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Sensitivity analysis of unsaturated infiltration flow using head based finite element solution of Richards' equation

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Abstract Sensitivity analysis is one of the tools available for analyzing the effects of soil parameter on the variably unsaturated flows in porous media, that may easily be implemented into existing conventional Galerkin finite element solution of Richards' equation based computational fluid dynamics codes. The sensitivity of the model is evaluated on the basis of vertical infiltration problem with time dependent boundary condition, sharp gradient in the infiltration front, and discontinuous derivatives in the soil hydraulic properties. Simulation results demonstrate the complicated nature of unsaturated porous media during redistribution water flow. The sample case presented highlights the different aspects of the performance of the algorithm and the different factors that can affect their convergence and efficiency, including temporal discretization, convergence error norm, conductivity and moisture content characteristics, boundary conditions, and the extent of fully unsaturated zones in the soil. From the preliminary assessment performed herein, consideration of the number of degrees of freedom used when performing a sensitivity analysis is shown to demand enormous concern, if predicted sensitivities are to have significant physical interpretations. The proposed model is capable of simulating preferential flow situations using parameters which can be related to soil hydraulic properties.

Keywords Sensitivity analysis; unsaturated infiltration flow; finite element; pressure head based Richards' equation.

AMS mathematics subject classification 76B07, 80M10

1 Introduction

The Richards' equation for soil moisture movement occupies a very important role in modern engineering science and applied hydrology. Analytical solution of this equation is practically impossible in unsaturated soil profiles with complex boundary conditions due to nonlinearities arising from pressure head dependencies in soil moisture and hydraulic conductivity, in combination with the non-trivial forcing conditions that are often encountered in engineering practice. The practical utility of analytic and semi-analytic solutions is limited by their restrictive assumptions, which are homogeneity of the soil medium and a simple mathematical form for the constitutive and forcing functions. Most commonly, tools for solving various hydrologic problems for saturated-unsaturated flows utilize numerical methods based on either the finite difference or finite element techniques [1, 2].

The numerical solution performances based on finite element and finite difference techniques for one-dimensional variably saturated-unsaturated flow problems have been thoroughly analyzed [3] and suggested a linear finite element solution was preferable to a higher order finite element solution, as well as noting that some improvements to the usual finite

difference solution were obtained with alternative inter-nodal hydraulic conductivity averaging schemes. A comprehensive description for using several finite element procedures to solve variably saturated, coupled flow and solute transport problems are developed [4].

Finite element and finite difference solutions based on pressure head formulation coupled with backward Euler time discretization suffer from mass balance error, convergence problems and poor CPU efficiency for the case of highly non-linear problems, such as infiltration into very dry heterogeneous soils. However, finite elements are generally inferior to finite differences. All unsaturated flow simulations use either the head-based or the moisture content-based formulation of Richards' equation. Several studies of finite difference and finite element techniques have been used with each of these equation forms [3,5–8] On the other hand, the finite difference and finite element approximations using mixed formulation of Richards' equation are perfectly mass conservative. This approach is show to be superior to the standard head-based approximations while requiring no more computational effort. However conservation of mass is shown to be inadequate to guarantee good numerical solutions.

To meet the stability criterion, an implicit time discretization requiring evaluation of the nonlinear coefficients at the current time level, is generally used to solve the equation numerically. The solution of the non-linear algebraic systems that arise in implicit discretizations of Richards' equation has been the subject of significant research. Iterative schemes, Newton and Picard iterative approximations are most commonly used to linearize the resulting discrete system of equations. Between these two iterative schemes, Picard scheme is more popular [5,8–11] and it is the most intuitive linearization of Richards' equation, computationally inexpensive on a per-iteration basis, and preserves symmetry of the discrete system of equations. However, this method may diverge under certain conditions is shown theoretically [5, 12], and verified theoretically [13]. On the other hand, the Newton scheme, also known as Newton-Raphson iteration, yields nonsymmetric system matrices and is more complex and expensive than Picard linearization, though it achieves a higher rate of convergence and can be more robust than Picard for certain types of problems. Base of the Newton scheme has been limited to one- and two-dimensional unsaturated flow models [5, 7, 14, 15]. Another drawback of Newton's method is that it is locally convergent and it involves the computations of derivatives. For unsaturated flow case, Newton iterations improve considerably the robustness of the method, and the convergence is ensured only when a regularization step and additional constraints on the discretization parameters are applied. While the Newton and the Picard schemes are sensitive to robustness, these iterative methods entail computational costs associated with having to evaluate and solve the system of equations repeatedly for each time step.

Sensitivity analysis is used for various reasons, such as decision making or development recommendations, communication, increasing understanding or quantification of system, and model development. In model development, it can be used for the purposes of model validation or accuracy, simplification, calibration, and coping with poor or missing data and even to identify important parameters for further studies. In a fundamental level, sensitivity analysis is a tool to assess the effect of changes in input parameter values on output values of a simulation model. Direct differentiation of the discretized Richards' equation with respect to parameters defining spatial variability leads to linear systems of equations for elementary sensitivities that are readily solved in conjunction with the original equation. These elementary sensitivities can be easily transformed into approximations of functional

sensitivities and into sensitivities of boundary fluxes. A numerical implementation of this technique in one space dimension yields results that are consistent with exact analytical solutions and with numerical calculations. Several numerical models have been developed, while many studies of steady-state seepage faces are performed, fewer transient problems have been investigated. Most of the transient unconfined drainage problems are restricted to one dimensional analyses [16–18]. A few studies have investigated two dimensional drainage processes using variably saturated flow models [19–21]. These studies reveal the importance of considering transient flow processes in the unsaturated zone and demonstrate the failure of fully saturated flow models for describing unconfined drainage problems include the observation that the fully saturated flow models predict slower responses in water tables than actually occur. However, in these studies, a sensitivity analysis on the effect of van Genuchten parameters for the infiltration of redistribution with time dependent boundary condition was not undertaken.

