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Fuzzy L-closed sets

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Abstract Our goal in this paper is to introduce the relatively new notions of fuzzy L–closed and fuzzy L-generalized closed sets. Several properties and connections to other well-known weak and strong fuzzy closed sets are discussed. Fuzzy L-generalized continuous and fuzzy L-generalized irresolute functions and their basic properties and relations to other fuzzy continuities are explored.

Keywords Fuzzy L-open set; fuzzy L-closed set; fuzzy L-generalized closed set; fuzzy L-generalized continuous function.

AMS mathematics subject classification 54C08, 54H05.

1 Introduction

For a set X, a fuzzy set in X is a function $\lambda : X \to [0.1]$. Here $\lambda(x)$ represents the degree of membership of x in the fuzzy subset A of X and by χ_A, we mean the fuzzy set that maps every element in A to 1 and every element outside A to 0. Fuzzy topological spaces (simply, spaces) were first introduced by [1, 2]. A fuzzy topology on a set X is a collection $\mathfrak{T}$ of subsets of X satisfying: $0, 1 \in \mathfrak{T}$, is closed under formation of finite intersections and is closed under formation of arbitrary unions. Fuzzy topological spaces were studied by several authors, see for example [1, 3–8]. Let $(X, \mathfrak{T})$ be a fuzzy topological space. If λ is a fuzzy set, then the closure of λ (the smallest fuzzy closed set containing λ) and the interior of λ (the largest fuzzy open set in λ) will be denoted by $Cl_{\mathfrak{T}}(\lambda)$ and $Int_{\mathfrak{T}}(\lambda)$, respectively. If no ambiguity appears, we use

$$\overline{\lambda} \text{ and } \overset{o}{\lambda}$$

instead, respectively. A fuzzy set λ is called *fuzzy semiopen* [6] if there exists a fuzzy open set μ such that $\mu \leq \lambda \leq Cl_{\mathfrak{T}}(\mu)$. Clearly λ is a fuzzy semiopen set if and only if $\lambda \leq Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda))$. A complement of a fuzzy semiopen set is called *fuzzy semiclosed*. The fuzzy semi-interior of λ is the union of all fuzzy semi-open subsets contained in λ and is denoted by $sInt(\lambda)$. λ is called *fuzzy preopen* if $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$.$\lambda$ is called fuzzy *$\alpha-$open* if $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda)))$ and fuzzy *$\beta-$open* if $\lambda \leq Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda)))$. λ is called *fuzzy regular open* if $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$. Complements of fuzzy regular open sets are called *fuzzy regular closed.* λ is called fuzzy *preclosed* if $Cl(Int(\lambda)) \leq \lambda$ and fuzzy *regular closed* if $\lambda = Cl(Int(\lambda))$. λ is a fuzzy *generalized closed* set if $\lambda \leq \mu$ and $\mu \in \mathfrak{T}$ implies that $\lambda \leq \mu$. A fuzzy function $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ is called *fuzzy generalized continuous* if $f^{-1}(\lambda)$ is fuzzy generalized closed in $(X, \mathfrak{T})$ for every fuzzy closed set λ of $(Y, \mathfrak{T}')$ and fuzzy *contra-semi-continuous* if $f^{-1}(\lambda)$ is fuzzy semiopen in $(X, \mathfrak{T})$ for every fuzzy closed set λ of $(Y, \mathfrak{T}')$. For more on the preceding notions, the reader is referred to [1, 2, 4, 6, 8].

We introduce the relatively new notions of fuzzy L−closed sets, which is closely related to the class of fuzzy closed subsets. We show that the collection of all fuzzy L−open subsets of a space $(X, \mathfrak{T})$ forms a fuzzy topology that is finer than $\mathfrak{T}$ and we investigate several characterizations of fuzzy L$-$open and fuzzy L$-$closed notions via the operations of interior and closure. In section 3, we introduce the notion of fuzzy L-generalized closed sets and study connections to other weak and strong forms of fuzzy generalized closed sets. In addition several interesting properties and constructions of fuzzy L-generalized closed sets are discussed. Section 4 is devoted to introducing and studying fuzzy L-generalized continuous and fuzzy L-generalized irresolute functions and connections to other similar forms of fuzzy continuity.

