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Harmonic starlike functions with respect to symmetric points

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Abstract Here a new class of complex-valued harmonic functions with respect to symmetric points is introduced. Coefficient bounds, distortion and various properties are obtained.

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1 Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply complex domain D is said to be harmonic in D. We call h the analytic part and g the co-analytic part of f if both u and v are real harmonic in D. In any simply connected domain, we can write $f = h+\overline{g}$ where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|,\ z \in D$(see Clunie [1]).

Denote by S_H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

Observe that S_H reduces to S, the class of normalized univalent analytic functions, if the co-analytic part of f is zero.

Hence

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}, |b_1| < 1. \tag{1.2}$$

We denote $\overline{H}$ the subclass of H consists of harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}, \quad |b_1| < 1. \tag{1.3}$$

A function $f = h + \overline{g}$ with h and g given by (1.1) is said to beharmonic starlike of order $\chi(0 \le \chi < 1)$ for $|z| = r < 1$, if

$$\frac{\partial}{\partial\theta}\left(\arg\ f(re^{i\theta})\right) = \mathrm{Im}\left\{\frac{\frac{\partial}{\partial\theta}\arg\ f(re^{i\theta})}{\arg\ f(re^{i\theta})}\right\} = \mathrm{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \ge \chi. \tag{1.4}$$

The class of all harmonic starlike functions of order χ is denoted by $S_H^*(\chi)$ and extensively studied by Jahangiri [2]. The cases $\chi = 0$ and $b_1 = 1$ were studied by Silverman and Silvia [3] and Silverman [4]. Other related works of the class H also appeared in [5, 6].

Lastly, let S_S^*denote the class of starlike functions with respect to symmetric points. This class was introduced by Sakaguchi [7] where f satisfying

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0, \quad z \in U. \tag{1.5}$$

Then, Ahuja and Jahangiri [8] studied the class of harmonic starlike functions of order $\chi(0 \le \chi < 1)$ with respect to symmetric points, $S_{HS}^*(\chi)$ and satisfying the condition

$$\operatorname{Im}\left\{\frac{2\frac{\partial}{\partial\theta}\ f(re^{i\theta})}{f(re^{i\theta})-\ f(-re^{i\theta})}\right\} = \operatorname{Re}\left\{\frac{2\left[zh'(z)-\overline{zg'(z)}\right]}{h(z)+\overline{g(z)}}\right\} \ge \chi. \tag{1.6}$$

Let the Hadamard product (or convolution) of two power series

$$\Phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$$

and

$$\varphi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n$$

be defined by

$$(\Phi * \varphi)(z) = (\varphi * \Phi)(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n.$$

Recently, Al Alamoush and Darus [9] derived a new operator as

$$D^k_{\alpha,\beta,\delta,\lambda} f(z) = z + \sum_{n=2}^{\infty}\ [\lambda(\alpha+\beta-1)(n-1)+1]^k\, C(\delta,n)\, a_n z^n \tag{1.7}$$

where

$$0 < \alpha \le 1, 0 < \beta \le 1,\ \lambda \ge 0,\ \delta, k \in N_0,\ z \in U, C(\delta,n) = \left(\begin{array}{c} n+\delta-1 \\ \delta \end{array}\right) = \frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(n+1)}$$

and $D^k_\delta f(z)$ is the Hadamard product (or convolution) between Salagean operator and Ruscheweyh operator (see Salagean [10], Ruscheweyh [11]).

The operator $D^k_{\alpha,\beta,\delta,\lambda} f(z)$ for harmonic functions $f = h + \overline{g}$ given by (1.1) is defined as

$$D^k_{\alpha,\beta,\delta,\lambda} f(z) = D^k_{\alpha,\beta,\delta,\lambda} h(z) + \overline{D^k_{\alpha,\beta,\delta,\lambda} g(z)},$$

where

$$D^k_{\alpha,\beta,\delta,\lambda} h(z) = z + \sum_{n=2}^{\infty} \Gamma^k_{\alpha,\beta,\delta,\lambda}\, a_n z^n,$$

and

$$D^k_{\alpha,\beta,\delta,\lambda}g(z) = \sum_{n=1}^{\infty} \Gamma^k_{\alpha,\beta,\delta,\lambda} b_n z^n,$$

where

$$\Gamma^k_{\alpha,\beta,\delta,\lambda} = \left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n). \tag{1.8}$$

