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Half-Sweep Geometric Mean Method for Solution of Linear Fredholm Equations

[1]**M.S. Muthuvalu** & [2]**J. Sulaiman**
School of Science and Technology, Universiti Malaysia Sabah
Locked Bag 2073, 88999 Kota Kinabalu, Sabah, Malaysia
e-mail: [1]sundaram_at2@yahoo.com, [2]jumat@ums.edu.my

Abstract The objective of this paper is to examine the application of the Half-Sweep Geometric Mean (HSGM) method by using the half-sweep approximation equation based on quadrature formula to solve linear integral equations of Fredholm type. The formulation and implementation of the Full-Sweep Geometric Mean (FSGM) and Half-Sweep Geometric Mean (HSGM) methods are also presented. Some numerical tests were carried out to show that the HSGM method is superior to the FSGM method in the sense of complexity and execution time.

Keywords Fredholm integral equations; quadrature formula; half-sweep geometric mean method.

1 Introduction

Generally, linear integral equations of the first kind have the form

$$\int_a^b K(x,t)y(t)\,dt = f(x), \quad a \le x \le b \tag{1}$$

where $y(x)$ is the unknown function, $K(x,t)$ is the kernel of the integral equation and $f(x)$ is a given function. $K(x,t)$ is called Fredholm kernel if the kernel in Eq. (1) is continuous on the square $S = \{a \le x \le b,\ a \le t \le b\}$ or at least square integrable on this square. Then, Eq. (1) with constant integration limits and Fredholm kernel are called Fredholm equations of the first kind (Polyanin & Manzhirov [12]).

Frequently, Fredholm integral equation of the first kind cannot be solved analytically for the unknown function $y(x)$. In many application areas, it is necessary to use the numerical approach to obtain an approximation solution for the problem. To solve a linear integral equation numerically, discretization of integral equation to the solution of system of linear algebraic equations is the basic concept used by researchers to solve integral equation problems. There are many discretization schemes can be used to gain a system of linear algebraic equations such as quadrature (Baker [5]; Polyanin & Manzhirov [12]; Abdou [1]; Laurie [10]), least squares (Ashour [4]), collocation (Maleknejad et al. [11]) and wavelet (Maleknejad et al. [11]) methods.

Consequently, the concept of the two-stage has been proposed widely to be one of the efficient methods for solving any system of linear algebraic equations. There are many two-stage iterative methods can be considered such as AGE (Evans & Sahimi [7]), IADE (Sahimi et al. [15]), RIADE (Sahimi & Khatim [16]), HSIADE (Sulaiman et al. [18]), QSIADE (Sulaiman et al. [19]), Asynchronous (Frommer & Szyld [8]), Block Jacobi (Allahviranloo et al. [3]) and Arithmetic Mean (AM) methods (Ruggiero & Galligani [14]).

Sulaiman et al. [20] has introduced Half-Sweep Arithmetic Mean (HSAM) method by combining the concept of half-sweep iteration and AM method. Half-sweep iterative method was introduced by Abdullah [2] via the Explicit Decoupled Group (EDG) to solve two-dimensional Poisson equation. Further studies of the HSAM method have been also conducted by Sulaiman et al. [21, 22] to solve Poisson equation. Apart from the HSAM method, Sulaiman et al. [23] also introduced Half-Sweep Geometric Mean (HSGM) method, combination of Geometric Mean (GM) method and half-sweep iteration. Sulaiman et al. [24] have been conducted a study to solve two-point boundary problems to verify the effectiveness of HSGM method. GM method can be also named as the Full-Sweep Geometric Mean (FSGM) method. In this paper, formulations of the FSGM and HSGM methods were developed to solve linear Fredholm integral equations of the first kind.

