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3-total edge product cordial labeling of wheel related graphs

[1]Roslan Hasni and [2]Almothana Azaizeh
[1]School of Informatics and Applied Mathematics
University Malaysia Terengganu
[2]College of Applied Studies and Community Service
University of Dammam, KSA
e-mail: [1]hroslan@umt.edu.my, [2]alm2410@yahoo.com

Abstract For a graph $G = (V(G), E(G))$, an edge labeling function $f : E(G) \to \{0, 1, \cdots, k-1\}$ where k is an integer, $2 \le k \le |E(G)|$, induces a vertex labeling function $f^* : V(G) \to \{0, 1, \cdots, k-1\}$ such that $f^*(v)$ is the product of the labels of the edges incident to v (mod k). This function f is called k-total edge product cordial (or simply k-TEPC) labeling of G if $|(v_f(i) + e_f(i)) - (v_f(j) + e_f(j))| \le 1$ for all $i, j \in \{0, 1, \cdots, k-1\}$. In this paper, 3-total edge product cordial labeling for wheel related graphs is determined.

Keywords Graph labeling; total edge product cordial labeling; wheel related graph

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1 Introduction

Let us begin with a graph that is simple, finite, connected and undirected $G = (V(G), E(G))$ with order p and size q. With regards to the standard terminology and notations in Graph Theory, we refer to West [1]. We also provide a brief summary of definitions that are invaluable for the present study.

Definition 1 A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

For a labeling function f, a vertex v (or an edge e) is an i-vertex (or an i-edge) if $f(v) = i$ (or $f(e) = i$) where $i \in Z$. Denote the number of i-vertices (or i-edges) of G under f by $v_f(i)$ (or $e_f(i)$), respectively and let $f(i) = v_f(i) + e_f(i)$.

Cahit [2] introduced cordial labeling and called a graph G cordial if there is a vertex labeling $f : V(G) \to \{0, 1\}$ such that the induced labeling $f^* : E(G) \to \{0, 1\}$, defined by $f^*(xy) = |f(x) - f(y)|$ for all edges $xy \in E(G)$ such that $|v_f(1) - v_f(0)| \le 1$ and, $|e_f(1) - e_f(0)| \le 1$ where $v_f(i)$ (respectively $e_f(i)$) is the number of vertices (respectively, edges) labeled with i.

Product cordial labeling was introduced by Sundaram *et al.* [3]. A product cordial labeling of a graph G with vertex set V is a function f from V to $\{0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a product cordial graph.

Recently, Vaidya and Barasara [4] proposed edge product cordial labeling. Given an edge labeling function defined as $f : E(G) \to \{0, 1\}$ and induced vertex labeling function

$f^* : V(G) \to \{0,1\}$ such that $f^*(v) = f(e_1)f(e_2)\cdots f(e_n)$ for edges $e_1, e_2, \cdots, e_n$ that are incident to v. The function f is called edge product cordial (or simply EPC) labeling of graph G if $|v_f(1) - v_f(0)| \leq 1$ and, $|e_f(1) - e_f(0)| \leq 1$. A graph G is called edge product cordial if it admits edge product cordial labeling. Vaidya and Barasara [5] used same labeling strategy and they called the function f a total edge product cordial (or simply TEPC) labeling of G if $|(v_f(0)+e_f(0))-(v_f(1)+e_f(1))| \leq 1$. A graph G is called total edge product cordial if it admits total edge product cordial labeling. For an extensive survey on graph labeling, we refer to Gallian [6].

Azaizeh *et al.* [7] introduced k-total edge product cordial labeling which is defined as follows.

Definition 2 Let f be a map from $E(G)$ to $\{0, 1, \cdots, k-1\}$ where k is an integer, $2 \leq k \leq |E(G)|$. For each vertex v, assign the label $f(e_1)f(e_2)\cdots f(e_n) \pmod k$ where $e_1, e_2, \cdots, e_n$ are the edges incident to vertex v. The function f is called a k-total edge product cordial (or simply k-TEPC) labeling of G if $|(v_f(i) + e_f(i)) - (v_f(j) + e_f(j))| \leq 1$, for $i, j \in \{0, 1, \cdots, k-1\}$. A graph G with a k-total edge product cordial labeling is called k-total edge product cordial graph.