Thus the objective of this study is to perform a modeling-based analysis of unsaturated flow with time dependent boundary condition commonly used in conjunction with vertical infiltration of redistribution for unsaturated flow. The analysis aims to expose the consequence of van Genuchten parameters choice on ground water modeling and thus provide a general guidance for the modeling community. The work is also focused on determining the method which would offer a stable solution, reduce the CPU time and maintain small truncation error. The performance of the algorithm is shown to be superior to the conventional pressure head-based form and can be easily used in layered porous media including multidimensional flow regime without any additional computational effort.

2 Numerical model for vertical unsaturated flow

Richards' equation is typically used to describe unsaturated flows and is derived from the continuity and Darcy equations. Assume that the densities of the fluid and solid phases are constant, ignore the effects of source and sink, assume no hysteresis in the hydraulic properties, and consider the effects of temperature, air pressure, and solute concentration on water flow to be negligible. One-dimensional vertical flow in unsaturated soils, pressure head-based Richards' equation is written as

$$C(\psi)\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z}\left(K(\psi)\left(\frac{\partial \psi}{\partial z} + 1\right)\right), \tag{1}$$

where, ψ is the pressure head $[L]$, t is time $[T]$, z denotes the vertical distance from the soil surface assumed positive downward $[L]$, $K(\psi)$ is the hydraulic conductivity $[LT^{-1}]$,

$$C(\psi) = \frac{d\theta}{d\psi}$$

is the specific moisture capacity $[L^{-1}]$, θ is the moisture content.

The pressure head of Richards' equation allows for both saturated and unsaturated zones as well as in layered and composite porous materials. However, in highly nonlinear problems such as infiltration into very dry heterogeneous soils, this formulation can suffers from mass-balance error, convergence problems and poor CPU efficiency unless very fine discretizations are used 10, 12, 22–27]. The time steps required for convergence are several

orders of magnitude smaller than is required for reasonable temporal discretization [23]. The reason for these problems is highly nonlinear nature of the saturation-pressure function under dry initial conditions, causing very high fluid pressure gradient near the wetting front and huge computational cost.

In order to solve equation (1) constitutive relationships between the dependent variable (such as pressure head) and the nonlinear terms (such as moisture content, moisture capacity and hydraulic conductivity) must be specified. The water retention characteristic equations used in the work reported here are of the van Genuchten [28] model. This model illustrated in detail as follows

$$\theta(\psi) = \theta_r + \frac{\theta_s - \theta_r}{\left[1 + |\alpha\psi|^n\right]^m} \quad \text{if} \quad \psi \leq 0 \tag{2}$$

$$\theta(\psi) = \theta_s \quad \text{if} \quad \psi > 0 \tag{3}$$

$$K(\psi) = K_s \left[\frac{\theta - \theta_r}{\theta_s - \theta_r}\right]^{\frac{1}{2}} \left\{1 - \left[1 - \left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^{\frac{1}{m}}\right]^m\right\}^2 \quad \text{if} \quad \psi \leq 0 \tag{4}$$

$$K(\psi) = K_s \quad \text{if} \quad \psi > 0 \tag{5}$$

$$C(\psi) = \alpha m n \frac{\theta_s - \theta_r}{\left[1 + |\alpha\psi|^n\right]^{m+1}} |\alpha\psi|^{n-1} \quad \text{if} \quad \psi \leq 0 \tag{6}$$

$$C(\psi) = 0 \quad \text{if} \quad \psi > 0. \tag{7}$$

2.1 Spatial and temporal approximations

A finite element Galarkin discretization in space and a finite difference discretization of the time derivative term are employed for solving the governing partial differential equation (10). The solution domain is divided into $M - 1$ elements where each of length is Δz and M represents the global nodes. Finite element approximation of the pressure head-based Richards' equation is usually generated using linear basis function and an approximating function is introduced [29]:

$$\psi(z,t) \approx \hat{\psi}(z,t) = \sum_{j=1}^{M} N_j(z)\,\psi_j(t), \tag{8}$$

$$K \approx \hat{K} = \sum_{j=1}^{M} K_j N_j(z), \tag{9}$$

$$C \approx \hat{C} = \sum_{j=1}^{M} C_j N_j(z), \tag{10}$$

where $\psi_j(t)$ are undetermined global nodal values of ψ and $N_j(z)$ are the corresponding linear Lagrangian basis functions. The weighted residual is used to set the criteria to solve for the unknown coefficients. In local coordinate space $-1 \leq \xi \leq 1$, the approximating function for each element (e) is

$$\hat{\psi}^{(e)} = \sum_{i=1}^{2} N_i^{(e)}(\xi)\psi_i^{(e)}(t) = \frac{1}{2}(1-\xi)\,\psi_1^{(e)}(t) + \frac{1}{2}(1+\xi)\,\psi_2^{(e)}(t)$$

which we can write in vector form as $\hat{\psi}^{(e)} = \left(\mathbf{N}^{(e)}(\xi)\right)^T \mathbf{\Psi}^{(e)}(t)$. The global function (8) becomes

$$\hat{\psi} = \sum_{e=1}^{M-1} \left(\mathbf{N}^{(e)}\right)^T \mathbf{\Psi}^{(e)} = \sum_{e=1}^{M-1} \hat{\psi}^{(e)}. \tag{11}$$

The symmetric weak formulation of Galerkin's method applied to (1) yields the system of ordinary differential equations [29]

$$\mathbf{A}(\mathbf{\Psi})\,\mathbf{\Psi} + \mathbf{F}(\mathbf{\Psi})\,\frac{\mathrm{d}\mathbf{\Psi}}{\mathrm{dt}} = \mathbf{q}(\mathrm{t}) - \mathbf{b}(\mathbf{\Psi}), \tag{12}$$

where $\mathbf{\Psi}$ is the vector of undetermined coefficients corresponding to the values of pressure head at each node, $\boldsymbol{q}$ contains the specified Darcy flux boundary conditions, and $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{F}$ are given over local subdomain element $\mathbf{\Omega}^{(e)}$ as

$$\mathbf{A}^{(e)} = \int_{\mathbf{\Omega}^{(e)}} \mathbf{K}_s^{(e)} K_r(\hat{\psi}^{(e)}) \frac{\mathbf{dN^{(e)}}}{\mathbf{dz}} \left(\frac{\mathbf{dN^{(e)}}}{\mathbf{dz}}\right)^T dz \tag{13}$$

$$\mathbf{b}^{(e)} = \int_{\mathbf{\Omega}^{(e)}} \mathbf{K}_s^{(e)} K_r(\hat{\psi}^{(e)}) \frac{\mathbf{dN^{(e)}}}{\mathbf{dz}} dz \tag{14}$$

$$\mathbf{F}^{(e)} = \int_{\mathbf{\Omega}^{(e)}} S(\hat{\psi}^{(e)}) \mathbf{N}^{(e)} \left(\mathbf{N}^{(e)}\right)^T dz. \tag{15}$$

Here $\mathbf{N}^T$ to denote the transpose of $\boldsymbol{N}$.

