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Numerical Comparisons of Blasius and Sakiadis Flows

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Abstract Momentum laminar boundary layers of an incompressible fluid either about a moving plate in a quiescent ambient fluid (Sakiadis flow) and the flow induced over a resting flat-plate by a uniform free stream (Blasius flow) are investigated simultaneously. The resulting similarity momentum equation is solved numerically, and the mechanical characteristics are also presented.

Keywords Laminar boundary layers; Blasius flow; Sakiadis flow.

1 Introduction

Steady flow of viscous incompressible fluids has attracted considerable attention in recent years due to its crucial role in numerous engineering applications. Numerical analysts encounter actually a wide variety of challenges in obtaining suitable algorithms for computing flow and heat transfer of viscous fluids. Since the pioneering work of Howarth [1] various aspects of the problem have been investigated by many authors. In [1], hand computations using the Runge-Kutta numerical method were performed for the flat-plate flow. Blasius solution for flow past a flat-plate was investigated by Abussita [2], and the existence of a solution was established. Asaithambi [3], presented a finite-difference method for the solution of the Falkner-Skan equation, and, recently, Wang [4] obtained an approximate solution for classical Blasius equation using Adomian decomposition method. Further, Asaithambi [5–7] studied numerically the Falkner-Skan equation by using shooting, finite-differences and finite elements, respectively. Blasius flow was also numerically analyzed by Cortell [8], and very recently a shooting procedure has been used by Zhang and Chen [9]in order to obtain interesting results of the Falkner-Skan equation.

On the other hand, different from Blasius [10], Sakiadis [11, 12]considered the boundary layer flow on a moving flat surface in a quiescent ambient fluid. He found the same ordinary differential equation (ODE) as Blasius, but the boundary conditions were different. Tsou et al. [13] made an experimental and theoretical treatment of this problem to prove that such a flow is physically realizable. Based on the fact that a single ODE (i.e., Blasius equation) governs both Blasius and Sakiadis flows, an innovative way of researching these two classical boundary-layer flows is to discuss both simultaneously in a single paper, and then interesting comparisons can be achieved, although unfortunately, there have only been a small number of studies in that direction. [14–16]However, although separately, different effects like suction/blowing, radiation, etc. on the above mentioned classes of flow are discussed in most recent papers by Ishak et al. [17, 18] ,Fang [19] and Cortell [20, 21]. Moreover, recent researches of the boundary layer flow and heat/mass transfer on a moving flat plate in a parallel stream have also been carried out by Cortell [22],Ishak et al. [23] and Ishak [24].Withal, significant differences were encountered (see Blasius [10], Sakiadis [11], Sakiadis [12], Tsou [13], Magyari [14], El-Arabawy [15],Cortell [16]) between Blasius and Sakiadis flows. For example, the skin friction is about 34% higher for the Sakiadis flow compared to the Blasius case.

The objective of the present paper is, therefore, to show that the aforementioned discrepancies exist when both classes of the flow are compared. The mechanical characteristics of such boundary-layer flows are investigated numerically by solving, for each case, a single initial value problem (IVP) via 4^{th} order Runge-Kutta algorithm along with shooting procedure.

2 The Problem to Solved and Its Solution Procedure

The governing equations of motion for the classical Blasius flat-plate flow problem can be summarized by the following boundary value problem (BVP) [8]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2}. \tag{2}$$

The boundary conditions for the velocity field are

$$u = v = 0 \text{ at } y = 0; u = U_\infty \text{ at } x = 0, \tag{3.a}$$

$$u \to U_\infty \text{ as } y \to \infty, \tag{3.b}$$

for the Blasius flat-plate flow problem, and

$$u = U_w;\ v = 0 \text{ at } y = 0, \tag{4.a}$$

$$u \to 0 \text{ as } y \to \infty, \tag{4.b}$$

for the classical Sakiadis flat-plate flow problem, respectively.

Introducing a similarity variable η and a dimensionless stream function $f(\eta)$ as

$$\eta = y\sqrt{\frac{U}{\upsilon x}} = \frac{y}{x}\sqrt{Re_x}, \tag{5}$$

$$\frac{u}{U} = f' \ ;\ v = \frac{1}{2}\sqrt{\frac{U\upsilon}{x}}(\eta f' - f), \tag{6}$$

where a prime denotes differentiation with respect to η and Re_x is the local Reynolds number $(= \frac{Ux}{\upsilon})$, we obtain by applying Eqs. (5) and (6)

$$\frac{\partial u}{\partial x} = -\frac{U}{2}\frac{\eta}{x}f''; \ \frac{\partial v}{\partial y} = \frac{U}{2}\frac{\eta}{x}f'' \tag{7}$$

and the equation of continuity (i.e., Eq. (1)) is satisfied identically.