Now,by using the operator $D^k_{\alpha,\beta,\delta,\lambda}$ and for $\chi(0 \le \chi < 1), S^*_{HS}(\alpha,\beta,\delta,\lambda;\chi)$ denote the class of harmonic univalent functions starlike of order with respect to symmetric points. The function $f \in S^*_{HS}(\alpha,\beta,\delta,\lambda;\chi)$ is satisfying

$$\begin{aligned} &\mathrm{Re}\left\{\frac{2z\left(D^k_{\alpha,\beta,\delta,\lambda}f(z)\right)'}{D^k_{\alpha,\beta,\delta,\lambda}f(z)-D^k_{\alpha,\beta,\delta,\lambda}f(-z)}\right\} \\ &= \mathrm{Re}\left\{\frac{2\left[z\left(D^k_{\alpha,\beta,\delta,\lambda}h(z)\right)' - \overline{z\left(D^k_{\alpha,\beta,\delta,\lambda}g(z)\right)'}\right]}{\left[D^k_{\alpha,\beta,\delta,\lambda}h(z)-D^k_{\alpha,\beta,\delta,\lambda}h(-z)\right]+\left[\overline{D^k_{\alpha,\beta,\delta,\lambda}g(z)-D^k_{\alpha,\beta,\delta,\lambda}g(-z)}\right]}\right\} \ge \chi \end{aligned} \tag{1.9}$$

where

$$\left(D^k_{\alpha,\beta,\delta,\lambda}f(z)\right)' = \frac{\partial}{\partial\theta}\left(D^k_{\alpha,\beta,\delta,\lambda}f(re^{i\theta})\right).$$

In this paper, we have obtained the coefficient conditions for the classes $S^*_{HS}(\alpha,\beta,\delta,\lambda;\chi)$ and $\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$. A representation theorem, inclusion properties and distortion bounds for the class $\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$ are also established.

2 Coefficient characterization

The sufficient coefficient bound for the harmonic functions in the subclass $S^*_{HS}(\alpha,\beta,\delta,\lambda;\chi)$ is deduced and presented.

Theorem 1 *Let a function $f = h + \overline{g}$ be given in (1.2) and $\Gamma^k_{\alpha,\beta,\delta,\lambda} \ge 1$. If*

$$\sum_{n=2}^{\infty}\left\{\frac{2n-\chi(1-(-1)^n)}{2(1-\chi)}\,|a_n| + \frac{2n+\chi(1-(-1)^n)}{2(1-\chi)}|b_n|\right\}\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right| \le 1 - \frac{1+\chi}{1-\chi}|b_1| \tag{2.1}$$

where $|b_1| < \frac{1+\chi}{1-\chi}$, $\chi(0 \le \chi < 1), \Gamma^k_{\alpha,\beta,\delta,\lambda}$ be defined by (1.7) and $z \in U$, then f is sense-preserving harmonic univalent in U and $f \in S^*_{HS}(\alpha,\beta,\delta,\lambda;\chi)$.

Proof To verify that f is orientation preserving, we show$|h'(z)| \ge |g'(z)|$,

$$\begin{aligned} |h'(z)| &= \left|1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right| \\ &\ge 1 - \sum_{n=2}^{\infty} n|a_n|\,|z|^{n-1} \\ &\ge 1 - \sum_{n=2}^{\infty} n|a_n|. \end{aligned}$$

By the hypothesis of the theorem, $\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right| \geq 1$ and by (2.1) give

$$\begin{aligned}
&\geq 1 - \sum_{n=2}^{\infty} \tfrac{2n-\chi(1-(-1)^n)}{2(1-\chi)}\, \Gamma^k_{\alpha,\beta,\delta,\lambda} |a_n| \\
&\geq \sum_{n=1}^{\infty} \tfrac{2n+\chi(1-(-1)^n)}{2(1-\chi)}\, \Gamma^k_{\alpha,\beta,\delta,\lambda} |b_n| \\
&\geq \sum_{n=1}^{\infty} n\ |b_n| \\
&\geq \sum_{n=1}^{\infty} n\ |b_n|\ |z|^{n-1} \\
&= |g'(z)|.
\end{aligned}$$

Thus, f is orientation preserving in U.