The nonlinear integrals in (13), (14), and (15) are evaluated by the second order Gaussian quadrature formula introducing an additional source of numerical error. The magnitude of this error will depend on the degree of nonlinearity in the $K_r(\psi)$ and $S(\psi)$ characteristic equations and can be minimized by using higher order numerical quadrature or a smaller mesh size Δz. The linear Lagrangian basis functions $\boldsymbol{A}$, and $\boldsymbol{F}$ have banded structures with band width of three.

Performing integration by parts to reduce the second derivative and using fully implicit backward Euler time-marching algorithm with the solution is assumed to be known at time level k and unknown at time level $k+1$, one can discretize the time derivative in (12) to yield [29]:

$$\mathbf{A}\left(\mathbf{\Psi}^{k+\lambda}\right)\mathbf{\Psi}^{k+\lambda} + \mathbf{F}\left(\mathbf{\Psi}^{k+\lambda}\right)\frac{\mathbf{\Psi}^{k+1} - \mathbf{\Psi}^{k}}{\Delta t} = \mathbf{q}\left(t^{k+\lambda}\right) - \mathbf{b}(\mathbf{\Psi}^{k+\lambda}), \tag{16}$$

where

$$\mathbf{\Psi}^{k+\lambda} = \lambda\mathbf{\Psi}^{k+1} + (1-\lambda)\mathbf{\Psi}^{k};\ 0 \le \lambda \le 1. \tag{17}$$

The equation (16) is a system of nonlinear equation in $\mathbf{\Psi}^{k+1}$ and when $\lambda = 0.5$ and $\lambda = 1$ which corresponds to the Crank-Nicolson and backward Euler implicit scheme respectively.

2.2 Picard scheme

To solve the system of nonlinear equations (16), iterative calculation and linearization are needed. From the practical point of view, the Picard method is used in this study due to its simplicity, and it also exhibits good performances in many problems [30]. Moreover, Picard

scheme is simpler to implement and less costly on a per-iteration basis. This technique also preserves the symmetry of the original discretization. This factor is very important in assessing the efficiency of the scheme. Moreover the Picard method conserves the tridiagonal banded structure of the system matrices. The simple formulation of Picard scheme [29] can be obtained directly from (12) with $\lambda = 0.5$ by iterating with all linear occurrences of $\mathbf{\Psi}^{k+1}$ taken at the current iteration level $m+1$ and all nonlinear occurrences at the previous level m. We get,

$$\left[\mathbf{A}\left(\lambda \mathbf{\Psi}^{k+1,\ m}\right) + \frac{1}{\Delta t}\mathbf{F}\left(\mathbf{\Psi}^{k+1,\ m}\right)\right]\mathbf{h} = -\mathbf{f}\left(\mathbf{\Psi}^{k+1,\ m}\right), \tag{18}$$

where

$$\mathbf{f}\left(\mathbf{\Psi}^{k+1}\right) = \mathbf{A}\left(\mathbf{\Psi}^{k+\lambda}\right)\mathbf{\Psi}^{k+\lambda} + \mathbf{F}\left(\mathbf{\Psi}^{k+\lambda}\right)\frac{\mathbf{\Psi}^{k+1} - \mathbf{\Psi}^{k}}{\Delta t} - \mathbf{q}\left(t^{k+\lambda}\right) - \mathbf{b}\left(\mathbf{\Psi}^{k+\lambda}\right) = 0. \tag{19}$$

and where $\mathbf{h} = \mathbf{\Psi}^{k+\lambda,\ m+1} - \mathbf{\Psi}^{k+\lambda, m}$.

3 Methodology

To investigate the effects of different values of soil fitting parameters for the one-dimensional vertical flow problem with time dependent boundary condition on both the outcome and simulation times of unsaturated flow models, a series of solutions were evaluated. These solutions were applied to soils with the most general unsaturated soil hydraulic properties used in unsaturated hydrology such as van Genuchten. As an example, this method for verification of codes for simulation of soil moisture flow in unsaturated zone, CATHY (CATchment HYdrology) that features elements of the sequential iterative coupling schemes. CATHY is a physically-based hydrological model where the surface module resolves the one-dimensional diffusion wave equation and the subsurface module solves the 3D Richards' equation. Coupling between these two equations is based on an extension of the boundary condition switching procedure used in some subsurface models for the handling of atmospheric inputs on the land surface boundary of the catchment. The main objective of this work is to assess, via sensitivity analysis, the accuracy, computational effort and mass balance limitations for the CATHY model over the frame of various values of soil hydraulic parameters which make soil retention functions are highly nonlinear. For the case of convergence criterion, dynamic time stepping strategies can be easily incorporated in the Picard iterative technique. The time stepsize can be increased by a factor of $\Delta t_{mag}\,(= 1.20)$ if convergence at the current time level is achieved in very few iterations $(= 5)$ and decreased by a reduction factor $\Delta t_{red}(= 0.5)$ to a minimum of Δt_{min}. Note that initial solution estimate can be improved by using small time stepsize in the iteration procedure, so as a minimum allowable time stepsize $\Delta t_{min} = 10^{-25}$ days is used in all simulation in this work If convergences not attained within the specified maximum number of iterations (=10) or exceeded this number, the solution at the current time level can be recomputed using a reduced time step-size to the minimum time step-size Δt_{min}. Back-stepping is also triggered if linear solver failed or if the convergence or residual errors become larger than maximum allowable convergence or residual error in the nonlinear solution. The iteration process continues until the difference between calculated values of the pressure head of the two successive iteration levels becomes less than the tolerances, until the inequality $\left\|\mathbf{\Psi}^{k+1,\ m+1} - \mathbf{\Psi}^{k+1,\ m}\right\| \leq Tol$ is satisfied for all grid points, where Tol is the nonlinear convergence tolerance, whose value is sufficiently

