MATEMATIKA, 2016, Volume 32, Number 2, 153–159
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A new paradigm of fuzzy control point in space curve

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Abstract The control point is the most important element in the production of spline curve or surface model. This is because any changes of control points in the spline model affect the shape of the resulting curve or surface. Wahab and colleagues have introduced fuzzy control points to solve the problem of uncertainty prevailing in the spline modeling. However, based on this concept, this paper will discusses a new type of fuzzy control point that can generates a spline space curve model in 3-dimensional. This is because the generated control point is a 3-dimensional that satisfies the basic concepts of fuzzy set was introduced by Zadeh. However, this paper only taking a B-Spline model as a numerical example in the discussed model.

Keywords Space curve; fuzzy set; spline; control point.

2010 Mathematics Subject Classification 03E72, 93C42, 94D05

1 Introduction

Fuzzy control point is a geometric coefficient that was hybridized with fuzzy approach to produce fuzzy geometric modeling. The fuzzy control point was blended with the spline basis functions to produce Fuzzy Spline Model such as Fuzzy Bezier, Fuzzy B-Spline and Fuzzy NURBS (Non-Uniform Rational B-Splines). Idea of fuzzy control point has been introduced by Wahab et al. to resolve the uncertainty prevailing in geometric modeling. Many previous studies related to fuzzy geometry have been discussed to solve the problem of uncertainty in industrial design and Geographic Information System (GIS) [1-3]. However, this paper will discuss a new paradigm of fuzzy control point which generates space curve in 3-dimensional space that meets the characteristics of fuzzy set was introduced by Zadeh in 1965. The resulting curve looks like floating in a 3-dimensional space that can be seen in various planes or axes.

2 Preliminaries

In this section, we briefly review the concept of fuzzy sets [4] and [5], the $\alpha-$cuts of fuzzy sets [6], the fuzzy number [7] and the fuzzy control points [8-12].

2.1 Fuzzy set

In 1965, Zadeh proposed the theory of fuzzy sets [4]. The fundamental of fuzzy set theory will be used in solving the problem of uncertainty data.

Definition 1 Let X be a nonempty set. A fuzzy set A denoted $\tilde{A}$ in X is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$ and $\mu_A(x)$ is called the degree of membership element x in $\tilde{A}$ for every $x \in X$. Fuzzy set is also a set of ordered pairs and can be written as $\tilde{A} = \{x, \mu_A(x) : x \in X\}$ [11]. If $X = \{x_1, x_2, \ldots, x_n\}$ is a finite set and $\tilde{A} \subseteq X$, then $\tilde{A}$ can be written as

$$\tilde{A} = \{(x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)), \ldots, (x_n, \mu_A(x_n))\} \subseteq X \tag{1}$$

Chen and Chang [13] represented the trapezoidal fuzzy set shown in Figure 1 by quadruple (a_1, a_2, a_3, a_4) where $A = (a_1, a_2, a_3, a_4)$. If $a_2 = a_3$, the trapezoidal fuzzy set shown in Figure 1 becomes a triangular fuzzy set, as shown in Figure 2, where $A = (a_1, a_2, a_3, a_4) = (a_1, a_3, a_3, a_4)$.