2 Full- and Half-sweep Quadrature Approximation Equations

Referring Fig. 1, the finite grid networks show the implementation of the full- and half-sweep iterative methods. Based on the Fig. 1, FSGM and HSGM methods will compute approximate values onto node points of type • only until the convergence criterion is reached. Then approximate values of other remaining points (points of the different type) are computed using the direct method, see Abdullah [2], Ibrahim & Abdullah [9] and Sulaiman & Abdullah [17].

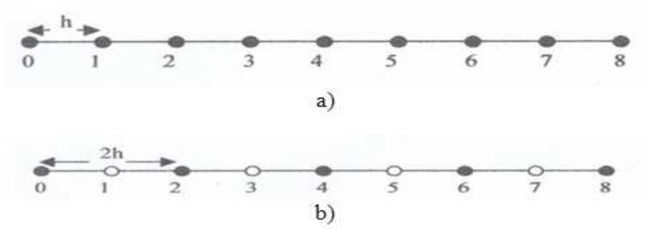

Figure 1: a) and b) Show the Distribution of Node Points for the Full- and Half-sweep Cases Respectively

In this paper, a discretization scheme based on method of quadrature was used to construct an approximation equation of an integral equation by replacing the integral to finite sums. Such formulas are called quadrature formulas and, in general have the form

$$\int_a^b y(t)dt = \sum_{j=0}^{n} A_j y(t_j) + \varepsilon_n(y) \tag{2}$$

where $t_j (j = 0, 1, \ldots, n)$ are the abscissas of the partition points of the integration interval $[a,b]$ or quadrature (interpolation) nodes, $A_j (j = 0, 1, \ldots, n)$ are numerical coefficients that do not depend on the function $y(t)$ and $\varepsilon_n(y)$ is the truncation error of Eq. (2). To facilitate in formulating the full- and half-sweep approximation equations for linear Fredholm equation of the first kind, further discussion will be restricted onto repeated trapezoidal rule, which is based on linear interpolation formulas with equally spaced data. Based on repeated trapezoidal rule, numerical coefficients A_j are satisfied by the following relation

$$A_j = \begin{cases} \frac{1}{2}h, & j = 0, n \\ h, & j = 1, 2, \cdots, n-1 \end{cases} \tag{3}$$

where the constant step size, h is defined as

$$h = \frac{b-a}{n} \tag{4}$$

and n is the number of subintervals in the interval $[a,b]$. In this paper, interval $[a,b]$ will be uniformly divided into $n = 2^m, m \geq 2$ and then consider the discrete set of points be given as $x_i = a + ih$.

By applying Eq. (2) into Eq. (1) and neglecting the error,$\varepsilon_n(y)$, a system of linear algebraic equations can be formed for approximation values $y(x)$ at the nodes $x_0, x_1, \ldots, x_n$. The following linear system generated either by the full- or half-sweep approximation equation can be easily shown as

$$M \underset{\sim}{y} = \underset{\sim}{f} \tag{5}$$

where

$$M = \begin{bmatrix} A_0 K_{0,0} & A_{1p} K_{0,1p} & \cdots & A_n K_{0,n} \\ A_0 K_{1p,0} & A_{1p} K_{1p,1p} & \cdots & A_n K_{1p,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_0 K_{n,0} & A_{1p} K_{n,1p} & \cdots & A_n K_{n,n} \end{bmatrix}_{((\frac{n}{p})+1) \times ((\frac{n}{p})+1)}$$

$$\underset{\sim}{y} = \begin{bmatrix} y_0 & y_{1p} & \cdots & y_{n-p} & y_n \end{bmatrix}^T$$

$$\underset{\sim}{f} = \begin{bmatrix} f_0 & f_{1p} & \cdots & f_{n-p} & f_n \end{bmatrix}^T$$

The value of p, which corresponds to 1 and 2, represents the full- and half-sweep cases respectively.