On the other hand, we get

$$\frac{\partial u}{\partial y} = Uf''\sqrt{\frac{U}{\upsilon x}}; \ \frac{\partial^2 u}{\partial y^2} = \frac{U^2}{\upsilon x}f'''. \tag{8}$$

Note that in Eqs.(5)-(8), $U = U_\infty$represents Blasius flow, whereas $U = U_w$indicates Sakiadis flow, respectively.

By inserting Eqs. (6)-(8) into Eq. (2) we get

$$f''' + \frac{1}{2} f f'' = 0, \tag{9}$$

and the transformed boundary conditions for the momentum Eq. (17) are

$$f = 0,\ f' = 0 \text{ at } \eta = 0, \tag{10.a}$$

$$f' \to 1 \text{ as } \eta \to \infty, \tag{10.b}$$

for the Blasius flow, and

$$f = 0,\ f' = 1 \text{ at } \eta = 0, \tag{11.a}$$

$$f' \to 0 \text{ as } \eta \to \infty, \tag{11.b}$$

for the Sakiadis case, respectively.

Without a break, we begin now the development of the numerical procedure for solving the momentum transfer problem for each case (i.e., the Blasius and Sakiadis flows). For this purpose, Equation (9) can easily be written as the equivalent first-order system

$$\begin{aligned} w_1' &= w_2, \\ w_2' &= w_3, \\ w_3' &= -\frac{w_1 w_3}{2}. \end{aligned} \tag{12}$$

Here $w_1 = f(\eta)$. In accordance with boundary conditions (10) (Blasius flow) we obtain

$$w_1(0) = 0,\ w_2(0) = 0, \tag{13}$$

whereas boundary conditions (11) (Sakiadis flow) give

$$w_1(0) = 0,\ w_2(0) = 1. \tag{14}$$

Using numerical methods of integration and disregarding temporarily the boundary conditions at infinity (10.b) and (11.b), a family of solutions of (12) can be obtained for arbitrarily chosen values of

$$w_3(0) = \left(\frac{d^2 f}{d\eta^2}\right)_{\eta=0}.$$

Tentatively one can assume in the Blasius case that a special value of $f''(0)$ yields a solution for which f vanishes at a certain $\eta = \eta_\infty$ and simultaneously satisfies the additional conditions

$$f' \to 1,\ f'' \to 0 \text{ as } \eta \to \eta_\infty. \tag{15}$$

Analogously, for the Sakiadis flow, we will estimate a special value of $|f''(0)|$ for which

$$f' \to 0,\ |f''| \to 0 \text{ as } \ \eta \to \eta_\infty. \tag{16}$$

In our shooting procedure, we guess $w_3(0)$ and integrate equations (12) along with either boundary conditions (13) and (15) (i.e., Blasius flow) or boundary conditions (14) and (16) (i.e., Sakiadis flow) as an IVP problem via fourth-order Runge-Kutta algorithm. It is obvious that, in each case, one missed value at $\eta = 0$ is guessed, that is, the dimensionless gradient velocity at the wall $f''(0) = w_3(0)$, and there is no necessity to select the η_∞ value before any calculation.

Unfortunately, standard numerical methods do not provide the additional conditions (15) or (16) automatically. Note that the $f'(\eta)$ values are essentially zero at $\eta = \eta_\infty$for the Sakiadis case. If our $f'(\eta)$computed values include some small negative results, these are not physically meaningful, and it is clear then that a non-negativity for $f'(\eta)$ over $[0, \eta_\infty]$ must be imposed. On the other hand, solutions with a low qualitative behaviour can be obtained as a consequence of the user specifying a $\Delta\eta$ too lax; however, there is no defect of the numerical scheme for guaranteeing non-negative results for $f'(\eta)$ over $[0, \eta_\infty]$. We will show that our approach can deal with all these difficulties in a satisfactory way.

Certainly other schemes are possible, and perhaps to be preferred for specific kinds of problems, but ours has proved successful for all the examples throughout the paper.

The results of the numerical solutions for $f(\eta)$ and its derivatives are given in Tables 1-2. Further, we obtain $w_3(0)_{Sakiadis}$ = -0.44374733, and $w_3(0)_{Blasius} = 0.332068884$. The difference between the present result for the Blasius flow (i.e.,$w_3(0)_{Blasius} = 0.332068884$) and the result obtained by Howarth [1] (i.e., $f''(0) = w_3(0) = 0.33206$) is negligible. In order to clarify our numerical procedure we also display in Figs. 1-2 the obtained results.