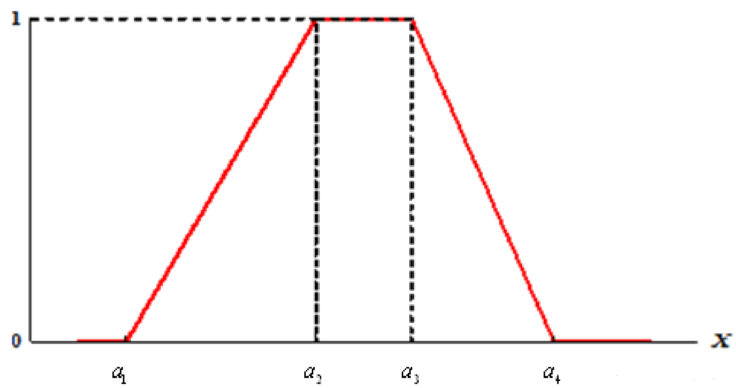

Figure 1: A trapezoidal fuzzy set A

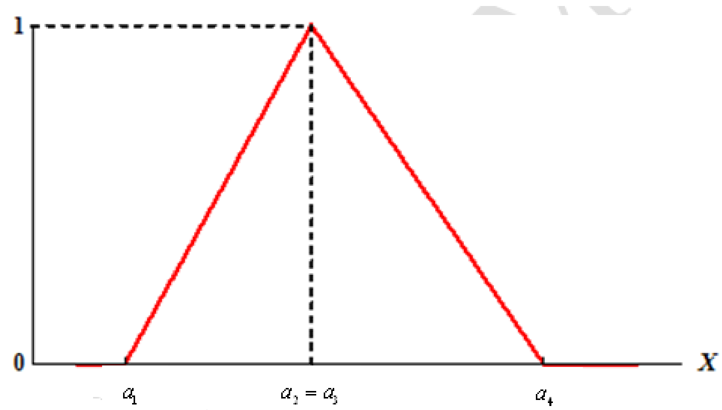

Figure 2: A triangular fuzzy set A

Definition 2 Let A be a fuzzy set in the universe of discourse X and let μ_A be the membership function of the fuzzy set A. The $\alpha-$cut A_α of the fuzzy set$A$ is an interval in the universe of discourse$X$ defined as follows [14]:

$$A_\alpha = \{x_i | x_i \in X \text{ and } \mu_A(x_i) \geq \alpha\} \tag{2}$$

where $\alpha \in (0, 1]$.

A fuzzy set in the universe of discourse X is convex if and only if for all x_1, x_2 in X,

$$\mu_A\left(\lambda x_1 + (1-\lambda)\, x_2\right) \geq \min\left(\mu_A\left(x_1\right), \mu_A\left(x_2\right)\right) \tag{3}$$

where $\lambda \in (0, 1]$. If $\exists x_i \in X$, such that $\mu_A\left(x_i\right) = 1$, then the fuzzy set is called a normal fuzzy set. A fuzzy number is a fuzzy set in the universe of discourse that is both convex and normal.

Definition 3 Let $X, Y \subseteq R$ be universal sets, then

$$\overleftrightarrow{R} = \left\{\left((x,y), \mu_{\overleftrightarrow{R}}(x,y)\right) | (x,y) \in X \times Y\right\} \tag{4}$$

is called a fuzzy relation on $X \times Y$ [15].

Definition 4 Let $X, Y \subseteq R$ and $\overleftrightarrow{A} = \{x, \mu_{\overleftrightarrow{A}}(x) | x \in X\}$ and $\overleftrightarrow{B} = \{y, \mu_{\overleftrightarrow{B}}(y) | y \in Y\}$ are two fuzzy sets. Then $\overleftrightarrow{R} = \left\{\left((x,y), \mu_{\overleftrightarrow{R}}(x,y)\right) | (x,y) \in X \times Y\right\}$ is a fuzzy relation on $\overleftrightarrow{A}$ and $\overleftrightarrow{B}$ if $\mu_{\overleftrightarrow{R}} \leq \mu_{\overleftrightarrow{A}}(x) \forall (x,y) \in X \times Y$ and $\mu_{\overleftrightarrow{R}} \leq \mu_{\overleftrightarrow{B}}(y) \forall (x,y) \in X \times Y$ [15].

Definition 5 Let $X, Y \subseteq R$ and $\overleftrightarrow{M} = \{x, \mu_{\overleftrightarrow{M}}(x) | x \in X\}$ and $\overleftrightarrow{N} = \{x, \mu_{\overleftrightarrow{N}}(y) | y \in Y\}$ represent two fuzzy data. Then the fuzzy relation between both fuzzy data is given by $\overleftrightarrow{C} = \left\{\left((x,y), \mu_{\overleftrightarrow{C}}(x,y)\right) | (x,y) \subseteq X \times Y\right\}$ [15].