3 Formulation of Geometric Mean Method

FSGM and HSGM methods are the two-stage iterative methods. Hence, the iterative process for these methods involves two levels of virtual time such as $\underset{\sim}{y}^{(1)}$ and $\underset{\sim}{y}^{(2)}$. To develop formulation of FSGM and HSGM methods, coefficient matrix M in Eq. (5) needs to be decomposed into

$$M = L + D + U \tag{6}$$

where L, D and U are strictly lower triangular, diagonal and strictly upper triangular matrices respectively. The general scheme for both GM methods is given by

$$\left.\begin{array}{l} (D+rL)\,\underset{\sim}{y}^{(1)} = (\,(1-r)D-rU)\,\underset{\sim}{y}^{(k)} + r\,\underset{\sim}{f} \\ (D+rU)\,\underset{\sim}{y}^{(2)} = (\,(1-r)D-rL)\,\underset{\sim}{y}^{(k)} + r\,\underset{\sim}{f} \\ \underset{\sim}{y}^{(k+1)} = \sqrt{\underset{\sim}{y}^{(1)}\underset{\sim}{y}^{(2)}} \end{array}\right\} \tag{7}$$

where r and $\underset{\sim}{y}^{(k)}$ represent as an acceleration parameter and an unknown vector at the k^{th} iteration respectively.

The value of r will be determined by implementing some numerical experiments by using computer programs. A value of r will be chosen by considering the smallest number of its iteration. By determining values of matrices L, D and U as stated in Eq. (6), the general algorithm for FSGM and HSGM schemes in Eq. (7) would be described in Algorithm 1. The FSGM and HSGM algorithms are explicitly performed by using all equations at level (1) and (2) alternatively until the specified convergence criterion is satisfied.

Algorithm 1: FSGM and HSGM schemes

i) Level (1)

For $i = 0,\ 1p,\ 2p,\ \cdots,\ n-p,\ n$

Calculate

$$y_i^{(1)} \leftarrow \begin{cases} \left((1-r)A_iK_{ii}y_i^{(k)} - r\sum\limits_{j=1p}^{n} A_jK_{ij}y_j^{(k)} + rf_i\right)/A_iK_{ii} & ,i=0 \\ \left((1-r)A_iK_{ii}y_i^{(k)} - r\sum\limits_{j=0}^{n-p} A_jK_{ij}y_j^{(1)} + rf_i\right)/A_iK_{ii} & ,i=n \\ \left((1-r)A_iK_{ii}y_i^{(k)} - r\sum\limits_{j=0}^{i-p} A_jK_{ij}y_j^{(1)} - r\sum\limits_{j=i+p}^{n} A_jK_{ij}y_j^{(k)} + rf_i\right)/A_iK_{ii} & ,others \end{cases}$$

ii) Level (2)

For $i = n,\ n-p,\ \cdots,\ 2p,\ 1p,\ 0$

Calculate

$$y_i^{(2)} \leftarrow \begin{cases} \left((1-r)A_iK_{ii}y_i^{(k)} - r\sum\limits_{j=1p}^{n} A_jK_{ij}y_j^{(2)} + rf_i\right)/A_iK_{ii} & ,i=0 \\ \left((1-r)A_iK_{ii}y_i^{(k)} - r\sum\limits_{j=0}^{n-p} A_jK_{ij}y_j^{(k)} + rf_i\right)/A_iK_{ii} & ,i=n \\ \left((1-r)A_iK_{ii}y_i^{(k)} - r\sum\limits_{j=0}^{i-p} A_jK_{ij}y_j^{(k)} - r\sum\limits_{j=i+p}^{n} A_jK_{ij}y_j^{(2)} + rf_i\right)/A_iK_{ii} & ,others \end{cases}$$

iii) For $i = 0,\ 1p,\ 2p,\ \cdots,\ n-p,\ n$

Calculate

$$y_i^{(k+1)} \leftarrow \sqrt{{y_i}^{(1)}{y_i}^{(2)}}$$

In comparison, the Full-Sweep Gauss-Seidel (FSGS) acts as the control of comparison of numerical results. In the implementation of the FSGS, FSGM and HSGM methods, the convergence test considered the tolerance error, $\varepsilon = 10^{-10}$.