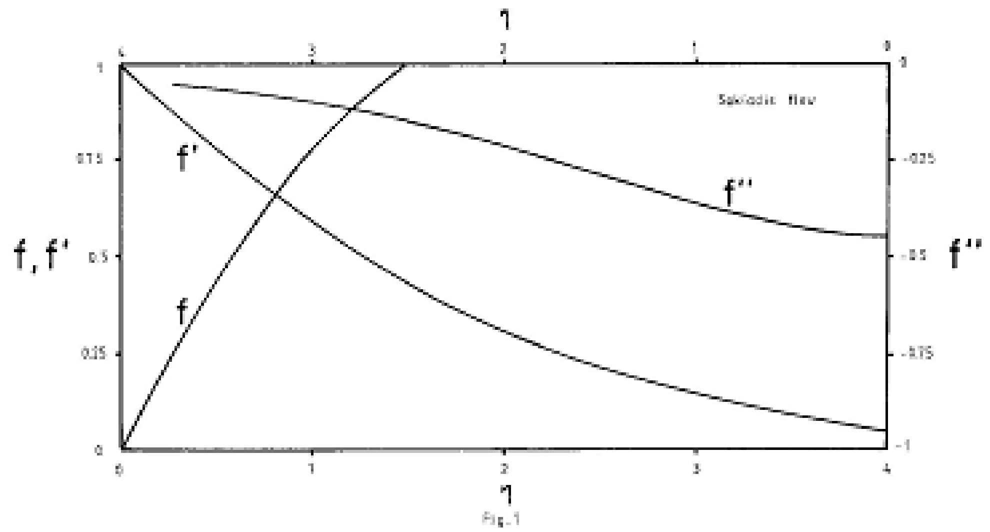

Figure 1: Plot of the Functions *f, f* and *f* for the Sakiadis Flow

Table 1: Sakiadis Momentum Transfer Solution with $\Delta\eta = 0.005$.

η	f	f'	f''
0.0	0.0	1.0	0.44374733
0.1	0.09778216	0.9556616	0.44265570
0.2	0.19113950	0.9115386	0.43946170
0.3	0.28010390	0.8678343	0.43430680
0.4	0.36472660	0.8247367	0.42735390
0.5	0.44507720	0.7824172	0.41878160
0.6	0.52124110	0.7410279	0.40877870
0.7	0.59331820	0.7007023	0.39753900
0.8	0.66142070	0.6615546	0.38525610
0.9	0.72567140	0.6236793	0.37211950
1.0	0.78620150	0.5871525	0.35831140
1.5	1.03801200	0.4262417	0.28477490
2.0	1.21855200	0.3017830	0.21450470
3.0	1.43273000	0.1440157	0.10983430
4.0	1.53308000	0.0662434	0.05215941
5.0	1.57884400	0.0299497	0.02392260
6.360	1.60365200	0.0100568	0.00809517
6.365	1.60370200	0.0100164	0.00806278
6.370	1.60375200	0.0099761	0.00803052
10.0	1.61546300	0.0005329	0.00043052
15.0	1.61611200	0.0000094	0.00000758
20.0	1.61611200	0.0000001	0.00000013

Table 2: Blasius Momentum Transfer Solution with $\Delta\eta = 0.001$
(Parenthesis indicates numerical results at $\eta = \eta_\infty$)

η	f	f'	f''
0.0	0.0	0.0	0.332068884
0.1	0.00166034	0.03320667	0.33205960
0.2	0.00664123	0.06641009	0.33199500
0.3	0.01494198	0.09960201	0.33182030
0.4	0.02656081	0.13276870	0.33148050
0.5	0.04149425	0.16589080	0.33092110
0.6	0.05973667	0.19894380	0.33008900
0.7	0.08127970	0.23189780	0.32893160
0.8	0.10611170	0.26471780	0.32739830
0.9	0.13421740	0.29736340	0.32544090
1.0	0.16557710	0.32979030	0.32301490
2.0	0.65004250	0.62978100	0.26675310
3.0	1.39684200	0.84605870	0.16135730
4.0	2.30579200	0.95552900	0.06423087
4.909	3.19316700	0.98997120	0.01843040
4.910	3.19415700	0.99000800	0.01840099
4.911	3.19514700	0.99002640	0.01837162
5.0	3.28332900	0.99154960	0.01590542
6.0	4.27968600	0.99897960	0.00240174
7.0	5.27933300	0.99992830	0.00022013
7.705	5.98428200	0.99999990	0.00003024
7.706	(5.98528200)	(1.00000000)	(0.00003014)
8.820	7.09920100	1.00000000	0.00000079

