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Stability of Charlie's Method on Linear Heat Conduction Equation

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Abstract Explicit schemes are attractive for obtaining finite difference solutions to partial differential equations because of their simplicity. However this feature is undermined by the severe restriction on stability that the schemes suffer. One method that appears to have better stability properties is Charlie's method. The stability region of this method applied to a one-dimensional heat conduction equation is discussed in this article.

Keywords Explicit schemes, Charlie's method, predictor-corrector, stability.

Abstrak *Kaedah tak tersirat sangat menarik dalam mendapatkan penyelesaian beza terhingga bagi persamaan terbitan separa kerana sifatnya yang ringkas. Akan tetapi batasan kestabilan yang teruk yang dialami oleh kaedah ini melemahkan ciri ini. Suatu kaedah yang mempunyai sifat kestabilan yang lebih baik ialah kaedah Charlie. Rantau kestabilan bagi kaedah ini untuk persamaan haba matra satu dibincangkan di sini.*

Katakunci *kaedah tak tersirat, kaedah Charlie, peramal-pembetul, kestabilan.*

1 Introduction

The simplest extrapolation techniques to solve partial differential equations are those explicit difference algorithms that normally suffer very stringent stability criteria. One method that has far better stability properties than the standard explicit scheme is Charlie's method. It is a predictor-corrector type algorithm in which the predictor is a transient (e.g. parabolic type) Euler forward approximation of the differential equation and the corrector is a convex-type operation.

The algorithm was developed in 1982 [2] to solve fluid flow problems. Although it has been around for almost two decades, it does not appear to be well-exposed to the computational community, perhaps because only a handful of literature is available written by the original authors [1, 2, 3]. The aim of this paper is to explore the stability properties of Charlie's method on a simple one-dimensional heat conduction equation in order to establish the computational credentials of the method.

2 The Algorithm

Consider the initial value problem

$$\frac{du}{dt} = f(u); \qquad u(t_0) = u_0, \tag{1}$$

to be approximated by Euler's method

$$U^{m+1} = U^m + \Delta t\, f(U^m), \quad U_0 = u_0. \tag{2}$$

Δt is the time step and U^m is the grid function corresponding to $u(t^m)$, $U^m \simeq u(t^m)$. Charlie's algorithm for solving equation (1) is a two-step procedure

$$\begin{aligned} &\text{predictor:} \quad \hat{U}^m = U^m + \Delta t\, f(U^m) \\ &\text{corrector:} \quad U^{m+1} = (1-\gamma)\hat{U}^m + \gamma[U^m + \Delta t\, f(\hat{U}^m)] \end{aligned} \tag{3}$$

where $0 < \gamma < 1$. If $\gamma = 0$ the original (explicit) method is recovered by the predictor.

The improved stability features of the method are closely related to an appropriate choice of γ. As an example of a partial differential equation consider the familiar linear heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{4}$$

The solution of equation (4) by Charlie's method may be obtained by applying the usual explicit FD formula that utilises a forward Euler step for the time derivative and a central difference replacement for the spatial derivative:

$$\begin{aligned} &\text{predictor:} \quad \hat{U}_i^m = U_i^m + r(U_{i-1}^m - 2U_i^m + U_{i+1}^m), \quad i = 1, 2, \ldots, n \\ &\text{corrector:} \quad U_i^{m+1} = (1-\gamma)\hat{U}_i^m + \gamma\left[U_i^m + r(\hat{U}_{i-1}^m - 2\hat{U}_i^m + \hat{U}_{i+1}^m)\right], \\ &\qquad\qquad i = 1, 2, \ldots, n \end{aligned} \tag{5}$$

where $U_i^m = U(x_i, t_m)$ is the grid function corresponding to $u(x_i, t_m)$, $r = \Delta t/h^2$, Δt is the time step and h is the spatial mesh size. If $\gamma = 0$ the corrector is not implemented and the algorithm again defaults to the underlying explicit method. A stability analysis is performed in order to determine the range of values of γ that may be used in (5).