In [7], the authors proved that paths P_n for $n \geq 4$, cycles C_n for $3 < n \neq 6$, some trees and some unicyclic graphs are 3-TEPC graphs. Azaizeh *et al.* in [8] showed that complete graph K_n for $n \geq 4$, bipartite graph $K_{m,n}$ for all $n \geq m \geq 2$ and generalized friendship graphs are 3-TEPC graphs. Recently, Azaizeh *et al.* [9] investigated the 3-TEPC labeling for more families of graphs namely, fans, double fans, triangular snake graph, double triangular snake graph, quadrilateral snake graph, double quadrilateral snake graph, cycle with one chord and cycle with two chords.

As a continuation of these results [7–9], the main aim of this paper is to determine the 3-total edge product cordial labeling for wheel related graphs.

2 Main result

Let $W_n = C_n + K_1 (n \geq 3)$ denotes the wheel of order n and a vertex which corresponds to C_n in W_n is called rim vertices and a vertex which corresponds to K_1 is called an apex vertex. We first give some definitions which are useful for our investigation.

The following definitions are due to Vaidya *et al.* [10].

Definition 3 Consider two wheels $W_n^{(1)}$ and $W_n^{(2)}$, then $G =< W_n^{(1)}, W_n^{(2)} >$ is the graph obtained by joining apex vertices of wheels to a new vertex x. Note that G has $2n+3$ vertices and $4n+2$ edges.

Definition 4 Consider k copies of wheels namely $W_n^{(1)}, W_n^{(2)}, W_n^{(3)}, \cdots, W_n^{(k)}$. Denote $G =< W_n^{(1)}, W_n^{(2)}, W_n^{(3)}, \cdots, W_n^{(k)} >$ be the graph obtained by joining apex vertices of each $W_n^{(p-1)}$ and $W_n^{(p)}$ to a new vertex x_{p-1} where $2 \leq p \leq k$. Note that G has $k(n+2)-1$ vertices and $2k(n+1)-2$ edges.

Theorem 1 *The graph* $< W_n^{(1)}, W_n^{(2)}, W_n^{(3)}, \cdots, W_n^{(k)} >$ *is 3-TEPC.*

Proof Let $V(< W_n^{(1)}, W_n^{(2)}, \cdots, W_n^{(k)} >) = \{u^{(j)}, v_i^{(j)}, 1 \leq j \leq k, 1 \leq i \leq n\} \cup \{x_m, 1 \leq m \leq k-1\}$ and $E(< W_n^{(1)}, W_n^{(2)}, \cdots, W_n^{(k)} >) = \{v_i^{(j)} v_{i+1}^{(j)}, 1 \leq j \leq k, 1 \leq i \leq n-1\} \cup \{v_n^{(j)} v_1^{(j)}, 1 \leq j \leq k\} \cup \{u^{(j)} v_i^{(j)}, 1 \leq j \leq k, 1 \leq i \leq n\} \cup \{u^{(j)} x_j, u^{(j+1)} x_j, 1 \leq j \leq k-1\}$. We consider the following cases.

Case 1: $k \equiv 0 \pmod 3$.

Case 1.1: $n \equiv 0 \pmod 3$. Define an edge labeling

$$f : E(< W_n^{(1)}, W_n^{(2)}, \cdots, W_n^{(k)} >) \to \{0, 1, 2\}$$

in the following way:

$$f(v_i^{(j)} v_{i+1}^{(j)}) = 0, 1 \leq j \leq k, 1 \leq i \leq \frac{(n-3)}{3}$$
$$f(v_i^{(j)} v_{i+1}^{(j)}) = 2, 1 \leq j \leq k, \frac{n}{3} \leq i \leq \frac{(2n-3)}{3}$$
$$f(v_i^{(j)} v_{i+1}^{(j)}) = 1, 1 \leq j \leq k, \frac{2n}{3} \leq i \leq n-1$$
$$f(v_n^{(j)} v_1^{(j)}) = 2, 1 \leq j \leq k$$
$$f(u^{(j)} v_i^{(j)}) = 0, 1 \leq j \leq k, 1 \leq i \leq \frac{n}{3}$$
$$f(u^{(j)} v_i^{(j)}) = 2, 1 \leq j \leq k, \frac{(n+3)}{3} \leq i \leq \frac{2n}{3}$$
$$f(u^{(j)} v_i^{(j)}) = 1, 1 \leq j \leq k, \frac{(2n+3)}{3} \leq i \leq n$$
$$f(u^{(3j-2)} x_{3j-2}) = 0, 1 \leq j \leq \frac{k}{3}$$
$$f(u^{(2)} x_1) = 0$$
$$f(u^{(3j+2)} x_{3j+1}) = 1, 1 \leq j \leq \frac{k-3}{3}$$
$$f(u^{(3j-1)} x_{3j-1}) = 1, 1 \leq j \leq \frac{k}{3}$$
$$f(u^{(3j)} x_{3j-1}) = 1, 1 \leq j \leq \frac{k}{3}$$
$$f(u^{(3j)} x_{3j}) = 0, 1 \leq j \leq \frac{k-3}{3}$$
$$f(u^{(3j+1)} x_{3j}) = 2, 1 \leq j \leq \frac{k-3}{3}.$$

Case 1.2: $n \equiv 1 \pmod 3$. Define an edge labeling

$$f : E(< W_n^{(1)}, W_n^{(2)}, \cdots, W_n^{(k)} >) \to \{0, 1, 2\}$$

in the following way:

$$f(v_i^{(j)} v_{i+1}^{(j)}) = 0, 1 \leq j \leq k, 1 \leq i \leq \frac{(n-4)}{3}$$
$$f(v_i^{(3j-2)} v_{i+1}^{(3j-2)}) = 1, 1 \leq j \leq \frac{k}{3}, i = \frac{(n-1)}{3}$$

$$f(v_i^{(2)}v_{i+1}^{(2)}) = 2, i = \frac{(n-1)}{3}$$
$$f(v_i^{(3j+2)}v_{i+1}^{(3j+2)}) = 0, 1 \leq j \leq \frac{(k-3)}{3}, i = \frac{(n-1)}{3}$$
$$f(v_i^{(3j)}v_{i+1}^{(3j)}) = 2, 1 \leq j \leq \frac{k}{3}, i = \frac{(n-1)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 2, 1 \leq j \leq k, \frac{(n+2)}{3} \leq i \leq \frac{(2n+1)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 1, 1 \leq j \leq k, \frac{(2n+4)}{3} \leq i \leq n-1$$
$$f(v_n^{(j)}v_1^{(j)}) = 1, 1 \leq j \leq k$$
$$f(u^{(j)}v_i^{(j)}) = 0, 1 \leq j \leq k, 1 \leq i \leq \frac{(n+2)}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 2, 1 \leq j \leq k, \frac{(n+5)}{3} \leq i \leq \frac{(2n-2)}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 1, 1 \leq j \leq k, \frac{(2n+1)}{3} \leq i \leq n$$
$$f(u^{(j)}x_j) = 2, 1 \leq j \leq k-1$$
$$f(u^{(j+1)}x_j) = 1, 1 \leq j \leq k-1.$$

Case 1.3: $n \equiv 2 \pmod 3$. Define an edge labeling

$$f : E(< W_n^{(1)}, W_n^{(2)}, \cdots, W_n^{(k)} >) \rightarrow \{0, 1, 2\}$$

in the following way:

$$f(v_i^{(j)}v_{i+1}^{(j)}) = 0, 1 \leq j \leq k, 1 \leq i \leq \frac{(n-5)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 2, 1 \leq j \leq k, \frac{(n-2)}{3} \leq i \leq \frac{(2n-1)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 1, 1 \leq j \leq k, \frac{(2n+2)}{3} \leq i \leq n-1$$
$$f(v_n^{(j)}v_1^{(j)}) = 1, 1 \leq j \leq k$$
$$f(u^{(j)}v_i^{(j)}) = 0, 1 \leq j \leq k, 1 \leq i \leq \frac{(n+1)}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 2, 1 \leq j \leq k, \frac{(n+4)}{3} \leq i \leq \frac{(2n-1)}{3}$$
$$f(u^{(3j+2)}x_{3j+1}) = 1, 1 \leq j \leq \frac{k-3}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 1, 1 \leq j \leq k, \frac{(2n+2)}{3} \leq i \leq n$$
$$f(u^{(3j-2)}x_{3j-2}) = 0, 1 \leq j \leq \frac{k}{3}$$
$$f(u^{(2)}x_1) = 0$$

