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Solution of fuzzy fractional differential equations using homotopy analysis method

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Abstract This paper presents an efficient analytical solution for fuzzy fractional differential equations (FDEs) using homotopy analysis method. HAM is the generalization of various methods like homotopy perturbation method and Adomian decomposition method. The main advantage is its rapid convergence to the solution without using perturbation or restrictive assumptions. Illustrative numerical exampls are presented for the proposed method.

Keywords Caputo's fractional derivative; fuzzy fractional differential equations; homotopy analysis method; fuzzy Bagley-Torvik equation.

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1 Introduction

Fractional differential equations (FDEs) are an equation where ordinary or partial differential equation is rewritten by replacing the integer order derivative by a fractional order derivative respectively. FDEs had stimulated scientist's interest, mainly due to its demonstrated applications in numerous seemingly diverse fields of science and engineering fluid mechanics, diffusive transport, viscoelasticity, etc., because many dynamical systems are better characterized using a non-integer order dynamic model based on the FDEs. The historical survey, theory and applications of FDEs can be found in books [1, 2]. Over the past decades, several analytical or numerical methods have been developed to solve FDEs. Therefore, the importance of obtaining an exact or approximate solution of FDEs is still significant challenge that needs new method to discover the solution. Among these methods that have been developed are Adomian decomposition method [3], homotopy perturbation method [4] and so on.

The homotopy analysis method (HAM), a basic concept in topology, was initially proposed by Liao [5], is a general approximate analytic approach for obtaining convergent series solutions of strongly nonlinear differential equations. Unlike perturbation methods, the HAM is independent of any small or large physical parameter, which is essentially required in perturbation techniques. The HAM provides us with greet flexibility in choosing proper initial guess and auxiliary linear operators is chosen to construct such kind of continuous mapping of an initial approximation to the exact solution of the given equations, and an auxiliary parameters a simple way to adjust and is used to ensure the convergence of the solution series. Therefore, a complicated nonlinear equation can be transformed into an infinite number of linear and simpler equations, which is the advantage of the method in this computer age. In recent decades, this method has been successfully applied to various linear and nonlinear problems in science and engineering [6–8]. All these successful applications verified the validity, effectiveness and flexibility of the HAM for solving linear and nonlinear problems.

A fuzzy system consists of linguistic IF-THEN rules that have fuzzy antecedent and consequent parts. It is a static nonlinear mapping from the input space to the output space. The inputs and outputs are crisp real numbers and not fuzzy sets. The fuzzification block converts the crisp inputs to fuzzy sets and then the inference mechanism uses the fuzzy rules in the rule-based to produce fuzzy conclusions or fuzzy aggregations and finally the defuzzification block converts these fuzzy conclusions into the crisp outputs. The fuzzy system with singleton fuzzifier, product inference engine, center average defuzzifier and Gaussian membership functions is called as standard fuzzy system [9].

Two main advantages of fuzzy systems for the control and modeling applications are (i) fuzzy systems are useful for uncertain or approximate reasoning, especially for the system with a mathematical model that is difficult to derive and (ii) fuzzy logic allows decision making with the estimated values under incomplete or uncertain information. Fuzzy controllers are rule-based nonlinear controllers; therefore their main application should be the control of nonlinear systems. However, since linear systems are good approximations of nonlinear systems around the operating points, it is of interest to study fuzzy control of linear systems. Additionally, fuzzy controllers due to their nonlinear nature may be more robust than linear controllers even if the plant is linear. Furthermore, fuzzy controllers designed for linear systems may be used as initial controllers for nonlinear adaptive fuzzy control systems where on-line turning is employed to improve the controller performance. Therefore, a systematic fuzzy controller for linear systems is of theoretical and practical interest. Stability and optimality are the most important requirements in any control system. Stable fuzzy control of linear systems has been studied by a number of researchers. It is well-known nowadays that fuzzy controllers are universal nonlinear controllers. All these studies are preliminary in nature and deeper studies can be done. For optimality, it seems that the field of optimal fuzzy control is totally open [10].