small ($Tol = 10^{-3}m$) enough to neglect. This represents a measure of absolute error, but it can also be used to measure relative error by selecting Tol to be a suitable multiple of some reference pressure head value [31]. In this report, the behavior of the convergence error using l_2 which is the square root of the sum of squares of pressure head differences over all nodes is examined in addition to the infinity norm (l_∞) The residual error ($\left\|\mathbf{f}\left(\mathbf{\Psi}^{k+1,\ m}\right)\right\|$) is also computed using l_∞ and l_2 norms. For the first time step of infiltration, transient simulation, or for steady state problems, the initial conditions are used as the first solution estimate for the iterative procedure. For subsequent time steps of a simulation, the pressure head solution from the previous step is used as the first estimate. Thus time step-size has a direct effect on convergence behavior via its influence on the quality of the initial solution estimate. The simulation begins with a time step-size of $\Delta t_0 = 10^{-10}$ days and proceeds until time $T_{max} = 3 \times 10^{-1}$ days.

Adequate conservation of global mass over the domain of interest is necessary but not sufficient condition for acceptability for an efficient numerical model. [12]. For the finite element approximation of Richards' equation, the global mass balance error (MBE) is calculated using the most widely used scheme [12], which is defined by

$$\text{MBE}= \left| 1- \frac{\text{Total additional mass in the domain}}{\text{Total net flux into the domain}} \right|,$$

where the total additional mass in the domain is the difference between the mass measured at any instant t and the initial mass in the domain, and the total net flux into the domain is the flux balance integrated in time up to t. The finite element mass balance is of the form [12]:

$$\text{MBE(t)}= \frac{\sum_{\text{i=1}}^{\text{E-1}} \left(\theta_i^{k+1} - \theta_i^0\right)(\Delta z) + \left(\theta_0^{k+1} - \theta_0^0\right)\left(\frac{\Delta z}{2}\right) + \left(\theta_E^{k+1} - \theta_E^0\right)\left(\frac{\Delta z}{2}\right)}{\sum_{j=1}^{k+1} \left\{ \left(q_o^j - q_N^j\right)(\Delta t) \right\}}, \tag{20}$$

with $N = E + 1$ nodes i.,e. $\{z_0,\ z_1,\ z_2,\ \ldots,\ z_N\}$, Δz is the constant nodal spacing and q_0 and q_N being the boundary fluxes calculated from the finite element equations associated with the boundary nodes z_0 and z_N.

To evaluate the performance of the proposed scheme, all numerical simulations were run on Dell INSPIRON, 2.56 GHz system.

4 Sensitivity discussions of numerical results

The mathematical accuracy, robustness and applicability of the proposed numerical scheme are verified through one-dimensional vertical infiltration with redistribution [32, 33], by comparing the simulation results with the recognized benchmark results that are available in the published literature. The examples presented below serve to illustrate the performance of the proposed scheme in simulating various physical situation, specifically, the pore-size density functions for soil that are parameterized by the following eight ($\alpha\,(m^{-1}),\ n$) pairs: (1.0, 4.264), (2.0, 4.264), (4.0, 4.264), (6.0, 4.264), (5.47, 1.8), (5.47, 2.239), (5.47, 3.0), (5.47, 4.5), which are varied over a wide range during infiltration flow to cover most of

the field soils and to show the sensitivity of the technique to selected parameters in the van Genuchten model. For the validation, stability and performance of the algorithm, different features such as the number of successful time steps, computational time, nonlinear iterations per time step, time-stepping behavior, and mass balance error are studied. The one-dimensional numerical test problem consists of *5 m* deep soil column is simulated by the numerical solution of Richards' equation for dense spatial grid ($\Delta z = 0.0250$ m). The porous medium is assumed to be homogeneous and values of the van Genuchten soil parameters used are: $\theta_s = 0.301$, $\theta_r = 0.093$, $\alpha = 5.47/m$, $n = 4.264.0$ and $K_s = 5.040\, m$/days. The numerical solutions obtained by these soil properties are defined as base solution (Base).

Constant head boundary condition $\psi(0,t) = 0.0$ at the bottom of the domain and a time dependent boundary condition $\psi(10,t) = -10(1.0 - 1.01e^{-t})$ at the top of the domain with hydrostatic equilibrium initial conditions $\psi(z,0) = -z$ are applied. These forcing conditions lead to the development of a sharp infiltration front and induce large gradients in the solution. Analytical differentiation of the soil characteristic curves is used for all runs.

Since the hydraulic conductivity and specific moisture equations of van Genuchten model are the function of n, larger pores which have less resistance to fluid transmission, tend to drain first, constraining water to flow through the remaining smaller pores. Therefore, as the value of n decreases, the hydraulic conductivity of the unsaturated zone decreases more rapidly with decreasing water content This decreases the net drainage rate in the unsaturated zone. Hence it is observed that the simulated water table falls more rapidly when the pore-size-density function is broad, rather than narrow [20, 21]. This sensitivity analysis is based upon the assumption that the saturated hydraulic conductivity K_s is independent of the pore sizes parameterized by α and n. Note that the hydraulic conductivity of a porous medium should be direct function of the pore-size-density function, which, of course, also governs the capilaric properties of the medium. Below, an attempt is made to perform a more sophisticated sensitivity analysis, treating the hydraulic conductivity as a function of the van Genuchten parameters, α and n.

In order to observe the analysis of sensitivity broadly, the eight pairs of pore-size density functions of soil are investigated. These parameters are used in the numerical code to generate tabular constitutive relationships through the van Genuchten functions, as illustrated in Figure 1 and Figure 2. Specific moisture content and hydraulic conductivity profiles are evaluated by taking fixed, that is base value of $\alpha = 5.47/m$ against $n = 1.8$, 2.239, 3.0 and 4.5. Similarly, for all the value of $\alpha = 1.0/m$, $2.0/m$, $4.0/m$ with the fixed value of $n = 4.264$ are computed. To measure the effects of the eight pairs of (αn) on the sensitivity of numerical simulator, base solution of moisture content and hydraulic conductivity of van Genuchten model are plotted together.

