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The method of lines solution of the Forced Korteweg-de Vries-Burgers equation

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Abstract In this paper, the application of the method of lines (MOL) to the Forced Korteweg-de Vries-Burgers equation with variable coefficient (FKdVB) is presented. The MOL is a powerful technique for solving partial differential equations by typically using finite-difference approximations for the spatial derivatives and ordinary differential equations (ODEs) for the time derivative. The MOL approach of the FKdVB equation leads to a system of ODEs. The solution of the system of ODEs is obtained by applying the Fourth-Order Runge-Kutta (RK4) method. The numerical solution obtained is then compared with its progressive wave solution in order to show the accuracy of the MOL method.

Keywords FKdVB Equation; partial differential equations; the method of lines; system of differential equation; Runge Kutta method.

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1 Introduction

There are many physical phenomena in engineering and physics which can be expressed by some nonlinear partial differential equations (PDEs) [1]. However, most of them do not have exact analytical solutions. Therefore, these nonlinear equations should be solved by using approximation method [2].

In literature, weakly nonlinear wave propagation in a prestressed fluid-filled stenosed elastic tube filled with a Newtonian fluid with variable viscosity fluid has been studied by [3] by applying the reductive perturbation method and long wave approximation, the governing equations. By employing the stretched coordinate of initial-value type and extending the field quantities into the asymptotic series of order ε, where ε is a small parameter, the nonlinear wave propagation in such medium is governed by the forced Korteweg-de Vries-Burgers (FKdVB) equation with variable coefficient. The FKdVB can be written as

$$U_\tau + \mu_1 U U_\xi - \mu_2 U_{\xi\xi} + \mu_3 U_{\xi\xi\xi} + \mu_4(\tau) U_\xi = \mu(\tau), \tag{1}$$

where ξ is a spatial variable, τ is a temporal variable, μ_1, μ_2, μ_3, $\mu_4(\tau)$ and $\mu(\tau)$ are the coefficients of nonlinear, dissipative, dispersive, variable coefficient and forcing term respectively. The presence of forcing terms $\mu(\tau)$ and variable coefficient term $\mu_4(\tau)$ show the presence of stenosis. The dissipative term $-\mu_2 u_{\xi\xi}$is caused by the effect of variable viscosity. The coefficients of μ_1, μ_2, μ_3, $\mu_4(\tau)$ and $\mu(\tau)$ are defined by [3] as

$$\begin{aligned} &\mu_1 = \frac{5}{2\lambda_\theta} + \frac{\beta_2}{\beta_1}, \quad \mu_2 = \frac{\nu}{2c}, \quad \mu_3 = \frac{m}{4\lambda_z} + \frac{\lambda_\theta^2}{16} - \frac{\beta_0}{2\beta_1}, \\ &\mu_4(\tau) = \frac{\lambda_\theta\gamma_2}{\beta_1}G(\tau) - \left[\frac{\beta_2}{\beta_1} + \frac{1}{2\lambda_\theta}\right]g(\tau), \quad \mu(\tau) = \frac{1}{2}g'(\tau) - \frac{\lambda_\theta\gamma_1}{2\beta_1}G'(\tau), \end{aligned} \tag{2}$$

where

$$\begin{aligned} \gamma_0 &= \frac{1}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)F(\lambda_\theta, \lambda_z), \\ \gamma_1 &= \frac{1}{\lambda_\theta\lambda_z}\left[\left(1 + \frac{3}{\lambda_\theta^4\lambda_z^2}\right) + 2\alpha\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2\right]F(\lambda_\theta, \lambda_z), \\ \gamma_2 &= \frac{1}{2\lambda_\theta\lambda_z}\left[-\frac{12}{\lambda_\theta^5\lambda_z^2} + 6\alpha\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)\left(1 + \frac{3}{\lambda_\theta^4\lambda_z^2}\right) + 4\alpha^2\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^3\right]F(\lambda_\theta, \lambda_z), \\ \beta_0 &= \frac{1}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right)F(\lambda_\theta, \lambda_z), \quad \beta_1 = \gamma_1 - \frac{\gamma_0}{\lambda_\theta}, \quad \beta_2 = \gamma_2 - \frac{\beta_1}{\lambda_\theta}. \end{aligned} \tag{3}$$