The proposed method is HAM, is effective for obtaining rapid convergence series solution of fuzzy FDEs. Illustrative example will be included to demonstrate the validity and applicability of the presented techniques to highlight the significant features of the HAM.

2 Materials and methods

In this section, we are going to present some definitions and properties about fractional calculus which is useful throughout in this paper.

Definition 1 Riemann-Liouville fractional integral operator of order α, of function $f(t)$ is defined as [1]:

$$ {}_0J_t^\alpha f(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau, $$

where $\alpha > 0$.

Definition 2 Caputo fractional derivative operator of order α, of function $f(t)$ is defined as [1]:

$$ {}_0D_t^\alpha f(t) = \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} \frac{d^n f(\tau)}{d\tau^n} d\tau, $$

where $n-1 \leq \alpha < n$ and $n = 1, 2, \ldots$.

Definition 3 A fuzzy fractional differential equation of order α, is a fractional differential equation of the form R^i. If $f_i(t)$ is $T_{ij}(m_{ij}, s_{ij})$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, r$, then

${}_0D_t^{\alpha} f(t) = a_i(t)f(t) + b_i(t)$, where $a_i(t) \in \mathrm{C}$, $b_i(t) \in \mathrm{C}$ and R^i denotes the i-th rule of the Takagi-Sugeno fuzzy model.

2.1 Homotopy analysis method

The principles of HAM and its applicability for various kinds of differential equations are given in [5]. For convenience of the reader, we will present a review of the HAM. To achieve our goal, we consider the following nonlinear equation:

$$N[f(t)] = 0, \tag{1}$$

where N is a nonlinear operator and $f(t)$ is an unknown function of the independent variable t.

2.1.1 Zeroth-order deformation equation

Liao [5] constructs the so-called zeroth-order deformation equation:

$$(1-q)L[\Phi(t;q) - f_0(t)] = qhH(t)N[\Phi(t;q)], \tag{2}$$

where $0 \leq q \leq 1$ is an embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $\Phi(t;q)$ is unknown function and $f_0(t)$ is an initial approximation of $f(t)$ that satisfies the initial conditions. It should be emphasized that one has great freedom to choose the initial guess $f_0(t)$, the auxiliary linear operator L, the auxiliary parameter h and the auxiliary function $H(t)$. According to the auxiliary linear operator and suitable initial conditions, when $q = 0$, we have

$$\Phi(t;0) = f_0(t), \tag{3}$$

and when $q = 1$, since $h \neq 0$ and $H(t) \neq 0$, the zeroth-order deformation equation (2) is equivalent to equation (1), hence

$$\Phi(t;1) = f(t). \tag{4}$$

Thus, as q increasing from 0 to 1, the solution $\Phi(t;q)$ various from $f_0(t)$ to $f(t)$.

Define

$$f_m(t) = \frac{1}{m!} \left. \frac{\partial^m \Phi(t;q)}{\partial q^m} \right|_{q=0}. \tag{5}$$

Expanding $\Phi(t;q)$ in a Taylor series with respect to the embedding parameter q, by using equation (3) and equation (5), we have:

$$\Phi(t;q) = f_0(t) + \sum_{m=1}^{\infty} f_m(t) q^m. \tag{6}$$

Assume that the auxiliary parameter h, the auxiliary function $H(t)$, the initial approximation $f_0(t)$ and the auxiliary linear operator L are properly chosen so that the series(6) converges at $q = 1$. Then at $q = 1$, from equation (4), the series solution (6) becomes

$$f(t) = f_0(t) + \sum_{m=1}^{\infty} f_m(t). \tag{7}$$

2.1.2 High-order deformation equation

For brevity, define the vector

$$\overrightarrow{f}_n(t) = \{f_0(t), f_1(t), \cdots, f_n(t)\}.$$

According to the definition (7), the governing equation of $f_m(t)$ can be derived from the zeroth-order deformation equation (2). Differentiating the zeroth-order deformation equation (2) m times with respective to the embedding parameter q and then dividing it by $m!$ and finally setting $q = 0$, we have the so-called mth-order deformation equation

$$L[f_m(t) - \chi_{m-1} f_{m-1}(t)] = hH(t)R_m[\overrightarrow{f}_{m-1}(t)], \tag{8}$$

where

$$\chi_{m-1} = \begin{cases} 0, & m = 1 \\ 1, & m \neq 1 \end{cases}$$

and

$$R_m[\overrightarrow{f}_{m-1}(t)] = \frac{1}{(m-1)!} \left. \frac{d^{m-1} N[\Phi(t;q)]}{dq^{m-1}} \right|_{q=0}.$$