Figure1 and Figure 2 show that, the effects of increasing α by one order of magnitude subject to fixed n or when n is increasing with constant $\alpha = 5.47/m$, the specific moisture content and hydraulic conductivity profiles shift down. Similar property preserves when α increases subject to fixed value of $n = 4.264$. Hence, by comparing of the pore-size-density functions of the soils parameterized by the variations of (α, n), one sees that the lower value of n corresponds to the broader density function with a somewhat lower, down shifted soil profiles are appeared. One remarkable difference between these media is the relative abundance of smaller pores which accompanies the smaller values of n. These smaller pores can serve as suffocation within porous media, and thus restrict the flow of fluid through

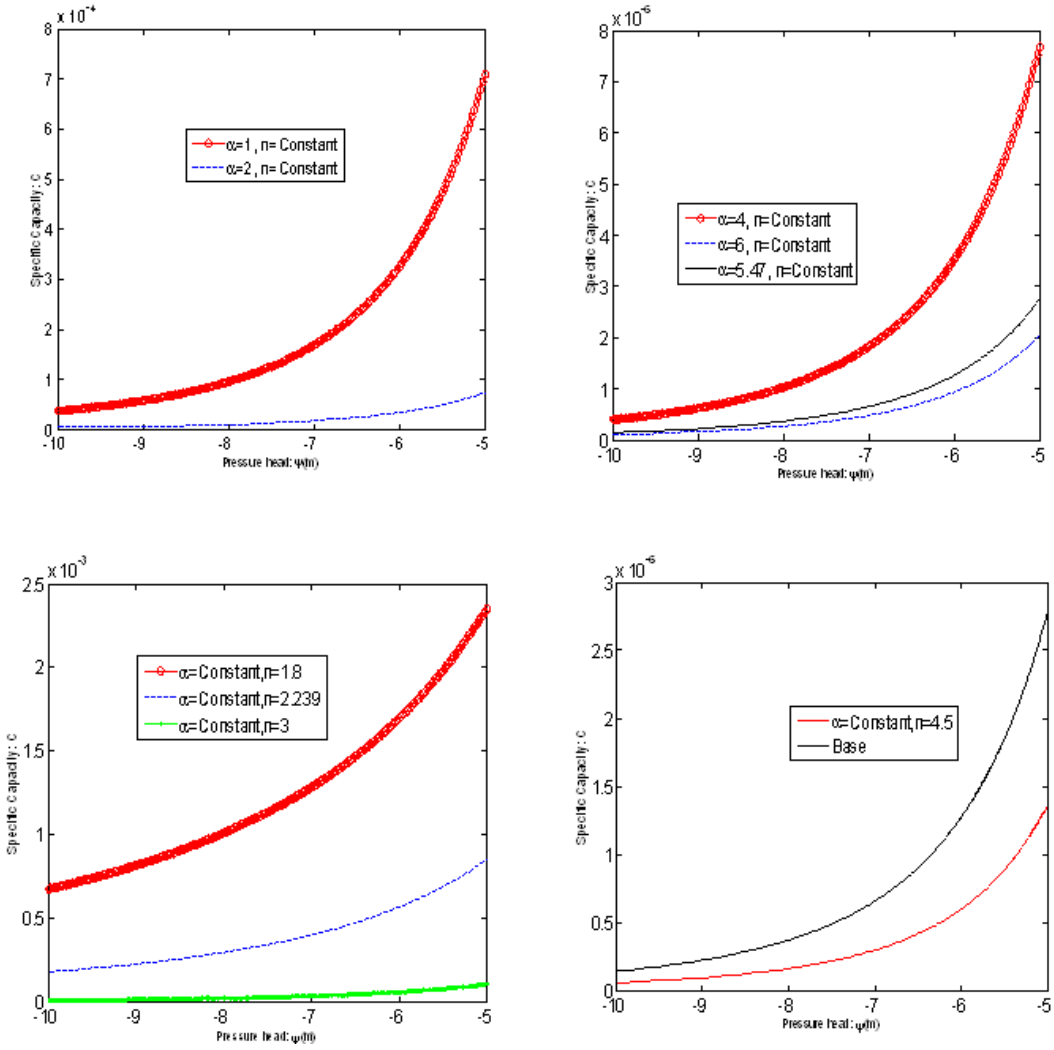

Figure 1: Sensitivity of the specific moisture content profiles of variably unsaturated model to variations in the van Genuchten parameters α with constant $n = 4.26$ and n with constant $\alpha = 5.47/m$