$$f(u^{(3j-1)}x_{3j-1}) = 1, 1 \leq j \leq \frac{k}{3}$$
$$f(u^{(3j)}x_{3j-1}) = 1, 1 \leq j \leq \frac{k}{3}$$
$$f(u^{(3j)}x_{3j}) = 0, 1 \leq j \leq \frac{k-3}{3}$$
$$f(u^{(3j+1)}x_{3j}) = 2, 1 \leq j \leq \frac{k-3}{3}.$$

In this case, we have

$$e(0) + v(0) = e(1) + v(1) = e(2) + v(2) = 3n + 3 + \frac{k-3}{3}(3n + 4).$$

Therefore $|f(0) - f(1)| = |f(0) - f(2)| = |f(1) - f(2)| = 0$. So $|f(i) - f(j)| = 0$ for $0 \leq i < j \leq 2$, therefore f is a 3-TEPC labeling.

Case 2: $k \equiv 1 \pmod 3$.

Case 2.1: $n \equiv 0 \pmod 3$. Define an edge labeling

$$f : E(< W_n^{(1)}, W_n^{(2)}, \cdots, W_n^{(k)} >) \rightarrow \{0, 1, 2\}$$

in the following way:

$$f(v_i^{(j)}v_{i+1}^{(j)}) = 0, 1 \leq j \leq k, 1 \leq i \leq \frac{(n-3)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 2, 1 \leq j \leq k, \frac{n}{3} \leq i \leq \frac{(2n-3)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 1, 1 \leq j \leq k, \frac{2n}{3} \leq i \leq n-1$$
$$f(v_n^{(j)}v_1^{(j)}) = 2, 1 \leq j \leq k$$
$$f(u^{(j)}v_i^{(j)}) = 0, 1 \leq j \leq k, 1 \leq i \leq \frac{n}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 2, 1 \leq j \leq k, \frac{(n+3)}{3} \leq i \leq \frac{2n}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 1, 1 \leq j \leq k, \frac{(2n+3)}{3} \leq i \leq n$$
$$f(u^{(3j-2)}x_{3j-2}) = 0, 1 \leq j \leq \frac{k-1}{3}$$
$$f(u^{(3j-1)}x_{3j-2}) = 1, 1 \leq j \leq \frac{k-1}{3}$$
$$f(u^{(3j-1)}x_{3j-1}) = 1, 1 \leq j \leq \frac{k-1}{3}$$
$$f(u^{(3j)}x_{3j-1}) = 1, 1 \leq j \leq \frac{k-1}{3}$$
$$f(u^{(3j)}x_{3j}) = 0, 1 \leq j \leq \frac{k-1}{3}$$
$$f(u^{(3j+1)}x_{3j}) = 2, 1 \leq j \leq \frac{k-1}{3}.$$

We have
$e(0)+v(0)=e(1)+v(1)=n+\frac{k-1}{3}(3n+4), e(2)+v(2)=n+1+\frac{k-1}{3}(3n+4)$.
Therefore, $|f(0)-f(1)|=0, |f(0)-f(2)|=|f(1)-f(2)|=1$. So $|f(i)-f(j)|\leq 1$ for $0\leq i<j\leq 2$, thus f is a 3-TEPC labeling.