Next, we prove $f \in S^*_{HS}(\alpha,\beta,\delta,\lambda;\chi)$. It suffices to show that the condition (1.8) is satisfied. Then, let

$$w = \frac{2\left[z\left(D^k_{\alpha,\beta,\delta,\lambda}h(z)\right)' - \overline{z\left(D^k_{\alpha,\beta,\delta,\lambda}g(z)\right)'}\right]}{\left[D^k_{\alpha,\beta,\delta,\lambda}h(z) - D^k_{\alpha,\beta,\delta,\lambda}h(-z)\right] + \left[\overline{D^k_{\alpha,\beta,\delta,\lambda}g(z) - D^k_{\alpha,\beta,\delta,\lambda}g(-z)}\right]} = \frac{A(z)}{B(z)}$$

where

$$\begin{aligned}
A(z) &= 2\left[z\left(D^k_{\alpha,\beta,\delta,\lambda}h(z)\right)' - \overline{z\left(D^k_{\alpha,\beta,\delta,\lambda}g(z)\right)'}\right] \\
&= 2\left[z + \sum_{n=2}^{\infty} n\, \Gamma^k_{\alpha,\beta,\delta,\lambda} a_n z^n - \sum_{n=1}^{\infty} n\, \overline{\Gamma^k_{\alpha,\beta,\delta,\lambda} b_n z^n}\right],
\end{aligned}$$

and

$$\begin{aligned}
B(z) = &\left[D^k_{\alpha,\beta,\delta,\lambda}h(z) - D^k_{\alpha,\beta,\delta,\lambda}h(-z)\right] + \left[\overline{D^k_{\alpha,\beta,\delta,\lambda}g(z) - D^k_{\alpha,\beta,\delta,\lambda}g(-z)}\right] \\
&2z + \sum_{n=2}^{\infty} [1-(-1)^n]\, \Gamma^k_{\alpha,\beta,\delta,\lambda} a_n z^n + \sum_{n=1}^{\infty} [1-(-1)^n]\, \overline{\Gamma^k_{\alpha,\beta,\delta,\lambda} b_n z^n}.
\end{aligned}$$

Using the fact that $Re\{w(z)\} > \chi$ if and only if $|1-\chi+w| \geq |1+\chi-w|$, it suffices to show that

$$|A(z) + (1-\chi)B(z)| - |A(z) - (1+\chi)B(z)| \geq 0. \tag{2.2}$$

Substituting for $A(z)$ and $B(z)$ we get

$$\begin{aligned}
&|[2+2(1-\chi)]z + \sum_{n=2}^{\infty} [2n + (1-\chi)(1-(-1)^n)]\, \Gamma^k_{\alpha,\beta,\delta,\lambda} a_n z^n \\
&\qquad - \sum_{n=1}^{\infty} [2n - (1-\chi)[1-(-1)^n]\, \Gamma^k_{\alpha,\beta,\delta,\lambda} \overline{b_n z^n}| \\
&\qquad -|[2-2(1+\chi)]z + \sum_{n=2}^{\infty} [2n - (1+\chi)(1-(-1)^n)]\, \Gamma^k_{\alpha,\beta,\delta,\lambda} a_n z^n \\
&\qquad - \sum_{n=1}^{\infty} [2n + (1+\chi)[1-(-1)^n]\, \Gamma^k_{\alpha,\beta,\delta,\lambda} \overline{b_n z^n}| \\
&= |[2(2-\chi)]z + \sum_{n=2}^{\infty} [2n + (1-\chi)(1-(-1)^n)]\, \Gamma^k_{\alpha,\beta,\delta,\lambda} a_n z^n \\
&\qquad - \sum_{n=1}^{\infty} [2n - (1-\chi)[1-(-1)^n]\, \Gamma^k_{\alpha,\beta,\delta,\lambda} \overline{b_n z^n}|
\end{aligned}$$