2 Fuzzy L-closed sets

We begin this section by introducing the notions of fuzzy L$-$open and fuzzy L$-$closed subsets.

Definition 1 *Let μ be a fuzzy subset of a space (X,τ). The fuzzy L-*interior *of μ is the union of all fuzzy subsets of X whose closures are contained in $Int(\mu)$, and is denoted by $Int_L(\mu)$. The fuzzy L$-$*closure *of μ is $Cl_L(\mu)$ is the smallest fuzzy closed set containing μ. μ is called fuzzy L-*open *if $\mu = Int_L(\mu)$. The complement of a fuzzy L-open subset is called fuzzy L-closed.*

Clearly $Int_L(\mu) \leq Int(\mu) \leq \mu$ and hence every fuzzy L-open set is fuzzy open and thus every fuzzy L-closed set is fuzzy closed, but the converses needs not be true.

Example 1 Let $X = \{a, b, c\}$ and $\tau = \{0, 1, \chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{a,b\}}\}$. Then $\chi_{\{a,c\}}$ is a fuzzy open set that is not a fuzzy L-open as $Int_L(Int_L(\chi_{\{a,c\}}) = 0$.

Next, we show that the collection of all fuzzy L-open subsets of a space (X,τ) forms a fuzzy topology τ_{FL} that is finer than τ.

Theorem 1 *If (X,τ) is a fuzzy space, then (X,τ_{FL}) is a fuzzy space such that $\tau \supseteq \tau_{FL}$.*

Proof We only need to show (X,τ_{FL}) is a fuzzy space. Clearly 0 and 1 are fuzzy L-open sets. If $\mu,\gamma \in \tau_{FL}$, then $\mu = Int_L(\mu)$ and $\gamma = Int_L(\gamma)$. Now $Int_L(\mu \cap \gamma) = \vee\{\theta \in \tau{:}Cl(\theta) \leq Int(\mu \wedge \gamma)\} = \vee\{\theta \in \tau{:}Cl(\theta) \leq Int(\mu) \wedge Int(\gamma)\}$. Thus $Int_L(\mu \wedge \gamma) \geq Int_L(\mu) \wedge Int_L(\gamma) = \mu \wedge \gamma$. Therefore, $\mu \wedge \gamma = Int_L(\mu \wedge \gamma)$ and so $\mu \wedge \gamma \in \tau_{FL}$.

If $\{\mu_\alpha : \alpha \in \Delta\}$ is a collection of fuzzy L-open subsets of X, then for every $\alpha \in \Delta$, $Int_L(\mu\alpha) = \mu_\alpha$. Hence

$$\begin{aligned} Int_L(\vee_{\alpha\in\Delta}\mu\alpha) &= \vee\{\theta \in \tau{:}Cl(\theta) \leq Int(\vee_{\alpha\in\Delta}\mu\alpha)\} \\ &\geq \vee\{\theta \in \tau{:}Cl(\theta) \leq \vee_{\alpha\in\Delta}Int(\mu\alpha)\} \\ &\geq \vee\{\theta \in \tau{:}Cl(\theta) \leq \mu\alpha\} \text{ for every } \alpha \in \Delta \\ &= Int_L(\mu\alpha) \text{ for every } \alpha \in \Delta \\ &= \mu\alpha \text{ for every } \alpha \in \Delta. \end{aligned}$$

Hence $\vee_{\alpha\in\Delta}\mu\alpha \leq Int_L(\vee_{\alpha\in\Delta}\mu\alpha)$ and thus $\vee_{\alpha\in\Delta}\mu\alpha$ is fuzzy L-open. □

In classical topology, a set is always contained in its closure, but this is not the case in τ_{FL}. Next we show that $\mu \leq Cl_L(\mu)$ needs not be true.