3 Stability

Fourier analysis is a standard method for analysing the stability of discrete models of PDEs on a uniform grid [4]. Let E_i^m be the computational point error at the node x_i defined by

$$E_i^m = U_i^m - V_i^m \tag{6}$$

where U is the exact value satisfying the explicit method and V is the computed value (arising most commonly from the existence of rounding on finite-precision machine). Assume that the error E_i^m (6) is a multiple of the initial error,

$$E_i^m = \xi^m e^{j\beta ih} \tag{7}$$

and $|\xi| < 1$ for stability. Since U and V share the same form of the linear finite difference representation, so does E. Therefore, substituting (7) into the predictor form of (5) for E leads to

$$\hat{E}_i^m = \xi^m e^{j\beta ih}(1+\theta) \tag{8}$$

where

$$\theta = -4r\sin^2\left(\frac{\beta h}{2}\right),$$

i.e, $\theta < 0$. Substituting equations (7) and (8) into the E corrector form of (5) gives

$$\begin{aligned}\xi^{m+1}e^{j\beta ih} &= (1-\gamma)\xi^m e^{j\beta ih}(1+\theta) + \gamma\,\xi^m e^{j\beta ih} \\ &\quad + \gamma\, r\Big(\xi^m e^{j\beta(i-1)h}(1+\theta) - 2\xi^m e^{j\beta ih}(1+\theta) \\ &\quad + \xi^m e^{j\beta(i+1)h}(1+\theta)\Big)\end{aligned}$$

which simplifies to

$$\xi = 1 + \theta + \gamma\,\theta^2. \tag{9}$$

When $\gamma = 0$, then $\xi = 1 + \theta$ and $|\xi| < 1 \Rightarrow -2 < \theta < 0$ and $0 < r \leq 0.5$ [4]. In other words, the usual stability result for the explicit method is recovered. Otherwise,

$$-2 < \theta + \gamma\,\theta^2 < 0.$$

The right hand inequality gives

$$-\frac{1}{\gamma} < \theta < 0. \tag{10}$$

Now consider the left hand inequality,

$$\gamma\,\theta^2 + \theta + 2 > 0. \tag{11}$$

The boundary of the region satisfying the inequality (11) is defined by the equation $\gamma\,\theta^2 + \theta + 2 = 0$, that is

$$\theta = \frac{-1 \pm \sqrt{1-8\gamma}}{2\gamma} = \{\theta_1, \theta_2\}$$

with $\theta_1 \leq \theta_2$. From inequality (11), $(\theta - \theta_1)(\theta - \theta_2) > 0$ which requires that

$$\theta < \theta_1 \quad \text{or} \quad \theta > \theta_2. \tag{12}$$

Combining inequalities (10) and (12) leads to the following two conditions:

$$-\frac{1}{\gamma} < \theta < 0 \quad \text{and} \quad \theta > \theta_2 = \frac{-1 + \sqrt{1-8\gamma}}{2\gamma}$$

$$-\frac{1}{\gamma} < \theta < 0 \quad \text{and} \quad \theta < \theta_1 = \frac{-1 - \sqrt{1-8\gamma}}{2\gamma}.$$

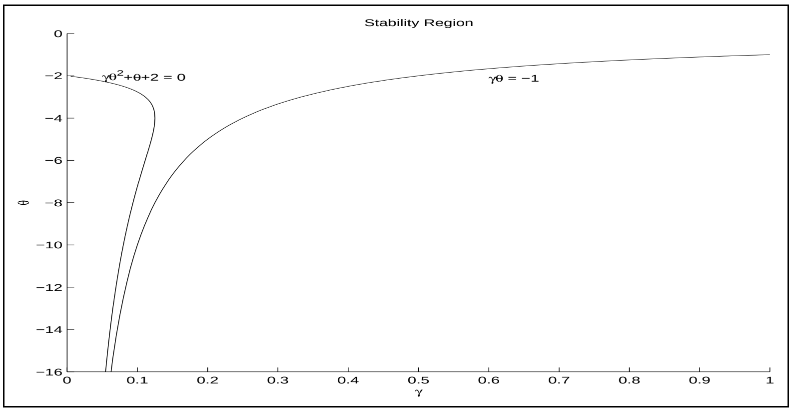

Figure 1: Stability region: $\gamma - \theta$ plane

By taking values $\gamma \in (0, 1/8)$ we see that only the first condition holds. Figure 1 shows the $\gamma - \theta$ stability region (the fourth quadrant of the plane) imposed by that condition.