Given that

$$F(\lambda_\theta, \lambda_z) = \exp\left[\alpha\left(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2\lambda_z^2} - 3\right)\right],$$

$\alpha = 1.948, \lambda_\theta = \lambda_z = 1.6, v = 1$, $c = 15.391$, $m = 0.1$, $G(\tau) = 0$ and $g(\tau) = \operatorname{sech}(0.01\tau)$. Here α refer to material constant, λ_θ is the initial circumferential stretch ratio, λ_z is the initial axial stretch ratio, ν is kinematic viscosity, m is a mass of an artery and c is the scale parameter.

The application of the MOL to the FKdVB equation (1) will be presented in this paper. It is shown that the MOL approach of the FKdVB equation leads to a system of ODEs. The solution of the system was obtained by applying the RK4 method. The solution of the FKdVB equation that is obtained by using the MOL with progressive wave solution conducted by Tay [3] is then compared in terms of its maximum absolute error at a certain time τ.

2 The MOL

The MOL is a powerful method used to solve PDEs. It involves making an approximation to the spatial derivatives and reducing the problem into a system of ODEs [4-6]. In addition, this system of ODEs can be solved by using time integrator. The most important advantage of the MOL approach is that it has not only the simplicity of the explicit methods [7] but also the superiority (stability advantage) of the implicit ones unless a poor numerical method for the solution of ODEs is employed. It is possible to achieve higher-order approximations in the discretization of spatial derivatives without significant increases in the computational complexity. This method has wide applicability to physical and chemical systems modeled by PDEs such as delay differential equations [8], two-dimensional sine-Gordon equation [9], the Nwogu one-dimensional extended Boussinesq equation [10], the fourth-order Boussinesq equation, the fifth-order Kaup–Kupershmidt equation and an extended Fifth-Order Korteweg-de Vries (KdV5) equation [11].

In this paper, the spatial derivatives are firstly discretized using central finite difference formulae as follows:

$$\begin{aligned} U_\xi &\approx \frac{U_{j+1}-U_{j-1}}{2\Delta\xi}, \\ U_{\xi\xi} &\approx \frac{U_{j+1}-2U_j+U_{j-1}}{(\Delta\xi)^2}, \\ U_{\xi\xi\xi} &\approx \frac{U_{j+2}-2U_{j+1}+2U_{j-1}-U_{j-2}}{2(\Delta\xi)^3}, \end{aligned} \tag{4}$$

where ξ is the spatial variable, τ is the temporal variable, j is the index denoting the spatial position along ξ-axis and $\Delta\xi$ is the step size along the spatial axis. The ξ-interval is divided into M points with $j = 1, 2, \ldots, M-1, M$. Therefore, the MOL approximation of equation (1) is given by

$$\begin{aligned} &\frac{\partial U_j}{\partial \tau} = -\frac{\mu_1}{2\Delta\xi} U_j\left(U_{j+1}-U_{j-1}\right) + \frac{\mu_2}{(\Delta\xi)^2}\left(U_{j+1}-2U_j+U_{j-1}\right) \\ &-\frac{\mu_3}{2(\Delta\xi)^3}\left(U_{j+2}-2U_{j+1}+2U_{j-1}-U_{j-2}\right) - \frac{\mu_4(\tau)}{2\Delta\xi}\left(U_{j+1}-U_{j-1}\right) + \mu(\tau) \\ &\equiv f(U_j)\ . \end{aligned} \tag{5}$$