4 Numerical Results

In order to verify the effectiveness of the proposed methods, several numerical tests were conducted. Three criteria will be considered in comparison for FSGM and HSGM methods such as number of iterations, execution time and maximum absolute error. As mentioned above, repeated trapezoidal method is used to discretize and to form a system of linear algebraic equation for the following examples.

Example 1 (Dobner, [6])

$$\int_0^1 K(x,t)\,y(t)\,dt = \frac{1}{6}(x^3 - x) \quad , \quad 0 \leq x \leq 1,$$

with kernel

$$K(x,t) = \begin{cases} t(x-1), & t < x \\ x(t-1), & x \leq t \end{cases}$$

and the exact solution of the problem is given by

$$y(x) = x.$$

Results of numerical experiments, which were obtained from implementations of the FSGS, FSGM and HSGM methods for Example 1, have been recorded in Table 1. Figs. 2 and 3 show number of iterations and execution time versus mesh size respectively for Example 1.

Example 2 (Rashed, [13])

$$\int_0^1 K(x,t)\,y(t)\,dt = e^x + (1-e)x - 1 \quad , \quad 0 \leq x \leq 1,$$

with kernel

$$K(x,t) = \begin{cases} t(x-1), & t \leq x \\ x(t-1), & x < t \end{cases}.$$

Exact solution of the problem is

$$y(x) = e^x.$$

For Example 2, numerical results of FSGS, FSGM and HSGM methods have been recorded in Table 2. Graphs of number of iterations and execution time versus mesh size for Example 2 shown in Figs. 4 and 5.

Table 1: Comparison of a Number of Iterations, Execution Time (Seconds) and Maximum Absolute Error for the Iterative Methods (Example 1)

Methods	Number of iterations				
	Mesh size				
	64	128	256	512	1024
FSGS	185	243	309	381	461
FSGM	102	103	104	104	104
HSGM	96	102	103	104	104
Methods	Execution time (seconds)				
	Mesh size				
	64	128	256	512	1024
FSGS	0.17	0.98	5.05	25.04	115.79
FSGM	0.16	0.88	3.52	14.22	56.35
HSGM	0.05	0.28	1.26	5.11	20.55
Methods	Maximum absolute error				
	Mesh size				
	64	128	256	512	1024
FSGS	5.0222 E-10	6.6421 E-10	6.4440 E-10	6.8225 E-10	8.3429 E-10
FSGM	5.6496 E-10	1.0102 E-9	1.0537 E-9	1.2998 E-9	1.4637 E-9
HSGM	7.3854 E-10	5.6496 E-10	1.0102 E-9	1.0537 E-9	1.2998 E-9

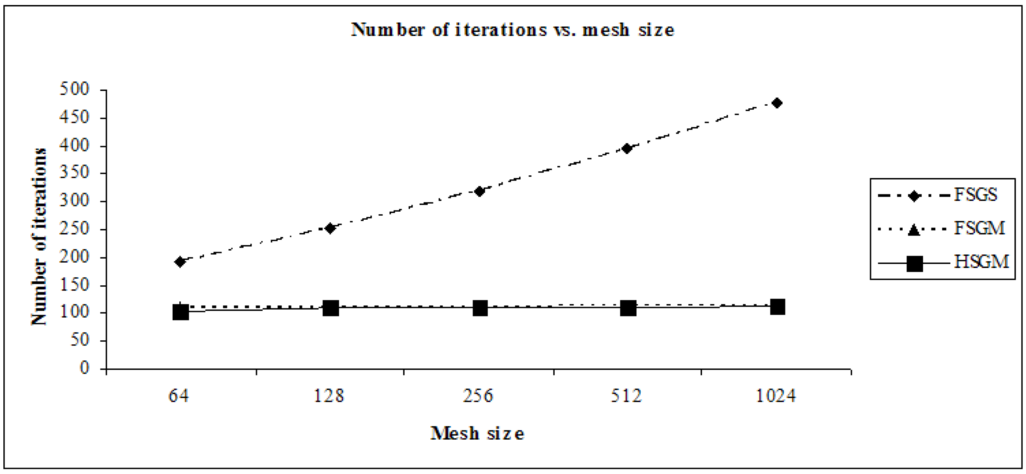

Figure 2: Number of Iterations Versus Mesh Size of the FSGS, FSGM and HSGM Methods for Example 1