Example 2 Let $X = \{a, b, c, d\}$ and $\tau = \{0, 1, \chi_{\{a\}}, \chi_{\{a,b\}}, \chi_{\{c,d\}}, \chi_{\{a,c,d\}}\}$. Then $\chi_{\{c\}} \leq \chi_{\{a,b,c\}}$, but $\chi_{\{c\}} \nleq Cl_{FL}(\chi_{\{a,b,c\}})$.

One might think that a fuzzy subset μ of a fuzzy space X is fuzzy L- closed if and only if $\mu = Cl_L(\mu)$, but this is not true as shown in the next example.

Example 3 Consider the space in Example 2 and cosider $\mu = \chi_{\{b,c\}}$. Since $Cl(\chi_{\{a\}}) = \chi_{\{a,c\}}$, $\chi_{\{a\}} \not\leq Cl_L(\mu)$. Since $Cl_L(\mu) = \mu$, but μ is not a fuzzy L-open set.

Lemma 1 *For any fuzzy set μ of a fuzzy space X,*
(i) $Int(\mu) \leq Cl_L(\mu)$.
(ii) $Int(\mu) = 0$ *if and only if* $Cl_L(\mu) = 0$.

Proof
(i) $\lambda \not\leq Cl_L(\mu)$ implies that there exists a fuzzy open set θ containing λ such that $Cl(\theta) \wedge Int(\mu) = 0$. Hence $\lambda \not\leq Int(\mu)$.
(ii) If $\lambda \leq Cl_L(\mu)$, then for every fuzzy open subset θ containing λ, $Cl(\theta) \wedge Int(\mu) \neq 0$. Hence there exists $\gamma \leq Cl(\theta) \wedge Int(\mu)$ and as $Int(\mu)$ is fuzzy open, $\theta \wedge Int(\mu) \neq 0$. Therefore $Int(\mu) \neq 0$.
Conversely if $Cl_L(\mu) = 0$, then by (i) as $Int(\mu) \leq Cl_L(\mu)$, $Int(\mu) = 0$. □

Lemma 2 *The union of a fuzzy open set with a fuzzy L-open-set is fuzzy open.*

Proof Let μ be an open set and η be a fuzzy L-open set. For all $\gamma \leq \mu \vee \eta$, $\gamma \leq \mu$ or $\gamma \leq \eta$ and so $\gamma \leq Int(\mu) \leq Int(\mu \vee \eta)$ or $\gamma \leq Int_L(\eta) \leq Int_L(\mu \vee \eta) \leq Int(\mu \vee \eta)$. □

Corollary 1 *The intersection of a fuzzy closed set with a fuzzy L–closed set is fuzzy closed.*

Lemma 3 *If λ is a fuzzy semiopen set of a fuzzy space X, $Cl_L(\mu) = Cl(\mu)$.*

Proof If θ is a fuzzy open set containing λ such that $Cl(\theta) \wedge Int(\mu) \neq 0$, then there exists $\gamma \leq Cl(\theta) \wedge Int(\mu)$. Thus $\theta \wedge Int(\mu) \neq 0$ and so $\theta \wedge \mu \neq 0$. Therefore $Cl_L(\mu) \leq Cl(\mu)$.

Conversely if for every fuzzy open set θ containing μ we have $\theta \wedge \mu \neq 0$, $\theta \wedge Int(Cl(\mu)) \neq 0$, since μ is fuzzy semiopen. Thus there exists $\gamma \leq \theta \wedge Int(Cl(\mu))$ and so $\theta \wedge Int(\mu) \neq 0$ which implies that $Cl(\theta) \wedge Int(\mu) \neq 0$. Hence $Cl(\mu) \leq Cl_L(\mu)$. □

Corollary 2
(i) *For any fuzzy subset μ of X, $Cl_L(\mu) \leq Cl(\mu)$.*
(ii) *If μ is a fuzzy semiopen subset of a space X, then $\mu \leq Cl_L(\mu)$.*

Lemma 4 *If μ is a fuzzy L-closed set in a fuzzy space X, then $Cl_L(\mu) \leq \mu$.*

Proof If μ is a fuzzy L-closed subset, then μ is fuzzy closed and thus by Corollary 2 (i), $Cl_L(\mu) \leq \mu$. □

Next, we show that a fuzzy preclosed set that is also fuzzy semiopen equals its fuzzy L-closure.