Case 2.2: $n\equiv 1 \pmod 3$. Define an edge labeling

$$f: E(< W_n^{(1)}, W_n^{(2)}, \cdots, W_n^{(k)} >) \rightarrow \{0,1,2\}$$

in the following way:

$$f(v_i^{(j)}v_{i+1}^{(j)})=0, 1\leq j\leq k, 1\leq i\leq \frac{(n-4)}{3}$$
$$f(v_i^{(3j-2)}v_{i+1}^{(3j-2)})=2, 1\leq j\leq \frac{(k+2)}{3}, i=\frac{(n-1)}{3}$$
$$f(v_i^{(3j-1)}v_{i+1}^{(3j-1)})=1, 1\leq j\leq \frac{(k-1)}{3}, i=\frac{(n-1)}{3}$$
$$f(v_i^{(3j)}v_{i+1}^{(3j)})=0, 1\leq j\leq \frac{(k-1)}{3}, i=\frac{(n-1)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)})=2, 1\leq j\leq k, \frac{(n+2)}{3}\leq i\leq \frac{(2n+1)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)})=1, 1\leq j\leq k, \frac{(2n+4)}{3}\leq i\leq n-1$$
$$f(v_n^{(j)}v_1^{(j)})=1, 1\leq j\leq k$$
$$f(u^{(j)}v_i^{(j)})=0, 1\leq j\leq k, 1\leq i\leq \frac{(n+2)}{3}$$
$$f(u^{(j)}v_i^{(j)})=2, 1\leq j\leq k, \frac{(n+5)}{3}\leq i\leq \frac{(2n-2)}{3}$$
$$f(u^{(j)}v_i^{(j)})=1, 1\leq j\leq k, \frac{(2n+1)}{3}\leq i\leq n$$
$$f(u^{(j)}x_j)=2, 1\leq j\leq k-1$$
$$f(u^{(j+1)}x_j)=1, 1\leq j\leq k-1.$$

We have $e(0)+v(0)=n+1+\frac{k-1}{3}(3n+4), e(1)+v(1)=e(2)+v(2)=n+\frac{k-1}{3}(3n+4)$. Therefore, $|f(0)-f(1)|=|f(0)-f(2)|=1, |f(1)-f(2)|=0$. So $|f(i)-f(j)|\leq 1$ for $0\leq i<j\leq 2$ and therefore f is a 3-TEPC labeling.

Case 2.3: $n\equiv 2 \pmod 3$. Define an edge labeling

$$f: E(< W_n^{(1)}, W_n^{(2)}, \cdots, W_n^{(k)} >) \rightarrow \{0,1,2\}$$

in the following way:

$$f(v_i^{(j)}v_{i+1}^{(j)})=0, 1\leq j\leq k, 1\leq i\leq \frac{(n-5)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)})=2, 1\leq j\leq k, \frac{(n-2)}{3}\leq i\leq \frac{(2n-1)}{3}$$

$$f(v_i^{(j)}v_{i+1}^{(j)}) = 1, 1 \le j \le k, \frac{(2n+2)}{3} \le i \le n-1$$
$$f(v_n^{(j)}v_1^{(j)}) = 1, 1 \le j \le k$$
$$f(u^{(j)}v_i^{(j)}) = 0, 1 \le j \le k, 1 \le i \le \frac{(n+1)}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 2, 1 \le j \le k, \frac{(n+4)}{3} \le i \le \frac{(2n-1)}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 1, 1 \le j \le k, \frac{(2n+2)}{3} \le i \le n$$
$$f(u^{(3j-2)}x_{3j-2}) = 0, 1 \le j \le \frac{k-1}{3}$$
$$f(u^{(3j-1)}x_{3j-2}) = 1, 1 \le j \le \frac{k-1}{3}$$
$$f(u^{(3j-1)}x_{3j-1}) = 1, 1 \le j \le \frac{k-1}{3}$$
$$f(u^{(3j)}x_{3j-1}) = 1, 1 \le j \le \frac{k-1}{3}$$
$$f(u^{(3j)}x_{3j}) = 0, 1 \le j \le \frac{k-1}{3}$$
$$f(u^{(3j+1)}x_{3j}) = 2, 1 \le j \le \frac{k-1}{3}.$$

We have $e(0) + v(0) = e(1) + v(1) = n + \frac{k-1}{3}(3n+4), e(2) + v(2) = n + 1 + \frac{k-1}{3}(3n+4)$. Therefore, $|f(0) - f(1)| = 0, |f(0) - f(2)| = |f(1) - f(2)| = 1$. So $|f(i) - f(j)| \le 1$ for $0 \le i < j \le 2$ and therefore f is a 3-TEPC labeling.