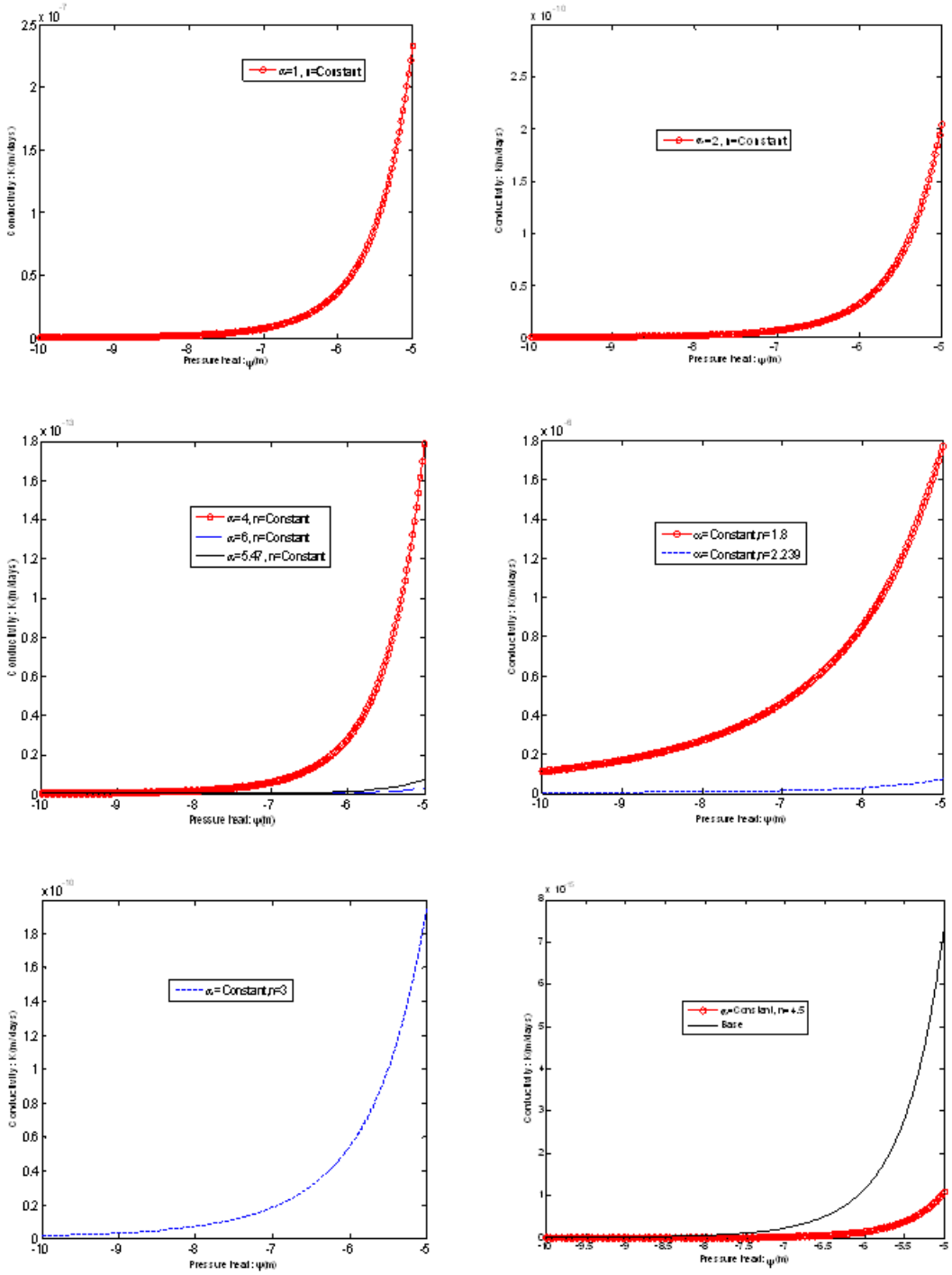

Figure 2: Sensitivity of the hydraulic conductivity profiles of variably unsaturated model to variations in the van Genuchten parameters α with constant $n = 4.26$ and n with constant $\alpha = 5.47/m$

it. In fact, it is the interconnectivity of larger pores which facilitates permeability for a given medium; the smallest pore through which water must flow when traversing a porous medium limits the permeability [34].

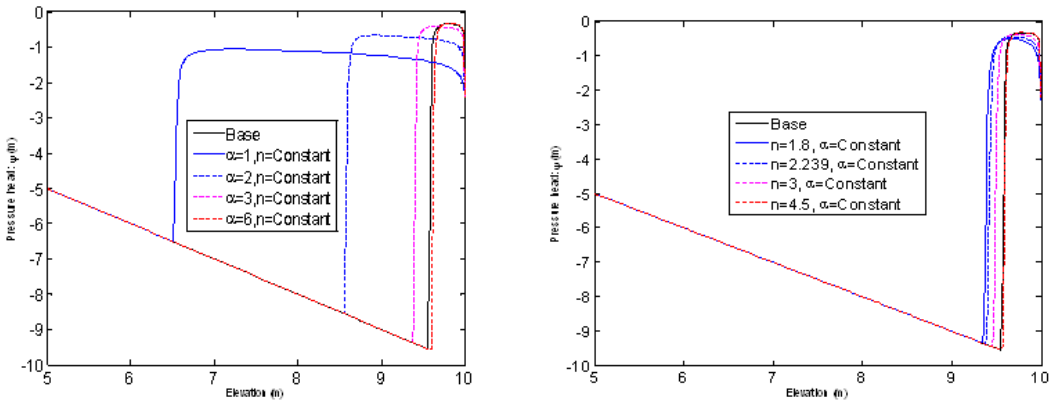

Figure 3: Sensitivity of the pressure head profiles of variably unsaturated model at time $t = 0.3$days to variations in the van Genuchten parameters α with constant $n = 4.26$ (left graph) and n with constant $\alpha = 5.47/m$ (right graph)

Figure 3 shows sensitivity comparison of pressure head solution profiles in unsaturated flow model at time $t = 0.3$ days. These profiles are evaluated for the different values of α with constant n which is plotted on the left including base solution and on the right is the effect of the pore size distribution parameter n by varying from 1.8 to 4.5 for fixed α with the base solution. These solution profiles are shown expected rapid infiltration of water from the surface for all pairs of (α, n), followed by a period of redistribution of the water due to the dynamic boundary condition at the top of the domain in unsaturated water flow. It is also seen that, as the value of α increases, the wetting front moves at slower rates, since the hydraulic conductivity and specific moisture capacity decreases when increases α. Moreover, the computed solutions indicate that the wetting front has moved to a greater depth when α is $1\,m^{-1}$ compared to when α is $3\,m^{-1}$ during the full simulation. It is noted that, moisture content gradients are non-uniform over the entire soil column for all values of n with steeper gradients occurring near the soil surface. Furthermore, the values of n decreases when the width of the pore size distribution increases. These are difficult to drain due to their large viscous effects. This results in slower rate of gravity drainage for low n values as compared to soils with high n values as can be seen from Figure 3. Therefore, the dependence of hydraulic conductivity and moisture capacity on the capillary fringe thickness α and the pore size distribution n during the vertical infiltration process plays an important role on the movement of the wetting front. It is also observed that the base solution of this numerical simulation is well agreed with the previous studies [32, 35].