Theorem 2 *If μ is a fuzzy regular closed subset of a fuzzy space X, then $Cl_L(\mu) \leq \mu$.*

Proof $Cl_L(\mu) \leq Cl(\mu) \leq Cl(Cl(Int(\mu))) = Cl(Int(\mu)) \leq \mu$. This together with Corollary 2 implies that $\mu = Cl_L(\mu)$. □

3 Fuzzy L-generalized closed sets

In this section, we introduce the notion of fuzzy L-generalized closed set. Moreover, several interesting properties and constructions of these subsets are discussed.

Definition 2 *A fuzzy subset μ of a fuzzy space X is called fuzzy L-generalized closed set if whenever θ is a fuzzy open subset containing μ, we have $Cl_L(\mu) \leq \theta$. μ is fuzzy L-generalized open if $1-\mu$ is fuzzy L-generalized closed set.*

Theorem 3 *A subset μ of (X, τ) is fuzzy L-generalized open if and only if $\eta \leq Int_L(\mu)$, whenever $\eta \leq \mu$ and η is fuzzy closed in (X, τ).*

Proof Let μ be a fuzzy L-generalized open set and η be a fuzzy closed subset such that $\eta \leq \mu$. Then $1-\mu \leq 1-\eta$. As $1-\mu$ is fuzzy L-generalized closed set and as $1-\eta$ is fuzzy open, $Cl_L(1-\mu) \leq 1-\eta$. So $\eta \leq 1 - Cl_L(1-\mu) = Int_L(\mu)$.

Conversely if $1-\mu \leq \theta$ where θ is fuzzy open, then the fuzzy closed set $1-\theta \leq \mu$. Thus $1-\theta \leq Int_L(\mu) = 1 - Cl_L(1-\mu)$ and so $Cl_L(1-\mu) \leq \theta$. □

Next we show that every fuzzy L- closed set is fuzzy L- generalized closed . Moreover, the class of fuzzy generalized closed sets is properly placed between the classes of fuzzy semiopen sets that are fuzzy closed and fuzzy L-generalized closed sets. Clearly every fuzzy closed set that is fuzzy semiopen, by Lemma 1, is a fuzzy L- closed set. A fuzzy closed set is trivially fuzzy generalized closed and every fuzzy generalized closed set is fuzzy L-generalized closed by Corollary 2 (i).

The following result follows from Corollary 2 (i) and the fact that every fuzzy L-closed set is fuzzy closed:

Lemma 5 *Every fuzzy L-closed-set is fuzzy L-generalized closed.*

The converse of the preceding result needs not be true.

Example 4 Let $X = \{a, b, c, d\}$ and $\tau = \{0, 1, \chi_{\{b\}}, \chi_{\{c\}}, \chi_{\{b,c\}}, \chi_{\{a,b\}}, \chi_{\{a,b,c\}}\}$. Then as $Cl_L(\chi_{\{a\}}) = 0$, $\chi_{\{a\}}$ is fuzzy L-generalized closed, but it is not fuzzy L-closed and not fuzzy generalized closed and hence not fuzzy closed. Also $\chi_{\{b,d\}}$ is an fuzzy generalized closed set that is not fuzzy closed.

The following is an immediate result from Lemma 1:

Theorem 4 *If μ is a fuzzy semiopen subset of a space X, the following are equivalent:*
(1) *μ is fuzzy L-generalized closed*;
(2) *μ is fuzzy generalized closed.*

Its clear that if $\mu \leq \gamma$, then $Int_L(\mu) \leq Int_L(\gamma)$ and $Cl_L(\mu) \leq Cl_L(\gamma)$.