The so-called high-order deformation equation (8) is a linear which can be easily solved by symbolic computation software such as Maple, Mathematica and Matlab. In this way, the nonlinear equation is transferred into an infinite number of linear equation and $f_m(t)$ can be obtained easily. When $m \to \infty$, then we have an approximate power series solution (7) of equation (1).

3 Results and discussion

In order to demonstrate the effectiveness of the proposed method, we consider the following initial values problem in the case of the inhomogeneous fuzzy Bagley-Torvik equation [11].

$$a_i \frac{d^2 f(t)}{dt^2} + b_{i0} D_t^{3/2} f(t) + c_i f(t) = 1 + t,$$

where $t \geq 0$. For fuzzy rule $i = 1$, $a_1 = b_1 = c_1 = 1$, the exact solution is $f(t) = 1 + t$ and subject to initial conditions $f(0) = 1$ and $\dot{f}(0) = 1$.

In view of the HAM presented above, according to equation (8), if we take the auxiliary function $H(t) = 1$ and the auxiliary linear operator $L[\Phi(t;q)] = \Phi(t;q)$, the mth-order deformation equation for fuzzy rule $i = 1$ is defined as

$$f_m(t) = (\chi_{m-1} + h) f_{m-1}(t) + h \left[\frac{d^2 f_{m-1}(t)}{dt^2} + {}_0D_t^{3/2} f_{m-1}(t) - (1 - \chi_{m-1})(1 + t) \right].$$

Using the initial guess approximation $f_0(t) = 0$, then we obtain the various iterates as below:

$$\begin{aligned} f_1(t) &= -h(1+t) \\ f_2(t) &= -h(1+h)(1+t) \\ f_3(t) &= -h(1+h)^2(1+t) \\ &\vdots \end{aligned}$$

Proceeding in this manner, the rest of components $f_m(t)$ for $m \geq 4$ can be obtained and the series solution is thus entirely determined. With the aid of Matlab software, 20^{th} order series solution is generated to evaluate the solution. The h curves of $\dot{f}(0)$ are given in Figure 1, which shows that the valid region of h is the horizontal line segment.