For $\gamma > 0.125$, the quadratic curve no longer controls the method's stability. For this reason, the $0 < \gamma < 1$ is sub-divided into two intervals as follows:

$$\text{(i)} \quad \text{for} \quad 0 < \gamma < \frac{1}{8}, \quad \frac{-1 + \sqrt{1 - 8\gamma}}{2\gamma} < \theta < 0 \tag{13}$$

$$\text{(ii)} \quad \text{for} \quad \frac{1}{8} \leq \gamma < 1, \quad -\frac{1}{\gamma} \leq \theta < 0. \tag{14}$$

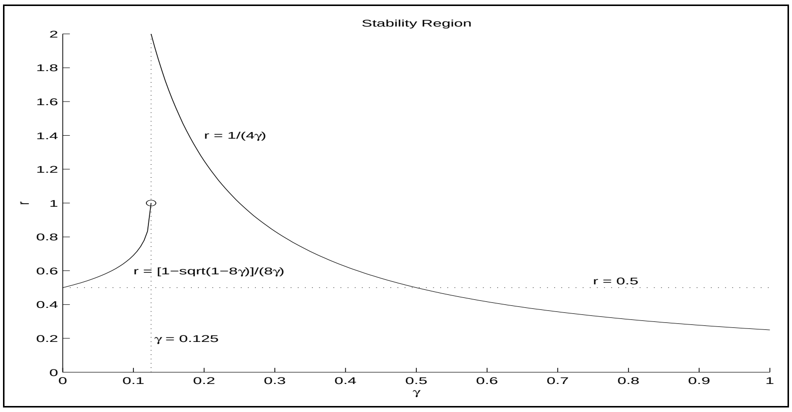

Figure 2: Stability region $\gamma - r$ plane

Because r defines a required computational parameter, a more practical way to look at the stability region is to transform the $\gamma - \theta$ plane to the $\gamma - r$ plane. Since $\theta = -4r\psi$ ($\psi = \sin^2(\beta h/2)$), inequality (13) becomes

$$-1 + \sqrt{1 - 8\gamma} < -8r\gamma\psi < 0$$

or

$$0 < r < \frac{1 - \sqrt{1 - 8\gamma}}{8\gamma\psi}.$$

This result must hold for all $0 \le \psi \le 1$, thus (taking the maximum value of ψ)

$$0 < r < \frac{1 - \sqrt{1 - 8\gamma}}{8\gamma}, \qquad 0 < \gamma < \frac{1}{8}, \tag{15}$$

(see Figure 2). Likewise inequality (14) becomes

$$-1 \le -4r\gamma\psi < 0,$$

that is

$$0 < r < \frac{1}{4\gamma}, \qquad \frac{1}{8} \le \gamma < 1. \tag{16}$$

To identify the limiting value of r as $\gamma \to 0$, we may expand the upper bound of r in condition (15) in a power series,

$$\begin{aligned} \lim_{\gamma \to 0} r &= \lim_{\gamma \to 0} \frac{1 - \sqrt{1 - 8\gamma}}{8\gamma} \\ &= \lim_{\gamma \to 0} \frac{1}{8\gamma}\{1 - (1 - 4\gamma + O(\gamma^2))\} \\ &= \frac{1}{2}. \end{aligned}$$

In other words, as $\gamma \to 0$ and Charlie's method 'approaches' the underlying numerical technique so the stability analysis recovers the correct restriction on r. Other significant limiting values, particularly at $\gamma = 1/8$ (critical value separating the stability region), are

$$\lim_{\gamma \to (1/8)^-} r = 1, \qquad \lim_{\gamma \to (1/8)^+} r = 2, \qquad \lim_{\gamma \to (1/2)} r = 1/2.$$

Note that for $\gamma = 1/2$, the time step used in Charlie's method is the same as that used in the underlying explicit method.

Figure 2 shows the stability region in the $\gamma - r$ plane. It appears that Charlie's method can increase the typical maximum Δt used in the explicit FD scheme by a factor of 4 from $r = 0.5$ to $r = 2$ as stated by previous authors [1, 3]. Note that for a fixed value of $r = 0.5$ (the basic method), the stability region in Figure 2 clearly shows that only values of γ in the range $[0, 0.5]$ can be used. Beyond that, the numerical solutions would be invalid as the fixed time step $\Delta t = h^2/2$ would lie outside the stability region. However, the whole range of $0 \le \gamma \le 1$ is applicable for a variable r (Δt variable). Values $\gamma > 1/2$ are less practical since the computational effort begins to increase since the time step is reduced below the basic value.

4 Conclusions

Stability region for a one-dimensional heat conduction equation using Charlie's method was explored. The range of γ that can be used for a fixed r implementation is $[0, 1/2]$ while all values of $0 \leq \gamma \leq 1$ are applicable for a variable r implementation. Employing $\gamma = 0$ or $\gamma = 1/2$ gives the same time step with $\gamma = 1/2$ requires more effort as one more computation (the corrector) is required. The maximum time step that Charlie's method can offer is by utilising $\gamma = 1/8$ when the time step obtained from the explicit scheme is improved by a factor of 4 from $\Delta t = h^2/2$ to $\Delta t = 2h^2$.

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