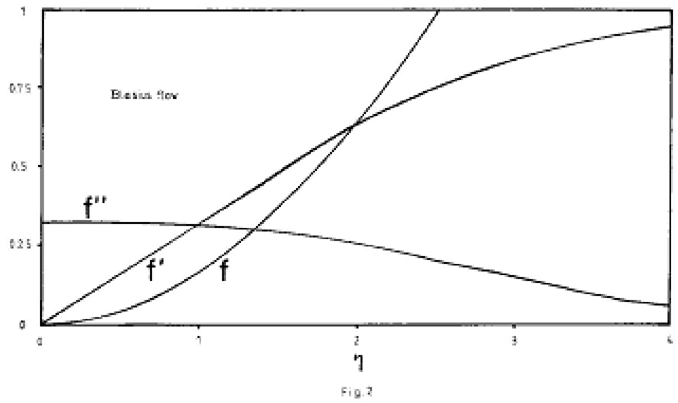

Figure 2: Plot of the Functions f, f' and f'' for the Blasius Flow

In order to clarify the numerical procedure in a semi-infinite domain where the solution for $f(\eta)$ decays to zero at infinity, we have showed in Table 3 some computed values for some "sufficiently large" η_∞, which must be obtained as part of the solution. To this purpose, Table 3 displays and compares (see [9])present results for the Blasius case at η = 8.820. Table 3 shows how our numerical results are in excellent accordance with earlier existing ones in the literature, and it is worth citing that our approaches have been applied in different studies since early 1990s [25].

Table 3: Far Field Boundary Conditions and Results Obtained with $\Delta\eta = 0.001$ for the Blasius Momentum Transfer Solution at $\eta = 8.820$ for Different Values of $f''(0)$

η	$f''(0)$	f	f'	f''
8.820	0.3320573[9]	7.099060	0.9999770	0.00000079
	0.332057336[9]	7.099063	0.9999772	0.00000079
	0.332059835	7.099095	0.9999821	0.00000079
	0.332068830	7.099198	0.9999999	0.00000079
	0.332068884	7.099201	1.0000000	0.00000079

Also, one can see from Table 4 that the integration domain is larger for the Sakiadis flow compared to the Blasius case. As far as flow kinematics is concerned and taking into consideration Tables 1-2, we also reflect in this table the non-dimensional thickness $\eta_\delta$as metricconverterProductID4.910 in4.910 in the Blasius case (i.e., the value of η for which$f' = 0.99$) as well as η_δ= metricconverterProductID6.365 in6.365 in the Sakiadis case (the value of η for which $f' = 0.01$). It is worth citing that in Sakiadis flow and from second

Table 4: Mechanical Characteristics of the Boundary Layer Flows Treated in This Work

	$f''(0)$	η_δ	η_∞	f_∞
Blasius Flow	0.332068884	4.910	7.706	–
Sakiadis Flow	-0.44374733	6.365	–	1.6161120

Eq. (6) at $\eta = \eta_\infty$ we have

$$v_\infty = -\frac{1}{2}\sqrt{\frac{Uv}{x}} f_\infty, \tag{17}$$

and from Table 4 will result

$$v_\infty = -0.808056\sqrt{\frac{Uv}{x}}, \tag{18}$$

from which the amount of fluid dragged by the moving sheet can be evaluated. Further, the values of $f(0)$ presented in Table 4 can be reduced to those of Blasius and Sakiadis cases if the factor$\frac{1}{2}$in Eq. (9) is neglected to give$f''(0)\sqrt{2} = 0.469616andf''(0)\sqrt{2} = -0.627553$,respectively. See, for example, [26].

3 Conclusion

In summary, we have used in this paper a numerical approach to give suitable solutions to two classical boundary-layer problems in fluid mechanics. The non-linear ODE (9) satisfying the boundary conditions either (10) or (11) has been solved numerically using a 4^{th} order Runge-Kutta shooting method. In order to obtain physically well-performed numerical results, the behaviour of $f''(\infty)$ for each numerical solution has been accounted throughout the paper. For the case of static flat-plate (i.e., $U_w = 0$, Blasius flow) the magnitude of the wall shear $f''(0) = 0.332068884$ is smaller in comparison to the situation of moving flat-plate (i.e., $U_\infty = 0$, Sakiadis flow), in which case we found $|f''(0)| = 0.44374733$. This trend was already predicted by Sakiadis [11] theoretically. His conclusion (later corroborated by Tsou et al. [13]experimentally) dealt with an increase in the wall shear about of 34% (see also) [27].Our numerical result for this prediction is easily seen to be

$$33.63\% = \frac{0.44374733 - 0.332068884}{0.332068884} \times 100.$$

All of our computations verify that the proposed procedure offers an effective tool for solving this nonlinear problem in fluid mechanics.

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