$$-|-2\chi z+\sum_{n=2}^{\infty}[2n-(1+\chi)(1-(-1)^n)]\,\Gamma^k_{\alpha,\beta,\delta,\lambda}\,a_nz^n$$
$$-\sum_{n=1}^{\infty}[2n+(1+\chi)[1-(-1)^n]\,\Gamma^k_{\alpha,\beta,\delta,\lambda}\,\overline{b_nz^n|}$$
$$\geq 4(1-\chi)|z|-\sum_{n=2}^{\infty}[4n-2\chi(1-(-1)^n)]\,\Gamma^k_{\alpha,\beta,\delta,\lambda}\,|a_n|\,|z|^n$$
$$-\sum_{n=1}^{\infty}[4n+2\chi[1-(-1)^n]\,\Gamma^k_{\alpha,\beta,\delta,\lambda}\,|b_n|\,|z|^n$$
$$=4(1-\chi)|z|\{1-\sum_{n=2}^{\infty}\left(\tfrac{2n-\chi(1-(-1)^n)}{2(1-\chi)}\right)\Gamma^k_{\alpha,\beta,\delta,\lambda}\,|a_n|\,|z|^{n-1}$$
$$-\sum_{n=1}^{\infty}[\left(\tfrac{2n+\chi(1-(-1)^n)}{2(1-\chi)}\right)\;\Gamma^k_{\alpha,\beta,\delta,\lambda}\,|b_n|\,|z|^{n-1}\}$$
$$\geq 4(1-\chi)|z|\{1-\sum_{n=2}^{\infty}\left(\tfrac{2n-\chi(1-(-1)^n)}{2(1-\chi)}\right)\Gamma^k_{\alpha,\beta,\delta,\lambda}\,|a_n|$$
$$-\sum_{n=1}^{\infty}[\left(\tfrac{2n+\chi(1-(-1)^n)}{2(1-\chi)}\right)\;\Gamma^k_{\alpha,\beta,\delta,\lambda}\,|b_n|\,|\}$$
$$=4(1-\chi)|z|$$
$$\left\{1-\tfrac{1+\chi}{1-\chi}|b_1|-\left\{\sum_{n=2}^{\infty}\left[\;\left(\tfrac{2n-\chi(1-(-1)^n)}{2(1-\chi)}\right)\;|a_n|\;-\left(\tfrac{2n+\chi(1-(-1)^n)}{2(1-\chi)}\right)\;|b_n|\;\right]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|\right\}\right\}$$

This last expression is nonnegative by (2.1), and thus $f\in S^*_{HS}(\alpha,\beta,\delta,\lambda;\chi)$.

For

$$\sum_{n=2}^{\infty}|x_n|\;\;+\sum_{n=1}^{\infty}|y_n|=1,$$

the harmonic univalent functions

$$\begin{aligned}f(z)=z+\sum_{n=2}^{\infty}\left(\frac{2(1-\chi)}{2n-\chi(1-(-1)^n)\Gamma^k_{\alpha,\beta,\delta,\lambda}}\right)x_nz^n\\+\sum_{n=1}^{\infty}\left(\frac{2(1-\chi)}{2n+\chi(1-(-1)^n)\Gamma^k_{\alpha,\beta,\delta,\lambda}}\right)\overline{y_nz}^n,\end{aligned}\tag{2.3}$$

shows the equality in the coefficient bound given by (2.1) is sharp. □

The following result proves the hypothesis in Theorem 1 is a necessary and sufficient condition for f to be in the class$\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$.

Theorem 2 *Let a functionf* $=h+g$ *be given in* (1.3). *Then* $f\in\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$ *if and only if*

$$\sum_{n=2}^{\infty}\left\{\frac{2n-\chi(1-(-1)^n)}{2(1-\chi)}\;|a_n|+\frac{2n+\chi(1-(-1)^n)}{2(1-\chi)}|b_n|\right\}\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|\;\leq 1-\frac{1+\chi}{1-\chi}|b_1|\tag{2.4}$$

where $|b_1|<\frac{1+\chi}{1-\chi},\ \chi(0\leq\chi<1),\Gamma^k_{\alpha,\beta,\delta,\lambda}$ be defined by (1.7) and $z\in U$.

