 1, \chi_{\{b\}}, \chi_{\{c\}}, \chi_{\{b, c\}}, \chi_{\{a, b\}}, \chi_{\{a, b, c\}}\} \). Then \( \lambda = \chi_{\{a, c, d\}} \) and \( \mu = \chi_{\{b, c, e\}} \) are fuzzy L-generalized closed sets since the only super fuzzy open set of both is 1. But \( \lambda \land \mu = \chi_{\{c\}} \) is not fuzzy L-generalized closed.

Theorem 5 The intersection of a fuzzy L-generalized closed set with a fuzzy L-closed set is fuzzy L-generalized closed.

Proof Let \( \mu \) be a fuzzy L-generalized closed set and \( \eta \) be a fuzzy L-closed set. Let \( \theta \) be a fuzzy open set containing \( \mu \land \eta \). Then \( \mu \leq \theta \lor 1 - \eta \). Since \( 1 - \eta \) is fuzzy L-open, by Lemma 3, \( \theta \lor 1 - \eta \) is fuzzy open and since \( \mu \) is fuzzy L-generalized closed set, \( Cl_L(\mu \land \eta) \leq Cl_L(\mu) \land Cl_L(\eta) \) and by Lemma 6, \( Cl_L(\mu \land \eta) \leq Cl_L(\mu) \land Cl_L(\eta) \leq (\theta \lor 1 - \eta) \land \eta = \theta \land \eta \leq \theta \). □

4 Fuzzy L-generalized continuous and fuzzy L-generalized irresolute functions

Definition 3 A fuzzy function \( f : (X, \tau) \to (Y, \tau') \) is called
1. Fuzzy L-generalized continuous if \( f^{-1}(\lambda) \) is fuzzy L-generalized closed set in \( (X, \tau) \) for every fuzzy closed set \( \lambda \) of \( (Y, \tau') \).
2. Fuzzy L-generalized irresolute if \( f^{-1}(\lambda) \) is fuzzy L-generalized closed set in \( (X, \tau) \) for every fuzzy L-generalized closed set \( \lambda \) of \( (Y, \tau') \).

Lemma 7 Let \( f : (X, \tau) \to (Y, \tau') \) be a fuzzy generalized continuous. Then \( f \) is fuzzy L-generalized continuous, but not conversely.

Proof Follows from the fact that every fuzzy generalized closed set is fuzzy L-generalized closed. □

Example 6 Let \( X = \{a, b, c, d, e\} \) and \( \tau = \{0, 1, \chi_{\{b\}}, \chi_{\{c\}}, \chi_{\{b, c\}}, \chi_{\{a, b\}}, \chi_{\{a, b, c\}}\} \) and \( \tau' = \{0, 1, \chi_{\{d\}}\} \). Let \( f : (X, \tau) \to (X, \tau') \) be the identity function. Since \( f^{-1}(\chi_{\{a, b, c\}}) = \chi_{\{a, b, c\}} = Cl_L(\chi_{\{a, b, c\}}) \), \( f \) is fuzzy L-generalized continuous, but \( f \) is not fuzzy generalized continuous and hence not fuzzy continuous.

Even the composition of fuzzy L-generalized continuous functions needs not be fuzzy L-generalized continuous.

Example 7 Let \( f \) be the fuzzy function in Example 6 and \( g : (X, \tau') \to (X, \tau') \) be the identity function. It is easy to see that \( g \) is also a fuzzy L-generalized continuous function, but \( g \circ f \) is not fuzzy L-generalized continuous as \( \chi_{\{c\}} \) is fuzzy closed in \( (X, \tau') \), but not fuzzy L-generalized continuous in \( (X, \tau) \).

Corollary 4 If \( f : (X, \tau) \to (Y, \tau') \) is a fuzzy continuous and fuzzy contra-semi-continuous, then \( f \) is fuzzy L-generalized continuous.

Proof If \( \lambda \) is a fuzzy closed subset of \( Y \), then as \( f \circ \lambda \) is fuzzy continuous \( f^{-1}(\lambda) \) is fuzzy closed and as \( f \) is fuzzy contra-semi-continuous, \( f^{-1}(\lambda) \) is fuzzy semiopen. Thus \( f^{-1}(\lambda) \) is fuzzy L-generalized closed set.

We end this section by giving a necessary condition for a fuzzy L-generalized irresolute function to be fuzzy L-generalized continuous.

Theorem 6 If \( f : (X, \tau) \to (Y, \tau') \) is bijective, fuzzy open and fuzzy L-generalized irresolute, then \( f \) is fuzzy L-generalized closed.
Proof  Let \( \lambda \) be a fuzzy closed subset of \( Y \) and let \( f^{-1}(\lambda) \leq \gamma \), where \( \gamma \in \tau \). Clearly, \( \lambda \leq f(\gamma) \). Since \( f(\gamma) \in \tau' \) and since \( \lambda \) is fuzzy \( L \)-generalized closed set, \( Cl_L(\lambda) \leq f(\gamma) \) and thus \( f^{-1}(Cl_L(\lambda)) \leq \gamma \). Since \( f \) is fuzzy \( L \)-generalized irresolute and since \( Cl_L(\lambda) \) is fuzzy \( L \)-generalized closed set in \( Y \), \( f^{-1}(Cl_L(\lambda)) \) is fuzzy \( L \)-generalized closed set. \( f^{-1}(Cl_L(\lambda)) \leq Cl_L(f^{-1}(Cl_L(\lambda))) = f^{-1}(Cl_L(\lambda)) \leq \gamma \). Therefore, \( f^{-1}(\lambda) \) is fuzzy \( L \)-generalized closed set and hence, \( f \) is fuzzy \( L \)-generalized continuous.

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