Time step-size plays an essential accountability in the convergence behavior of the iterative method. A small Δt means a short step in the infiltration flow which corresponds to small variations in pressure heads. For the time varying boundary condition, one can obtain unpredictably large or small pressure changes over the course of a simulation. In such

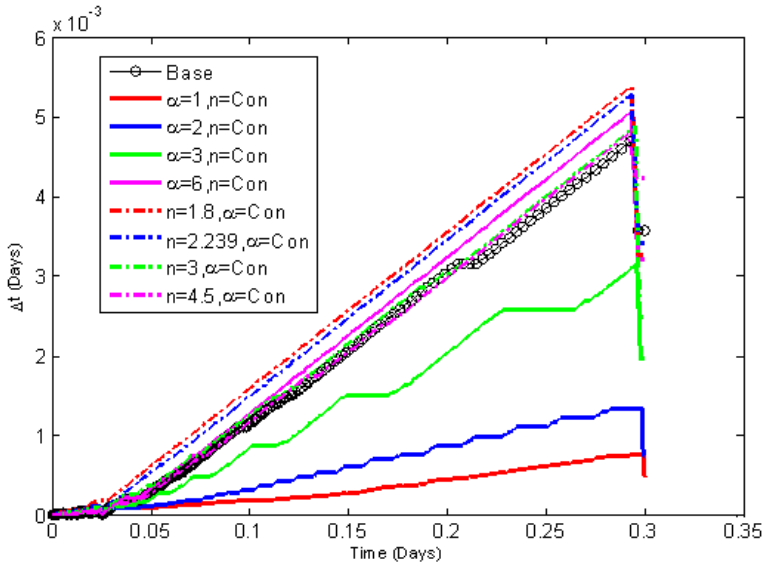

Figure 4: Sensitivity of the pressure head profiles of variably unsaturated model at time $t = 0.3$ days to variations in the van Genuchten parameters α with constant $n = 4.26$ (left graph) and n with constant $\alpha = 5.47/m$ (right graph)

case, dynamic time step adaptation is influential, since large time steps may be acceptable during certain periods, whereas at other times, the small time size is needed to stabilize the technique. Time step control in these cases should be based on the nonlinear characteristics of the problem instead of estimation of time truncation errors. To estimate the optimal truncation error by the Newton iterative scheme, generally, there is no restriction on time stepsize and time step control. On the other hand, on using a Picard approximation, time step restrictions are introduced by the linearization scheme, and control must be based on the nonlinear behavior. Figure 4 exhibited the sensitivities of the time step Δt for all values of van Genuchten parameters α and n It is clear that the behavior of time stepsize is significantly affect by the changes of α. Time stepping profiles also explain, as the values of the soil parameter α increases, the time stepsize is gradually increasing with the base run but the other parameter n is little sensitive to Δt.

Under highly nonlinear conditions, convergence evidently requires time step-sizes that are much smaller than those dictated by accuracy considerations alone. The results in Figure 5 reflect the convergence behavior of the infiltration redistribution simulation into a soil column initially at hydrostatic equilibrium. The effect of variations in the van Genuchten parameters is most evident in modeling infiltration of time dependent boundary condition. Numerical performance showed that the different number of iterations is required to complete simulation for different values of the parameters. The nonlinear iterations are evaluated at each time step based on the Picard iterative scheme. Most of the cases, convergence of the scheme is achieved within 3 to 8 iterations per time step throughout the simulation period, making the method is efficient.

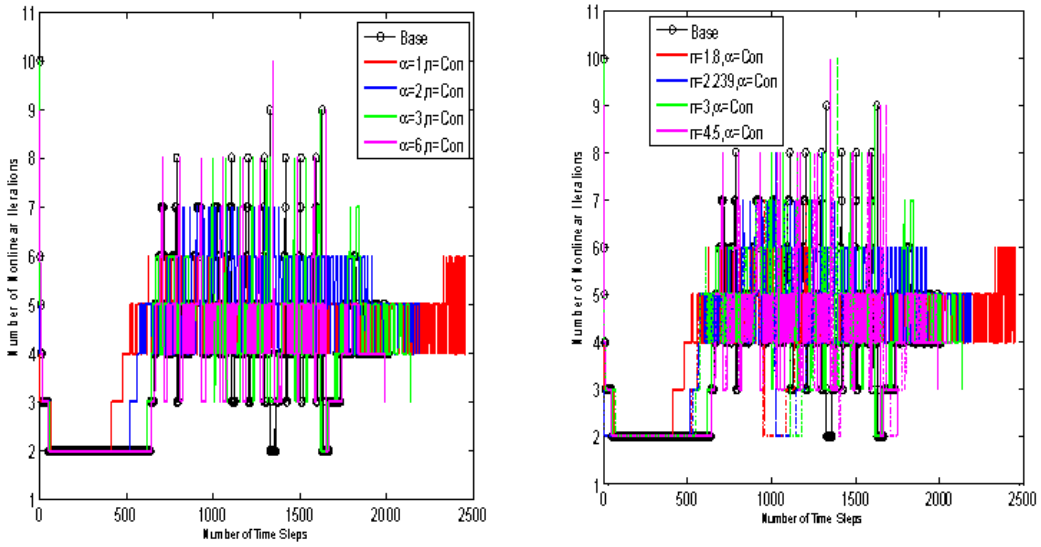

Figure 5: Sensitivity of the nonlinear iterations profiles of variably unsaturated model at time $t = 0.3$ days to variations in the van Genuchten parameters α with constant $n = 4.26$ (left graph) and n with constant $\alpha = 5.47/m$ (right graph)

To meet the criterion of robustness and efficiency of the numerical simulator, cumulative mass balance errors profiles for all pairs of van Genuchten parameters are presented in Figure 6. The mass balance error at any given time step is calculated as the absolute difference the changes in water storage during that time. The change in water storage is calculated in two ways, as the difference between incoming and outgoing water volumes, and from the changes in volumetric moisture content caused differences in pressure head between the current and the previous time level. It is apparent from Figure 6 that the maximum error is approximately $1.5 \times 10^{-3}\, m$ for the case of $\alpha = 1$ and $\alpha = 2$. The other runs show that slightly lower mass balance error with satisfactory numerical performances.