Lemma 6 *If μ and γ are fuzzy sets in a fuzzy space X, then $Cl_L(\mu) \vee Cl_L(\gamma) = Cl_L(\mu \vee \gamma)$ and $Cl_L(\mu \wedge \gamma) \leq Cl_L(\mu) \wedge Cl_L(\gamma)$.*

Proof Since μ and γ are subsets of $\mu \vee \gamma$, $Cl_L(\mu) \vee Cl_L(\gamma) \leq Cl_L(\mu \vee \gamma)$. On the other hand, if $\eta \leq Cl_L(\mu \vee \gamma)$ and θ is a fuzzy open set containing η, then $Cl(\theta) \wedge Int(\mu \vee \gamma) \neq 0$. Hence either $Cl(\theta) \wedge Int(\mu) \neq 0$ or $Cl(\theta) \wedge Int(\gamma) \neq 0$. Thus $\eta \leq Cl_L(\mu) \vee Cl_L(\gamma)$.

Finally since $\mu \wedge \gamma$ is a fuzzy subset of μ and γ, $Cl_L(\mu \wedge \gamma) \leq Cl_L(\mu) \wedge Cl_L(\gamma)$. □

Corollary 3 *Finite union of fuzzy L-generalized closed sets is fuzzy L-generalized closed.*

While the finite intersection of fuzzy L-generalized closed sets needs not be fuzzy L-generalized closed.

Example 5 Let $X = \{a, b, c, d, e\}$ and $\tau = \{0, 1, \chi_{\{b\}}, \chi_{\{c\}}, \chi_{\{b,c\}}, \chi_{\{a,b\}}, \chi_{\{a,b,c\}}\}$. Then $\lambda = \chi_{\{a,c,d\}}$ and $\mu = \chi_{\{b,c,e\}}$ are fuzzy L-generalized closed sets since the only super fuzzy open set of both is 1. But $\lambda \wedge \mu = \chi_{\{c\}}$ is not fuzzy L-generalized closed.

Theorem 5 *The intersection of a fuzzy L-generalized closed set with a fuzzy L-closed set is fuzzy L-generalized closed.*

Proof Let μ be a fuzzy L-generalized closed set and η be a fuzzy L-closed set. Let θ be a fuzzy open set containing $\mu \wedge \eta$. Then $\mu \leq \theta \vee 1 - \eta$. Since $1 - \eta$ is fuzzy L-open, by Lemma 3, $\theta \vee 1 - \eta$ is fuzzy open and since μ is fuzzy L-generalized closed set, $Cl_L(\mu \wedge \eta) \leq Cl_L(\mu) \wedge Cl_L(\eta)$ and by Lemma 6, $Cl_L(\mu \wedge \eta) \leq Cl_L(\mu) \wedge \eta \leq (\theta \vee 1 - \eta) \wedge \eta = \theta \wedge \eta \leq \theta$.□

4 Fuzzy L-generalized continuous and fuzzy L-generalized irresolute functions

Definition 3 *A fuzzy function $f : (X, \tau) \to (Y, \tau')$ is called*

(1) Fuzzy L-generalized continuous *if $f^{-1}(\lambda)$ is fuzzy L-generalized closed set in (X, τ) for every fuzzy closed set λ of (Y, τ'),*
(2) Fuzzy L-generalized irresolute *if $f^{-1}(\lambda)$ is fuzzy L-generalized closed set in (X, τ) for every fuzzy L-generalized closed set set λ of (Y, τ').*

Lemma 7 *Let $f : (X, \tau) \to (Y, \tau')$ be a fuzzy generalized continuous. Then f is fuzzy L-generalized continuous, but not conversely.*

Proof Follows from the fact that every fuzzy generalized closed set is fuzzy L-generalized closed. □

Example 6 Let $X = \{a, b, c, d, e\}$ and $\tau = \{0, 1, \chi_{\{b\}}, \chi_{\{c\}}, \chi_{\{b,c\}}, \chi_{\{a,b\}}, \chi_{\{a,b,c\}}\}$ and $\tau^{'} = \{0, 1, \chi_{\{d\}}\}$. Let $f : (X, \tau) \to (X, \tau')$ be the identity function. Since $f^{-1}(\chi_{\{a,b,c\}}) = \chi_{\{a,b,c\}} = Cl_L(\chi_{\{a,b,c\}})$, f is fuzzy L-generalized continuous, but f is not fuzzy generalized continuous and hence not fuzzy continuous.