Case 3: $k \equiv 2 \pmod 3$.

Case 3.1: $n \equiv 0 \pmod 3$. Define an edge labeling

$$f : E(< W_n^{(1)}, W_n^{(2)}, \cdots, W_n^{(k)} >) \to \{0, 1, 2\}$$

in the following way:

$$f(v_i^{(j)}v_{i+1}^{(j)}) = 0, 1 \le j \le k, 1 \le i \le \frac{(n-3)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 2, 1 \le j \le k, \frac{n}{3} \le i \le \frac{(2n-3)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 1, 1 \le j \le k, \frac{2n}{3} \le i \le n-1$$
$$f(v_n^{(j)}v_1^{(j)}) = 2, 1 \le j \le k$$
$$f(u^{(j)}v_i^{(j)}) = 0, 1 \le j \le k, 1 \le i \le \frac{n}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 2, 1 \le j \le k, \frac{(n+3)}{3} \le i \le \frac{2n}{3}$$

$$f(u^{(j)}v_i^{(j)}) = 1, 1 \le j \le k, \frac{(2n+3)}{3} \le i \le n$$
$$f(u^{(3j-2)}x_{3j-2}) = 0, 1 \le j \le \frac{k+1}{3}$$
$$f(u^{(3j-1)}x_{3j-2}) = 1, 1 \le j \le \frac{k+1}{3}$$
$$f(u^{(3j-1)}x_{3j-1}) = 0, 1 \le j \le \frac{k-2}{3}$$
$$f(u^{(3j)}x_{3j-1}) = 2, 1 \le j \le \frac{k-2}{3}$$
$$f(u^{(3j)}x_{3j}) = 1, 1 \le j \le \frac{k-2}{3}$$
$$f(u^{(3j+1)}x_{3j}) = 1, 1 \le j \le \frac{k-2}{3}.$$

Case 3.2: $n \equiv 1 \pmod 3$. Define an edge labeling

$$f : E(< W_n^{(1)}, W_n^{(2)}, \cdots, W_n^{(k)} >) \to \{0, 1, 2\}$$

in the following way:

$$f(v_i^{(j)}v_{i+1}^{(j)}) = 0, 1 \le j \le k, 1 \le i \le \frac{(n-4)}{3}$$
$$f(v_i^{(1)}v_{i+1}^{(1)}) = 2, i = \frac{(n-1)}{3}$$
$$f(v_i^{(3j-1)}v_{i+1}^{(3j-1)}) = 2, 1 \le j \le \frac{k+1}{3}, i = \frac{(n-1)}{3}$$
$$f(v_i^{(3j)}v_{i+1}^{(3j)}) = 1, 1 \le j \le \frac{k-2}{3}, i = \frac{(n-1)}{3}$$
$$f(v_i^{(3j+1)}v_{i+1}^{(3j+1)}) = 0, 1 \le j \le \frac{k-2}{3}, i = \frac{(n-1)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 2, 1 \le j \le k, \frac{(n+2)}{3} \le i \le \frac{(2n+1)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 1, 1 \le j \le k, \frac{(2n+4)}{3} \le i \le n-1$$
$$f(v_n^{(j)}v_1^{(j)}) = 1, 1 \le j \le k$$
$$f(u^{(j)}v_i^{(j)}) = 0, 1 \le j \le k, 1 \le i \le \frac{(n+2)}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 2, 1 \le j \le k, \frac{(n+5)}{3} \le i \le \frac{(2n-2)}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 1, 1 \le j \le k, \frac{(2n+1)}{3} \le i \le n$$
$$f(u^{(j)}x_j) = 2, 1 \le j \le k-1$$
$$f(u^{(j+1)}xj) = 1, 1 \le j \le k-1.$$