The algorithm efficiency can be accessed on the basis of successful number of time steps, smallest and largest time step-size during the simulation, cumulative mass balance errors, the nonlinear iterations per time steps, and number of back-stepping for the various runs. The influence of the two van Genuchten parameters on computational efficiency and accuracy are presented in Table 1 and Table 2. If the nonlinear solver does not converge within the prescribed iterations, the back-stepping mechanism is activated whereby the current time step is repeated with a reduced time step, obtained by multiplying Δt with Δt_{red}. The smaller Δt might result in a low truncation error, setting the adaptive time-stepping magnification factor for calculating the next time stepsize almost always equal to its maximum value. Due to this sudden increase in Δt, the Picard scheme will probably again not converge, and the back-stepping mechanism is activated again. It is clear from Table 1 and Table 2 that significantly fewer back-stepping is occurred implying the computational performance increases of the simulator. The reason for this is that the time step-sizes predicted by the adaptive time stepping scheme. The total number of iterations can be used as the measure of computational effort since the CPU time is governed by the total

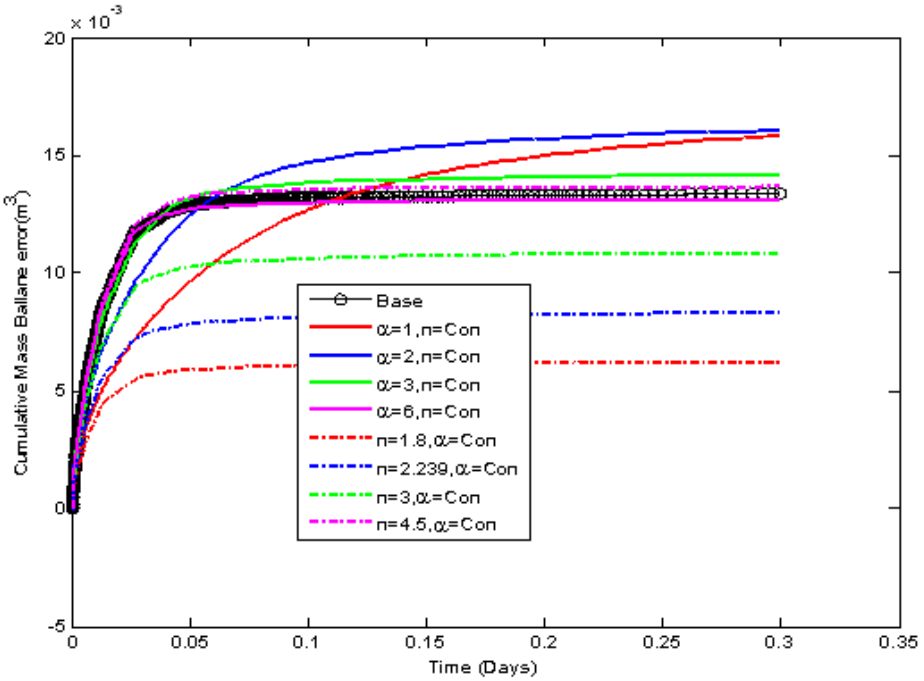

Figure 6: Sensitivity of the cumulative mass balance error profiles of variably unsaturated model at time $t = 0.3$ days to variations in the van Genuchten parameters α with constant $n = 4.26$ and n with constant$\alpha = 5.47/m$

number of matrix inversions, rather than by the number of time steps. The simulation time for all the cases is comparable predictions that agree well with the homogeneous trend. CPU time, average nonlinear (Avg. NL) iterations, total number of times, and cumulative mass balance error (CMBE) for all runs are confirming the sensitivity of the parameters and attesting to the consistency efficiency and accuracy constraints of the model.

5 Conclusions

A practical numerical sensitivity approach based on Richards' equation that able to accurately simulate for the various values of two effective van Genuchten parameters are have been presented. The effects of variations in unsaturated soil parameters α and n on the infiltration of redistribution processes are studied by means of functional sensitivities. It is observed that, during infiltration, as the value of α increases, the wetting front moves at a slower rate due to decreasing values of hydraulic conductivity. The effect of n on infiltration is complex since the movement of the wetting front depends on the initial pressure head. Therefore sensitivity analysis was carried out to determine the sensitivity response for all input parameters and use sensitivity analysis results to assess the model simulation based on the assumption that possibly the cumulative effect of input parameters, in terms of significant digits approximation, could be contributing to the underprediction of the spe-

Table 1: Sensitivity of computational statistics of variably unsaturated model at time $t = 0.3$ days to variations in the van Genuchten parameters α with constant $n = 4.26$

Values of α (1/m)	1.0	2.0	4.0	5.47(base case)	6.0
No. of steps	2454	2189	2140	2010	1995
Largest Δt (days)	7.895e-4	1.347e-3	3.155e-3	4.689e-3	1.504e-4
Smallest Δt (days)	1.000e-10	5.000e-11	2.500e-11	2.500e-11	2.500e-11
Avg. NL/steps	4.38	4.45	3.91	3.77	3.68
No. of back steps	0	0	3	2	2
CMBE (m^3)	1.592e-2	1.614e-2	1.418e-2	1.345e-2	1.321e-2
CPU (s)	1680.67	1429.86	1178.22	1067.21	1021.30

Table 2: Sensitivity of computational statistics of variably unsaturated model at time $t = 0.3\,days$ to variations in the van Genuchten parameters n with constant $\alpha = 5.47/m$

Values of α (1/m)	1.8	2.239	3.0	4.264(base case)	4.5
No. of steps	1392	1534	1681	2010	2076
Largest Δt (days)	5.388e-3	5.282e-3	4.880e-3	4.689e-3	4.782e-3
Smallest Δt (days)	1.000e-10	1.000e-10	1.000e-10	2.500e-11	2.500e-11
Avg. NL/steps	3.28	3.41	3.80	3.77	3.68
No. of back steps	5	4	4	2	2
CMBE (m^3)	6.331e-3	8.414e-3	1.092e-2	1.345e-2	1.380e-2
CPU (s)	644.79	739.23	841.76	1067.21	1093.25

cific moisture content or hydraulic conductivity of the simulation. In addition, sensitivity analysis is one of the most important steps in evaluating the effect of input parameters on simulation results, and it is also used by others for model validation. It can be conclude that the proposed model provides a conceptual and numerical framework for studying unsaturated flow systems using different parameter values which can be related to hydraulic properties of the medium.

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