Definition 6 Set $\tilde{D} = \left\{\tilde{D}_i\right\}$ in a space $\psi\left(\tilde{D}_i\right)$ considered as uncertainty data (fuzzy data) if and only if for every $\tilde{D}_i$ with $i = 0, 1, 2, \ldots, n$, its grade membership that defined by function

$$\mu_D\left(D_i\right) = \begin{cases} f\left(D_i\right) = \alpha \in (0,1] & \text{for} \quad D_i \tilde{\in} [a,b), \quad (f \text{ increasing}) \\ 1 & \text{for} \quad D_i \in [b,c], \quad (\text{at peak}) \\ g\left(D_i\right) = \beta \in (0,1] & \text{for} \quad D_i \tilde{\in} (c,d], \quad (f \text{ decreasing}) \\ 0 & \text{for} \quad D_i < a \text{ or } D_i > d, \quad f\left(D_i\right) = g\left(D_i\right) = 0 \end{cases} \tag{5}$$

Then, $\left\{\tilde{D}_i\right\}_{i=0}^{n} = \left\{\tilde{D}_0 = \tilde{D}_0^{\prec}, D_0, \tilde{D}_0^{\succ}, \tilde{D}_1^{\prec}, \tilde{D}_1, \tilde{D}_1^{\succ}, \ldots, \tilde{D}_n^{\prec}, D_n, \tilde{D}_n^{\succ}\right\}$ are 3-tuple set interval forms of fuzzy data. If $\alpha_i \in (0,1]$, then $\tilde{D}_{\alpha_i} = \left\{\tilde{D}_i^{\prec} - \varepsilon, D_i, \tilde{D}_i^{\succ} + \delta\right\}$ in $\psi\left(\tilde{D}_{\alpha_i}\right)$ with $D_i \in [b,c]$. If $f\left(D_i\right) = g\left(D_i\right)$, then $\epsilon = \delta$, and if $f\left(D_i\right) \neq g\left(D_i\right)$, then $\epsilon \neq \delta$.

Definition 7 Fuzzy set $\tilde{P}$ in a space of S said set of fuzzy control point if and only if for every $\alpha-$level set was chosen, there exist pointed interval which is $P = \tilde{P}_i^{\prec}, P_i, \tilde{P}_i^{\succ}$ in S with every P_i is crisp point and membership function $\mu_P : S \rightarrow (0,1]$ which is defined as $\mu_P\left(P_i\right) = 1$ [13], [14] and [15].

$$\mu_P\left(P_i^{\prec}\right) = \begin{cases} 0 & \text{if} \quad P_i^{\prec} \notin S \\ c \in (0,1) & \text{if} \quad P_i^{\prec} \tilde{\in} S \\ 1 & \text{if} \quad P_i^{\prec} \in S \end{cases} \quad \text{and} \quad \mu_P\left(P_i^{\succ}\right) = \begin{cases} 0 & \text{if} \quad P_i^{\succ} \notin S \\ k \in (0,1) & \text{if} \quad P_i^{\succ} \tilde{\in} S \\ 1 & \text{if} \quad P_i^{\succ} \in S \end{cases} \tag{6}$$

with $\mu_P\left(P_i^{\prec}\right)$ and $\mu_P\left(P_i^{\succ}\right)$ are left membership grade value and right membership grade value respectively and generally written as

$$\tilde{P} = \left\{\tilde{P}_i : i = 0, 1, 2, \ldots, n\right\} \tag{7}$$

for every $\tilde{P}_i = \tilde{P}_i^{\prec}, P_i, \tilde{P}_i^{\succ}$ with $\tilde{P}_i^{\prec}, P_i$ and $\tilde{P}_i^{\succ}$ are left fuzzy control point, crisp control point and right control point respectively [8], [9], [16] and [17].

3 Fuzzy system of linear equation

The $n \times n$ fuzzy system of linear equations with $m-$ degree polynomial parametric form may be written as

$$\begin{array}{l} \tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 + \ldots + \tilde{a}_{1n}x_n = \tilde{b}_1 \\ \tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 + \ldots + \tilde{a}_{2n}x_n = \tilde{b}_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \tilde{a}_{n1}x_1 + \tilde{a}_{n2}x_2 + \ldots + \tilde{a}_{nn}x_n = \tilde{b}_n. \end{array} \tag{8}$$

In matrix, notation the above system may be written as $\left[\tilde{A}\right]\{X\} = \left\{\tilde{b}\right\}$, where the coefficient matrix $\left[\tilde{A}\right] = (\tilde{a}_{kj}) = \left(a_{kj}^{\prec}, a_{kj}, a_{kj}^{\succ}\right)$, $1 \leq k, j \leq n$ is a fuzzy $n \times n$ matrix $\left\{\tilde{b}\right\} = \left(\tilde{b}_k\right) = (b_k^{\prec}, b_k, b_k^{\succ})$, $1 \leq k$ is a column vector of fuzzy number and $\{X\} = \{x_j\}$ is a vector of crisp unknown. For positive integer m, all $\tilde{a}_{kj}$ and $\tilde{b}_k$ are fuzzy numbers with $m-$ degree polynomial form.