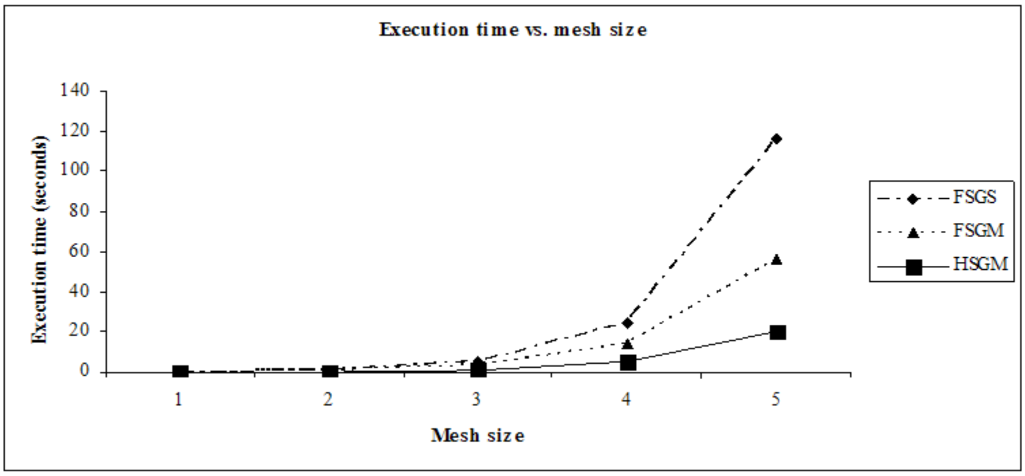

Figure 3: The Execution Time (Seconds) Versus Mesh Size of the FSGS, FSGM and HSGM Methods for Example 1

Table 2: Comparison of a Number of Iterations, Execution Time (Seconds) and Maximum Absolute Error for the Iterative Methods (Example 2)

Methods	Number of iterations				
	Mesh size				
	64	128	256	512	1024
FSGS	193	253	321	394	478
FSGM	110	112	112	113	113
HSGM	105	110	112	112	113
Methods	Execution time (seconds)				
	Mesh size				
	64	128	256	512	1024
FSGS	0.22	1.10	5.82	28.78	124.18
FSGM	0.17	0.99	4.06	14.67	58.72
HSGM	0.06	0.17	0.88	3.51	14.29
Methods	Maximum absolute error				
	Mesh size				
	64	128	256	512	1024
FSGS	5.4447 E-5	1.3718 E-5	3.4430 E-6	8.6243 E-7	2.1582 E-7
FSGM	5.4446 E-5	1.3718 E-5	3.4425 E-6	8.6183 E-7	2.1553 E-7
HSGM	5.2775 E-4	1.3506 E-4	3.4162 E-5	8.5903 E-6	2.1534 E-6