Equation (5) is written as an ODE since there is only one independent variable, which is τ. Also, equation (5) represents a system of M equations of ODEs. The initial condition for equation (5) after discretization is given by

$$U(\xi_j, \tau = 0) = U_0(\xi_j), \qquad j = 1, 2, ..., M-1, M. \tag{6}$$

For the time integration, the RK4 method is applied. Thus, the numerical solution at time τ_{i+1} is

$$U_{i+1,j} = U_{i,j} + \frac{1}{6}\left(a_{i,j} + 2b_{i,j} + 2c_{i,j} + d_{i,j}\right)\ , \tag{7}$$

where

$$\begin{aligned} a_{i,j} &= \Delta\tau\, f(U_{i,j}), \\ b_{i,j} &= \Delta\tau\, f\left(U_{i,j} + \frac{1}{2}a_{i,j}\right), \\ c_{i,j} &= \Delta\tau\, f\left(U_{i,j} + \frac{1}{2}b_{i,j}\right), \\ d_{i,j} &= \Delta\tau\, f(U_{i,j} + c_{i,j}). \end{aligned} \tag{8}$$

Here $\Delta\tau$ is the step size of the temporal coordinates.

3 Progressive wave solution

The progressive wave solution of the FKdVB equation as given by [3] is

$$U=\frac{a}{\mu_1}+\frac{3\mu_2^2}{25\mu_1\mu_3}\left(\operatorname{sech}^2\zeta-2\tanh\zeta\right)+\frac{1}{2}\left[g\left(\tau\right)-\frac{\lambda_\theta\lambda_1}{\beta_1}G\left(\tau\right)\right], \tag{9}$$

where a is a constant. The phase function ζ can be expressed as

$$\zeta=\frac{\mu_2}{10\mu_3}\left\{\xi-a\tau-\int_0^\tau\left[\left(\frac{3}{4\lambda_\theta}-\frac{\beta_2}{2\beta_1}\right)g\left(s\right)\quad+\frac{\lambda_\theta}{\beta_1}\left(\gamma_2-\frac{\mu_1\gamma_1}{2}\right)G\left(s\right)\right]ds\right\} \tag{10}$$

4 Results and discussion

To test MOL on the FKdVB equation, we need the initial condition as follows:

$$U\left(\xi,0\right)=\frac{a}{\mu_1}+\frac{3\mu_2^2}{25\mu_1\mu_3}\operatorname{sech}^2\left(\frac{\mu_2}{10\mu_3}\xi\right)-\frac{6\mu_2^2}{25\mu_1\mu_3}\tanh\left(\frac{\mu_2}{10\mu_3}\xi\right)+0.5 \tag{11}$$

Figure 1 (a) gives the MOL solution of the FKdVB equation (1) with spatial parameters at certain time τ, while Figure 1 (b) represents the progressive wave solution of the FKdVB equation (1) with spatial parameters at certain time τ. The solution of the FKdVB equation (1) with space ξ shows a decreasing shock profile propagating to the right with a decrease in wave amplitude as time τ increases.

We then compute the absolute error between the progressive wave and MOL solutions for each discretized spatial point at a certain time τ and later find the maximum absolute error. The maximum absolute errors between the progressive wave and MOL solutions are calculated based on the formula

$$L_\infty =_{\max}\left|U_{progressive}-U_{MOL}\right| \tag{12}$$

Table 1. gives the maximum absolute error between the progressive wave solution and MOL solution. It shows the maximum absolute errors are in order of 10^{-6}.

Table 1: Maximum absolute error of the FKdVB equation for different time τ at $\Delta\tau=\partial\xi^{-3}$

Time, τ	0	10	20
L_∞	0	0.76506×10^{-6}	1×10^{-6}

The computing time for progressive wave and MOL is found to be 3.435802 seconds and 14.38130 seconds, respectively. The progressive wave is straight forward since we can plot the function of progressive wave, U versus time, τ directly. It consists 2 steps of operation where the input formula U (9) and ζ (10). Comparatively, MOL will be a little bit complex. It consists 6 steps of operation. The derivatives have to be replaced by finite-difference approximation to reduce it to the system of ODEs and calculate a, b, c and d in equation (8) and new U (7).