The above system $\left[\tilde{A}\right]\{X\} = \left\{\tilde{b}\right\}$, can be written as

$$\sum_{j=1}^{n} \tilde{a}_{kj}x_j = \tilde{b}_k \text{ for } k = 1, 2, \ldots, n. \tag{9}$$

As per the parametric form we may write Equation (9) as

$$\sum_{j=1}^{n} \left(a_{kj}^{\prec}, a_{kj}, a_{kj}^{\succ}\right) x_j = \left(b_k^{\prec}, b_k, b_k^{\succ}\right) \text{ for } k = 1, 2, \ldots, n. \tag{10}$$

4 A new paradigm of fuzzy control point

As discussed above, this paper only focuses on polynomial in spline model. Wahab et al. redefined a fuzzy control point as in [10-12]. In this section, we propose a new paradigm of fuzzy control point (coefficient) which is control point in the model will produce a curve model in 3-dimensional space.

Definition 8 Let $\overleftrightarrow{P}_\alpha$ be a fuzzy control point in 3–dimensional space.

$\overleftrightarrow{P}_\alpha = \left\{(\tilde{x}, \tilde{y}, \tilde{z}) | (\tilde{x}, \tilde{y}) \in X \times Y, \tilde{z} \in (0, 1]\right\}$ where $(\tilde{x}, \tilde{y})$ is fuzzy data point, $\tilde{D}_i$.

$\tilde{D}_i = \left\{\tilde{D}_{i_\alpha}^{\min}, \tilde{D}_i, \tilde{D}_{i_\alpha}^{\max}\right\}$ for $i = 0, 1, 2, \ldots, n$ and $\alpha \in (0, 1]$.

If $\tilde{z} = \mu_{(x_i,y_i)}$ and $\tilde{D}_{i_\alpha}^{\min} = \left(\min\left(x_{i_\alpha}, y_{i_\alpha}\right), \mu_{\tilde{D}_{i_\alpha}^{\min}}\left(x_{i_\alpha}, y_{i_\alpha}\right)\right), \tilde{D}_i = \left(\left(x_{i_1}, y_{i_1}\right), \mu_{\tilde{D}_i}\left(x_{i_1}, y_{i_1}\right)\right)$ and $\tilde{D}_{i_\alpha}^{\mathrm{mak}} = \left(\max\left(x_{i_\alpha}, y_{i_\alpha}\right), \mu_{\tilde{D}_{i_\alpha}^{\max}}(x_{i_\alpha}, y_{i_\alpha})\right)$ respectively, then

$$\overset{\leftrightarrow}{P}_{i\alpha} = \begin{cases} \left(\min\left(x_{i_\alpha}, y_{i_\alpha}\right), \mu_{\tilde{D}_{i_\alpha}^{\min}}\left(x_{i_\alpha}, y_{i_\alpha}\right)\right) & \text{for left fuzzy number } (f \text{ increasing}), \\ \left(\left(x_{i_1}, y_{i_1}\right), \mu_{D_i}\left(x_{i_1}, y_{i_1}\right)\right) \quad \alpha = 1 & \text{for crisp number (at peak)}, \\ \left(\max\left(x_{i_\alpha}, y_{i_\alpha}\right), \mu_{\tilde{D}_{i_\alpha}^{\max}}(x_{i_\alpha}, y_{i_\alpha})\right) & \text{for right fuzzy number } (f \text{ decreasing}). \end{cases} \tag{11}$$

By Definition 1, a new type of fuzzy control point can be produced as below that satisfy the concept of fuzzy set.

$$\overset{\leftrightarrow}{P}_{i\alpha} = \left(\tilde{D}_i, \mu_{\overset{\leftrightarrow}{P}_{i\alpha}}\left(\tilde{D}_i\right)\right) \tag{12}$$