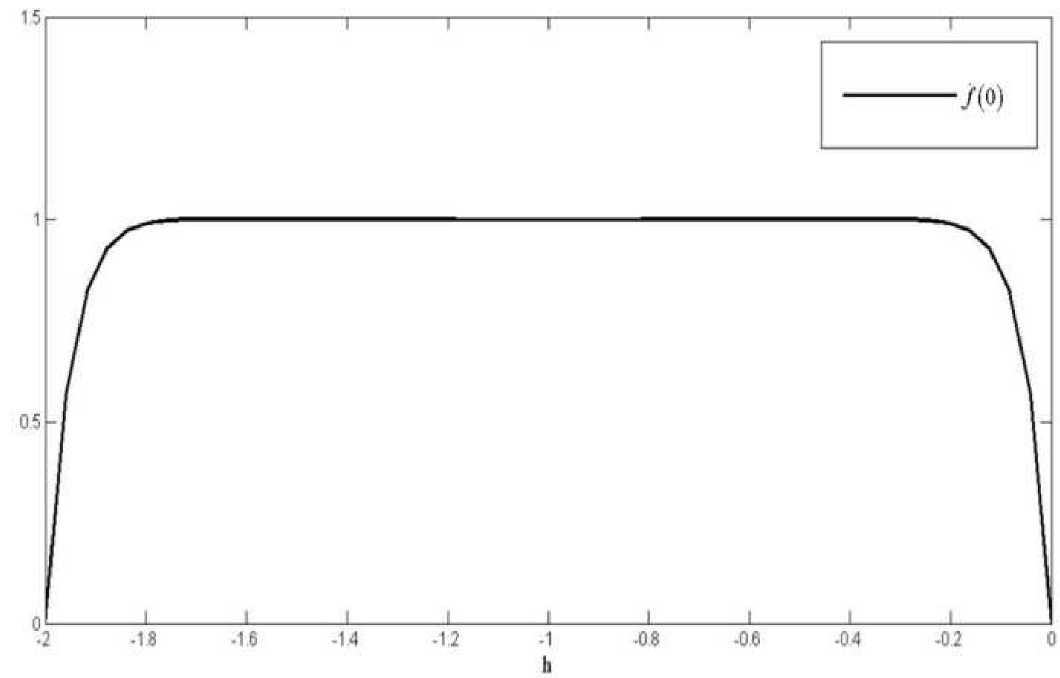

Figure 1: The h curves of $\dot{f}(0)$

Thus, the series solution is convergent for $-1.75 \leq h \leq -0.25$ at fuzzy rule $i = 1$. Also graphs of comparison between the exact and series solutions are plotted for different h values at fuzzy rule $i = 1$ to verify the convergence of the series solution are given in Figure 2.

If we choose $h = -1$, then clearly, we can conclude that the obtained series solution approximately converges to the exact solution. For fuzzy rule

$$i = 2, a_2 = b_2 = c_2 = 2,$$

similarly fuzzy Bagley-Torvik equation can be solved using the HAM.

4 Conclusion

The main aim of this research to illustrate HAM, which introduces a significant improvement in solving fuzzy systems of FDEs. This reliable approach leads to higher accuracy as evident from the result of the numerical example and the utilization is straightforward without using perturbation or restrictive assumptions. This proposed approach can be further developed and enhanced for fuzzy systems of FDEs in fractional calculus field as well.

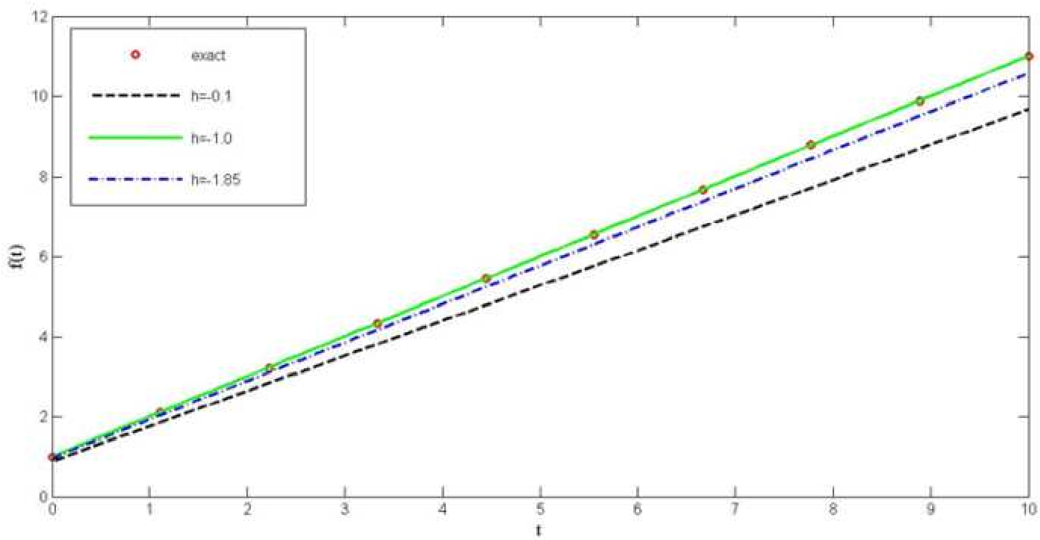

Figure 2: Comparison between the exact and series solutions are plotted for different h values

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