Even the composition of fuzzy L-generalized continuous functions needs not be fuzzy L-generalized continuous.

Example 7 Let f be the fuzzy function in Example 6 and $g : (X, \tau^{'}) \to (X, \tau'^{'})$ be the identity fuzzy function. It is easy to see that g is also a fuzzy L-generalized continuous function, but $g \circ f$ is not fuzzy L-generalized continuous as $\chi_{\{c\}}$ is fuzzy closed in $(X, \tau'^{'})$, but not fuzzy L-generalized continuous in (X, τ).

Corollary 4 *If $f : (X, \tau) \to (Y, \tau')$ is a fuzzy continuous and fuzzy contra-semi-continuous, then f is fuzzy L-generalized continuous.*

Proof If λ is a fuzzy closed subset of Y, then as f is fuzzy continuous $f^{-1}(\lambda)$ is fuzzy closed and as f is fuzzy contra-semi-continuous, $f^{-1}(\lambda)$ is fuzzy semiopen. Thus $f^{-1}(\lambda)$ is fuzzy L-generalized closed set. □

We end this section by giving a necessary condition for a fuzzy L-generalized irresolute function to be fuzzy L-generalized continuous.

Theorem 6 *If $f : (X, \tau) \to (Y, \tau')$ is bijective, fuzzy open and fuzzy L-generalized irresolute, then f is fuzzy L-generalized closed.*

Proof Let λ be a fuzzy closed subset of Y and let $f^{-1}(\lambda) \leq \gamma$, where $\gamma \in \tau$. Clearly, $\lambda \leq f(\gamma)$. Since $f(\gamma) \in \tau'$ and since λ is fuzzy L-generalized closed set, $Cl_L(\lambda) \leq f(\gamma)$ and thus $f^{-1}(Cl_L(\lambda)) \leq \gamma$. Since f is fuzzy L-generalized irresolute and since $Cl_L(\lambda)$ is fuzzy L-generalized closed set in Y, $f^{-1}(Cl_L(\lambda)$ is fuzzy L-generalized closed set. $f^{-1}(Cl_L(\lambda) \leq Cl_L(f^{-1}(Cl_L(\lambda))) = f^{-1}(Cl_L(\lambda)) \leq \gamma$. Therefore, $f^{-1}(\lambda)$ is fuzzy L-generalized closed set and hence, f is fuzzy L-generalized continuous. □

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References

[1] Chakrabarty, M. K. and Ahsanullah, T. M. Fuzzy topology on fuzzy sets and tolerance topology. *Fuzzy Set Syst.* 1992. 45: 103–108.

[2] Chang, C. L. Fuzzy topological spaces. *J. Math. Anal. Appl.* 1968. 24: 182–190.

[3] Al-Hawary, T. Fuzzy ω_0-open sets. *Bull. Korian Math. Soc.* 2008. 45(4): 749–755.

[4] Chaudhuri, A. K. and Das, P. Some results on fuzzy topology on fuzzy sets. *Fuzzy Set Syst.* 1993. 56: 331–336.

[5] Haydar Es, A. Almost compactness and near compactness in fuzzy topological spaces. *Fuzzy Sets and Syst.* 1987. 22: 289–295.

[6] Mahmoud, F. S. Fath Alla, M. A. and Abd Ellah, S. M. Fuzzy topology on fuzzy sets: fuzzy semicontinuity and fuzzy semiseparation axioms. *Appl. Math. Comput.* 2004. 153: 127–140.

[7] Wong, C. K. Covering properties of fuzzy topological spaces, J. Math. Anal. Appl. 43(1973), 697-704.

[8] Wong, C. K. Fuzzy points and local properties of fuzzy topology. *J. Math. Anal. Appl.* 1974. 46: 316–328.