Proof Since $\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)\subset S^*_{HS}(\alpha,\beta,\delta,\lambda;\chi)$., sufficiency part follows from Theorem 1. To prove the necessity part, assume that $f\in\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$. For functions f of the form

(1.3), the condition (1.8) is equivalent to

$$\mathrm{Re}\left\{\frac{2\left[z\left(D^k_{\alpha,\beta,\delta,\lambda}h(z)\right)' - \overline{z\left(D^k_{\alpha,\beta,\delta,\lambda}g(z)\right)'}\right]}{\left[D^k_{\alpha,\beta,\delta,\lambda}h(z) - D^k_{\alpha,\beta,\delta,\lambda}h(-z)\right] + \left[\overline{D^k_{\alpha,\beta,\delta,\lambda}g(z) - D^k_{\alpha,\beta,\delta,\lambda}g(-z)}\right]} - \chi\right\}$$

$$= \mathrm{Re}\left\{\frac{2z - \sum_{n=2}^{\infty} 2n\Gamma^k_{\alpha,\beta,\delta,\lambda} a_n z^n - \sum_{n=1}^{\infty} 2n\Gamma^k_{\alpha,\beta,\delta,\lambda}\overline{b_n z^n}}{2z - \sum_{n=2}^{\infty}(1-(-1)^n)\Gamma^k_{\alpha,\beta,\delta,\lambda} a_n z^n + \sum_{n=1}^{\infty}(1-(-1)^n)\Gamma^k_{\alpha,\beta,\delta,\lambda}\overline{b_n z^n}} - \chi\right\} \geq 0.$$

The condition should hold for all values of z, $|z| = r < 1$. Choosing the values of z on the real positive axis, $0 \leq z = r < 1$, and $\Gamma^k_{\alpha,\beta,\delta,\lambda}$ is real, we have

$$= \mathrm{Re}\left\{\frac{2(1-\chi) - \sum_{n=2}^{\infty}[2n - \chi(1-(-1)^n]\Gamma^k_{\alpha,\beta,\delta,\lambda} a_n r^{n-1} - \sum_{n=1}^{\infty}[2n + \chi(1-(-1)^n]\Gamma^k_{\alpha,\beta,\delta,\lambda} b_n r^{n-1}}{2 - \sum_{n=2}^{\infty}(1-(-1)^n)\Gamma^k_{\alpha,\beta,\delta,\lambda} a_n r^{n-1} + \sum_{n=1}^{\infty}(1-(-1)^n)\Gamma^k_{\alpha,\beta,\delta,\lambda} b_n r^{n-1}}\right\} \geq 0. \tag{2.5}$$

Letting $r \to 1^-$and if the condition (2.4) does not hold, then the numerator in (2.5) is negative. Thus the coefficient bound inequality (2.5) holds true when $f \in \overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$. This completes the proof of Theorem 2. □

3 Distortion theorem and extreme points

In the theorem below we give distortion bounds for f in the class $\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$.

Theorem 3 *Let the functions $f(z)$ defined by (1.3) be in the class $f \in \overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$, then for $|z| = r < 1$, we have*

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{[\lambda(\alpha+\beta-1)+1]^k C(\delta,2)}\left\{\frac{1-\chi}{2} - \frac{1+\chi}{2}|b_1|\right\} r^2,$$

and

$$|f(z)| \geq (1-|b_1|)r - \frac{1}{[\lambda(\alpha+\beta-1)+1]^k C(\delta,2)}\left\{\frac{1-\chi}{2} - \frac{1+\chi}{2}|b_1|\right\} r^2.$$

The result is sharp.

Proof We prove only the left hand inequality, let $f \in \overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$. Taking the absolute value of $f(z)$, we have

$$\begin{aligned}|f(z)| &\geq (1-|b_1|)r - \sum_{n=2}^{\infty}\{|a_n| + |b_n|\} r^n \geq (1-|b_1|)r - r^2\sum_{n=2}^{\infty}\{|a_n| + |b_n|\} \\ &= (1-|b_1|)r - \tfrac{1-\chi}{[\lambda(\alpha+\beta-1)+1]^k C(\delta,2)}\sum_{n=2}^{\infty}\tfrac{[\lambda(\alpha+\beta-1)(n-1)+1]^k C(\delta,n)}{1-\chi}\{|a_n| + |b_n|\} r^2 \\ &\geq (1-|b_1|)r - \tfrac{(1-\chi)r^2}{[\lambda(\alpha+\beta-1)+1]^k C(\delta,2)}\sum_{n=2}^{\infty}\{\tfrac{[2n-\chi(1-(-1)^n)]}{4(1-\chi)}|a_n| \\ &\qquad + \tfrac{[2n+\chi(1-(-1)^n)]}{4(1-\chi)}|b_n|\}\,\Gamma^k_{\alpha,\beta,\delta,\lambda}\end{aligned}$$