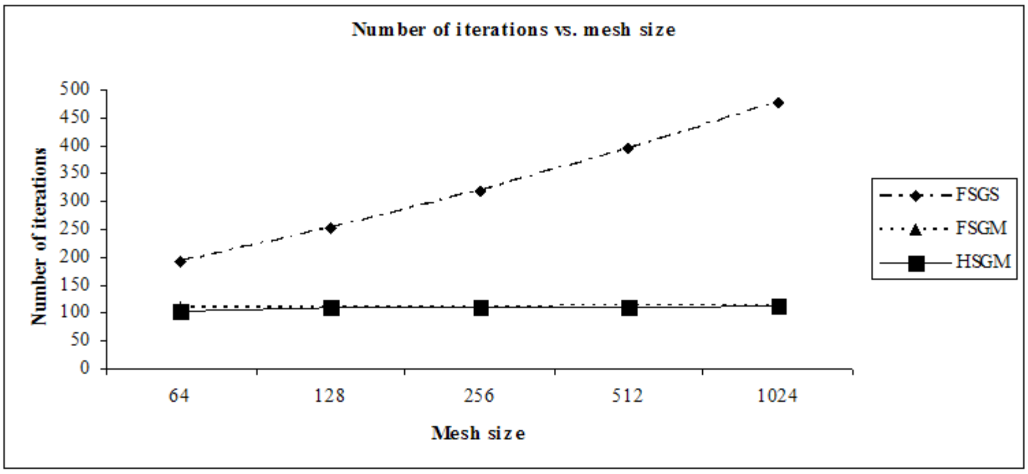

Figure 4: Number of Iterations Versus Mesh Size of the FSGS, FSGM and HSGM Methods for Example 2

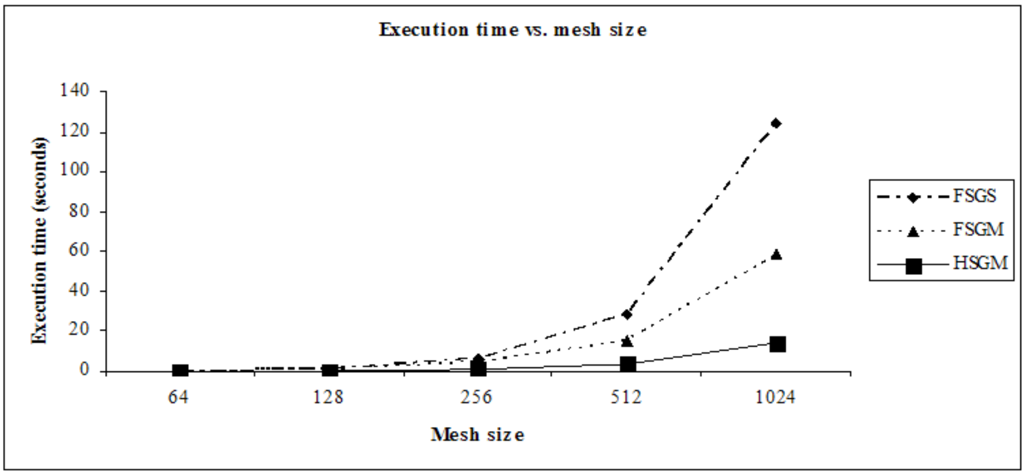

Figure 5: The Execution Time (Seconds) Versus Mesh Size of the FSGS, FSGM and HSGM Methods for Example 2

5 Conclusion

In the previous section, the formulation of full- and half-sweep quadrature approximation equations based on repeated trapezoidal rule can easily generate a system of linear algebraic equations as shown in Eq. (5). Through numerical results obtained for Example 1 (in Table 1), shows that number of iterations of FSGM and HSGM methods decreased approximately 44.86% - 77.44% and 48.11% - 77.44% respectively compared to the FSGS method. In terms of execution time, both the FSGM and HSGM methods are much faster than the FSGS method about 5.88% - 51.33% and 68.75% - 82.25% respectively. Number of iterations for FSGM and HSGM iterative methods for Example 2 as shown in Table 2 decreased approximately 43.01% - 76.36% and 45.60% - 76.36% compared with FSGS method. Through the observation in Table 2 and Fig. 5, show that execution time for FSGM and HSGM methods decreased about 22.73% - 52.71% and 72.73% - 88.49% respectively compared to the FSGS method.

Overall, the numerical results prove that the HSGM iterative method is a better method compared with the FSGS and FSGM methods. This is due to the computational complexity of the HSGM method is approximately 50% less than FSGM method.

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