So, Equation (12) can be rewrite as

$$\overset{\leftrightarrow}{P}_{i\alpha} = \left(\tilde{x}, \tilde{y}, \mu_{\overset{\leftrightarrow}{P}_{i\alpha}}\left(\tilde{x}, \tilde{y}\right)\right) \tag{13}$$

5 Discussion and numerical example

Fuzzy B-Spline curve was introduced by using fuzzy number concept which is used in modeling the uncertainty data point for curve modeling. Fuzzy B-Spline curve had been discussed in [10], [11] and [12]. Therefore, fuzzy B-Spline can be defined as

$$\overset{\longleftrightarrow}{BS}(t) = \sum_{i=1}^{k+n-1} \overset{\leftrightarrow}{P}_{i\alpha} B_{i,n}(t) \tag{14}$$

where $\overset{\leftrightarrow}{P}_{i\alpha}$ are fuzzy control point and $B_{i,n}(t)$ are B-Spline basic function with crisp knot sequences $t_1, t_2, t_{m=d+n+1}$ where d represents the degree of B-Spline function and n represents the numbers of control points.

Figure 3 shows the curve of a Fuzzy B-Spline Model in the space generated by B-Spline basis functions and fuzzy control point that is characterized by three dimensional. A new type of Fuzzy Control Point $\left(\tilde{x}, \tilde{y}, \mu_{\overset{\leftrightarrow}{P}_{i\alpha}}\left(\tilde{x}, \tilde{y}\right)\right)$ meets the basic theory of fuzzy set which were introduced by Zadeh as in Definition 1.

Fuzzy control point $\overset{\leftrightarrow}{P}_{i\alpha}$ has a degree of expertise in the z-axis to generate a new B-Spline curve which looked as if floating in a 3-dimensional space. The changes of the degree of membership in each control points $\overset{\leftrightarrow}{P}_{0\alpha}, \overset{\leftrightarrow}{P}_{1\alpha}, \overset{\leftrightarrow}{P}_{2\alpha}, \overset{\leftrightarrow}{P}_{3\alpha}, \overset{\leftrightarrow}{P}_{4\alpha}$ will change the position of curve as shown in Figure 3. In this example, we use the control points with membership value $\overset{\leftrightarrow}{P}_{i0.2}, \overset{\leftrightarrow}{P}_{i0.5}$ and $\overset{\leftrightarrow}{P}_{i0.7}$ to reflect the changes that occur on the curve. Figure 3 (b) and (c) each show Fuzzy B-Spline curves can be seen from the plan view (z-axis) and the front view (x-axis).

6 Conclusion

This paper discussed a new paradigm of fuzzy control point that generates fuzzy spline space curve. This new fuzzy control point was redefined and satisfies the fuzzy set theory

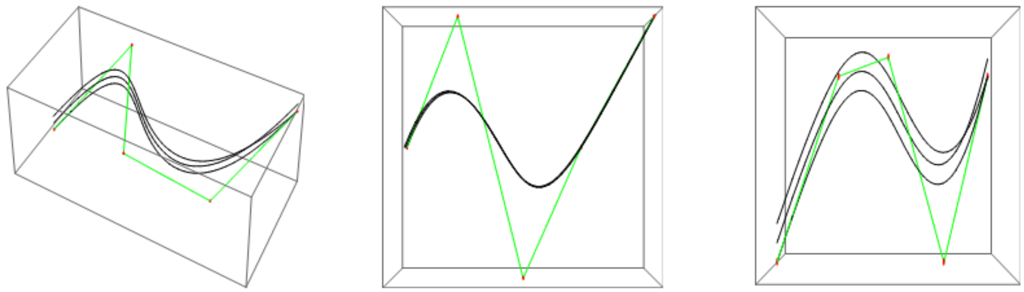

Figure 3: B-Spline space curve in (a) 3-dimesional x, y, z-axis; (b) top view (z-axis) and (c) front view (x-axis)

introduced by Zadeh. The proposed ideas can be extended to other spline model such as Bezier and NURBS model.

Acknowledgments

On this occasion, we would like to give our sincere appreciation to the Research Management Centre (RMC), Universiti Malaysia Terengganu and the Ministry of Education Malaysia for research grant ERGS (Vote 55099) to conduct this research.

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