$$= (1-|b_1|)r - \frac{(1-\chi)r^2}{2[\lambda(\alpha+\beta-1)+1]^k C(\delta,2)} \sum_{n=2}^{\infty} \{ \frac{[2n-\chi(1-(-1)^n)]}{2(1-\chi)}|a_n| + \frac{[2n+\chi(1-(-1)^n)]}{2(1-\chi)}|b_n|\} \Gamma^k_{\alpha,\beta,\delta,\lambda}$$
$$\geq (1-|b_1|)r - \frac{(1-\chi)r^2}{2[\lambda(\alpha+\beta-1)+1]^k C(\delta,2)} \left(1 - \frac{1+\chi}{1-\chi}|b_1|\right)$$
$$= (1-|b_1|)r - \frac{r^2}{[\lambda(\alpha+\beta-1)+1]^k C(\delta,2)} \left(\frac{1-\chi}{2} - \frac{1+\chi}{2}|b_1|\right).$$

The proof of the right hand inequality follows on lines similar to that of the left hand inequality which completes the proof of Theorem 3. □

Denote $clco\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$ as the closed convex hull of $\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$. The following result gives extreme points of $clco\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$.

Theorem 4 *A function $f = h + \overline{g} \in clco\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$ if and only if $f(z)$, can be expressed in the form*

$$f_n(z) = \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_n(z)) \tag{3.1}$$

where

$$h_1(z) = z, \quad h_n(z) = z - \frac{2(1-\chi)}{[2n-\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} z^n \quad (n = 2,3,...),$$
$$g_n(z) = z + \frac{2(1-\chi)}{[2n+\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} \overline{z^n} \quad (n = 1,2,3,...),$$

$\Gamma^k_{\alpha,\beta,\delta,\lambda}$ is given by (1.7) and $\sum_{n=1}^{\infty}(X_n+Y_n) = 1$, with $X_n \geq 0$, $Y_n \geq 0$. In particular the extreme points of $\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$ are h_n and g_n.

Proof. Let f be of the form (3.1). Then we have

$$f_n(z) = \sum_{n=1}^{\infty}(X_n+Y_n)z - \sum_{n=2}^{\infty} \frac{2(1-\chi)}{[2n-\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} X_n z^n + \sum_{n=1}^{\infty} \frac{2(1-\chi)}{[2n+\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} Y_n \overline{z}^n$$
$$= z - \sum_{n=2}^{\infty} \frac{2(1-\chi)}{[2n-\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} X_n z^n + \sum_{n=1}^{\infty} \frac{2(1-\chi)}{[2n+\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} Y_n \overline{z}^n.$$

Furthermore, let

$$|a_n| = \frac{2(1-\chi)}{[2n-\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} X_n z^n$$

and

$$|b_n| = \frac{2(1-\chi)}{[2n+\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} Y_n z^n.$$

Applying Theorem 2, gives

$$\sum_{n=2}^{\infty} \frac{[2n-\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|}{2(1-\chi)}|a_n| + \sum_{n=1}^{\infty} \frac{[2n+\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|}{2(1-\chi)}|b_n|$$

$$= \sum_{n=2}^{\infty} \frac{[2n-\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|}{2(1-\chi)} \frac{2(1-\chi)}{[2n-\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} X_n$$

$$+ \sum_{n=1}^{\infty} \frac{[2n+\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|}{2(1-\chi)} \frac{2(1-\chi)}{[2n+\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} X_n$$

$$= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1,$$

and so $f_n \in clco\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$.

Conversely, suppose that $f_n \in clco\overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi)$. Setting

$$X_n = \frac{[2n-\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right||a_n|}{2(1-\chi)} \quad (n = 2, 3, ...),$$

$$Y_n = \frac{[2n+\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right||a_n|}{2(1-\chi)} \quad (n = 1, 2, 3, ...)$$

and define $X_1 = 1 - \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n$.