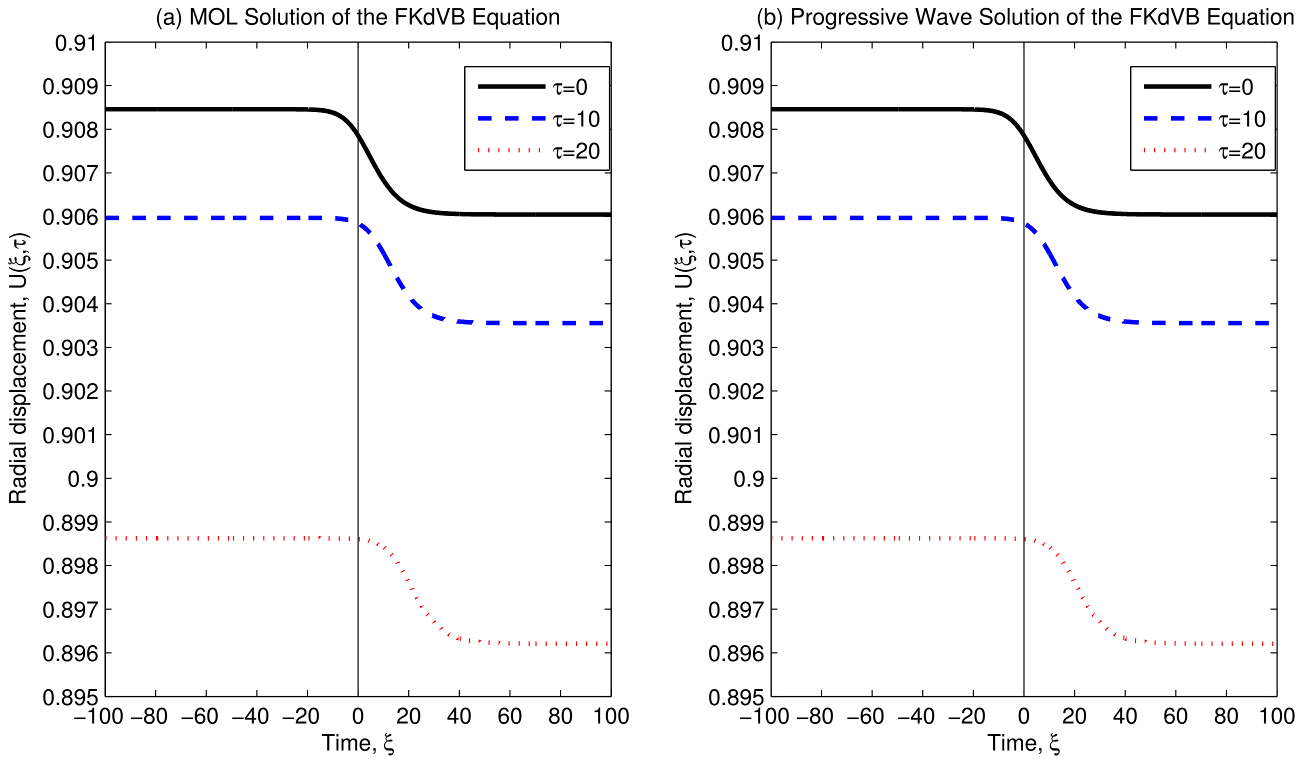

Figure 1: Solutions of the FKdVB equation versus space ξ for different time τ at $\Delta\xi = 0.1$

5 Conclusion

The MOL was employed to solve the FKdVB equation. It involved replacing the spatial derivatives in the PDE with finite-difference approximations and by doing that, the spatial derivatives are longer stated explicitly in terms of spatial independent variables. This led to a system of ODEs. The system was then solved by using the RK4 method. This paper has described the effect of computational effort with respect to the accuracy of results. The MOL solution of the FKdVB equation (1) was plotted versus its progressive wave solution. From the observation, it was found that there were no differences for both MOL and progressive wave solutions. The maximum absolute errors between both MOL and progressive wave solutions at a certain time τ were computed. Results revealed that the maximum absolute errors are in the order of 10^{-6} for $\Delta\xi = 0.1$ and $\Delta\tau = 1 \times 10^{-3}$. Hence, it can be concluded that the FKdVB equation can be solved successfully using the MOL.

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