Case 3.3: $n \equiv 2 \pmod 3$. Define an edge labeling

$$f : E(< W_n^{(1)}, W_n^{(2)}, \cdots, W_n^{(k)} >) \to \{0, 1, 2\}$$

in the following way:

$$f(v_i^{(j)}v_{i+1}^{(j)}) = 0, 1 \leq j \leq k, 1 \leq i \leq \frac{(n-5)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 2, 1 \leq j \leq k, \frac{(n-2)}{3} \leq i \leq \frac{(2n-1)}{3}$$
$$f(v_i^{(j)}v_{i+1}^{(j)}) = 1, 1 \leq j \leq k, \frac{(2n+2)}{3} \leq i \leq n-1$$
$$f(v_n^{(j)}v_1^{(j)}) = 1, 1 \leq j \leq k$$
$$f(u^{(j)}v_i^{(j)}) = 0, 1 \leq j \leq k, 1 \leq i \leq \frac{(n+1)}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 2, 1 \leq j \leq k, \frac{(n+4}{3} \leq i \leq \frac{(2n-1)}{3}$$
$$f(u^{(j)}v_i^{(j)}) = 1, 1 \leq j \leq k, \frac{(2n+2}{3} \leq i \leq n$$
$$f(u^{(3j-2)}x_{3j-2}) = 0, 1 \leq j \leq \frac{k+1}{3}$$
$$f(u^{(3j-1)}x_{3j-2}) = 1, 1 \leq j \leq \frac{k+1}{3}$$
$$f(u^{(3j-1)}x_{3j-1}) = 0, 1 \leq j \leq \frac{k-2}{3}$$
$$f(u^{(3j)}x_{3j-1}) = 2, 1 \leq j \leq \frac{k-2}{3}$$
$$f(u^{(3j)}x_{3j}) = 1, 1 \leq j \leq \frac{k-2}{3}$$
$$f(u^{(3j+1)}x_{3j}) = 1, 1 \leq j \leq \frac{k-2}{3}.$$

In this case, we have $e(0)+v(0) = e(2)+v(2) = 2n+2+\frac{k-2}{3}(3n+4)$, $e(1)+v(1) = 2n+1+\frac{k-2}{3}(3n+4)$. Therefore, $|f(0)-f(1)| = |f(1)-f(2)| = 1, |f(0)-f(2)| = 0$. So $|f(i) - f(j)| \leq 1$ for $0 \leq i < j \leq 2$ and therefore f is a 3-TEPC labeling.

Hence, this completes the proof. □

Example 1 The graph $< W_6^{(1)}, W_6^{(2)} >$ and its 3-TEPC labeling is shown in Figure 1.

3 Conclusion

We completely determined the 3-total edge product cordial labeling for wheel related graphs. The labeling pattern is elucidated by means of illustrations. To derive similar results for other families of graphs and in the context of different labeling problems is an open area of research.

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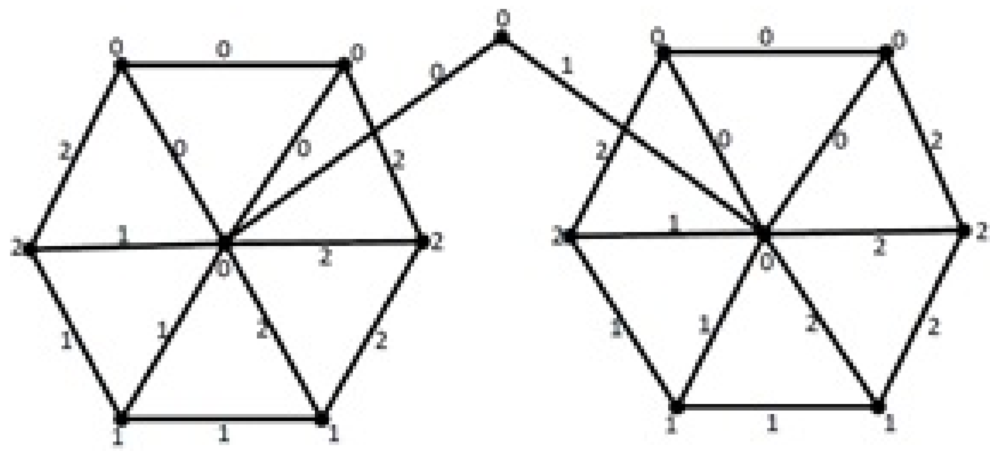

Figure 1: The graph $< W_6^{(1)}, W_6^{(2)} >$ and its 3-total edge product cordial labeling

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