Therefore,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n + \sum_{n=1}^{\infty} |b_n|\overline{z}^n$$

$$= z - \sum_{n=2}^{\infty} \frac{2(1-\chi)X_n}{[2n-\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} z^n + \sum_{n=1}^{\infty} \frac{2(1-\chi)Y_n}{[2n+\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|} \overline{z}^n$$

$$= z - \sum_{n=2}^{\infty} [X_n(h_n(z)-z)] + \sum_{n=1}^{\infty} [Y_n(g_n(z)-z)]$$

$$= \sum_{n=1}^{\infty} [X_n h_n(z) + Y_n g_n(z)].$$

as required. □

4 Convolution and convex combination

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n, \quad |b_1| < 1, \tag{4.1}$$

and

$$G(z) = z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \overline{z}^n, \quad (A_n \geq 0, B_n \geq 0), \tag{4.2}$$

we define the convolution of two harmonic functions f and G as

$$(f * G)(z) = f(z) * G(z) = z - \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n \overline{z}^n. \tag{4.3}$$

Using this definition, we show that the class $\overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \chi)$. is closed under convolution.

Theorem 5 *For* $0 \leq \psi \leq \chi < 1$, *let* $f \in \overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \chi)$ *and* $G \in \overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \psi)$. *Then* $f * G \in \overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \chi) \subset \overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \psi)$.

Proof. Let the function $f(z)$ defined by (4.1) be in the class $\overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \chi)$ and let the function $G(z)$ defined by (4.2) be in the class $\overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \psi)$. Then the convolution $f * G$ is given by (4.3). We wish to show that the coefficients of $f * G$ satisfy the required condition given in Theorem 1.

For $G \in \overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \psi)$ we note that $0 \leq A_n \leq 1$ and $0 \leq B_n \leq 1$. Now, for the convolution function $f * G$ we obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty} [2n - \chi(1 - (-1)^n] \left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right| |a_n| A_n + \sum_{n=1}^{\infty} [2n + \chi(1 - (-1)^n] \left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right| |b_n| B_n \\
&\quad \leq \sum_{n=2}^{\infty} [2n - \chi(1 - (-1)^n] \left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right| |a_n| + \sum_{n=1}^{\infty} [2n + \chi(1 - (-1)^n] \left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right| |b_n| \\
&\quad \leq 2(1 - \chi),
\end{aligned}$$

since $0 \leq \psi \leq \chi < 1$ and $f \in \overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \chi)$. Therefore $f * G \in \overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \chi) \subset \overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \psi)$, since the above inequality bounded by $2(1-\chi)$ while $2(1-\chi) \leq 2(1-\psi)$. This completes the proof of Theorem 5. □

Now, we show that the class $\overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \chi)$ is closed under convex combinations of its members.

Theorem 6 *The class* $\overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \chi)$ *is closed under convex combination.*

Proof. For $i = 1, 2, ...,$ let $f_i \in \overline{S^*_{HS}}(\alpha, \beta, \delta, \lambda; \chi)$, where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n_i}| z^n + \sum_{n=1}^{\infty} |b_{n_i}| \overline{z}^n, \quad (a_{n_i} \geq 0; b_{n_i} \geq 0; \, z \in U).$$

Then by using Theorem 1, we have

$$\sum_{n=2}^{\infty} \frac{[2n - \chi(1 - (-1)^n] \left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|}{2(1-\chi)} |a_{n_i}| + \sum_{n=1}^{\infty} \frac{[2n + \chi(1 - (-1)^n] \left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|}{2(1-\chi)} |b_{n_i}| \leq 1. \tag{4.4}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{n_i}| \right) \overline{z}^n.$$

Then, by using (4.4), we have

$$\sum_{n=2}^{\infty} \frac{[2n-\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|}{2(1-\chi)} \left(\sum_{i=1}^{\infty} t_i|a_{n_i}|\right) + \sum_{n=1}^{\infty} \frac{[2n+\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|}{2(1-\chi)} \left(\sum_{i=1}^{\infty} t_i|b_{n_i}|\right)$$
$$= \sum_{i=1}^{\infty} t_i \left[\sum_{n=2}^{\infty} \frac{[2n-\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|}{2(1-\chi)}|a_{n_i}| + \sum_{n=1}^{\infty} \frac{[2n+\chi(1-(-1)^n]\left|\Gamma^k_{\alpha,\beta,\delta,\lambda}\right|}{2(1-\chi)}|b_{n_i}|\right]$$
$$\leq \sum_{i=1}^{\infty} t_i = 1,$$

and this is the necessary and sufficient condition given by (2.4) and so

$$\sum_{i=1}^{\infty} t_i f_i(z) \in \overline{S^*_{HS}}(\alpha,\beta,\delta,\lambda;\chi).$$

This completes the proof of Theorem 6. □

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